Representing Integral Monoids By Inequalities

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ABSTRACT. It is shown how any integral monoid can be represented as the projection of the intersection of the solution set of a finite collection of linear inequalities, and a lattice, both in a possibly higher dimension. This in turn can be used to derive a known representation using Chvátal functions, in the same dimension as the monoid. Both representations can be regarded as discrete analogues of the classical theorems of Weyl and Minkowski, but applicable in non-polyhedral monoids.

1 Introduction

Definition. An *integral monoid* M is a set of integral vectors containing 0 that is closed under addition. M is *finitely generated* by $\{a_1, a_1, \ldots, a_n\} \subseteq \mathbb{Z}^m$ if $M = \{Ax \mid x \in \mathbb{Z}_+^n\}$ where A is the $m \times n$ matrix whose columns are the vectors a_j.

Finitely generated integral monoids are clearly a generalisation of ideals but because of the absence of the subtraction operation they lack the structure of ideals. Sometimes they are referred to as "quasi-ideals" (eg Bostock [2]).

Since an integral monoid lacks the richness of structure of other mathematical entities we show that by possibly "lifting" it into dimension n, if $n > m$, any integral monoid can be regarded as the intersection of a polyhedral cone and a lattice. Both these types of structures have well developed and understood properties (eg duality properties). Hopefully our treatment will enable further results to be obtained regarding integer monoids.

Our main interest in investigating integral monoids is that the set of feasible right-hand-side vectors for a Pure Integer Programme (which can, without loss of generality be assumed to have integral coefficients) forms a finitely generated integer monoid. This aspect is considered by Ryan [5].

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Integral monoids also play a part in the definition of the superadditive dual of an Integer Programme (see eg Blair and Jeroslow [1]).

Integral monoids can be regarded as the discrete analogue of convex cones. However, the analogues of the theorems of Weyl [8] and Minkowski [4] for convex cones do not hold. These theorems are stated below.

Definition. A *convex cone* S is the set of vectors in $\mathbb{R}^n$ containing 0 that is closed under the operation of taking non-negative linear combinations. It is *finitely generated* if it can be generated from a finite number of vectors (a *basis*).

Weyl's Theorem. *If a convex cone is finitely generated then it is the solution set of a finite set of homogeneous linear inequalities. Such a cone is known as polyhedral.*

Minkowski's Theorem. *If a cone is defined as the solution set of a finite set of homogeneous linear inequalities (eg it is polyhedral) then it can also be represented as the set of non-negative linear combinations of a finite set of vectors.*

For an integer monoid there may or may not be a finite set of homogeneous linear inequalities for which the integer solution set is the monoid. In the former case the monoid is said to be *polyhedral.*

Example 1. The (1-dimensional) integer monoid $\{0, 3, 6, 7, 9, 10, 12, 13, 14, \ldots\}$ generated by $\{3, 7\}$ is not polyhedral.

Trivially the associated convex cone (obtained by taking rational non-negative combinations of the generators) is $\{x \mid x \geq 0\}$ which is generated by 1.

Example 2. The integer monoid generated by $\left\{\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}7\\2\end{pmatrix}, \begin{pmatrix}9\\3\end{pmatrix}\right\}$ is polyhedral as it is the integer solûtion set of $\{-2x + 7y \geq 0, x - 3y \geq 0\}$. The associated polyhedral convex cone is the solution set of the inequalities obtained by ignoring the integrality requirements on x and y. It is generated by $\left\{\begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}7\\2\end{pmatrix}\right\}$.

It can be shown that a pointed (ie the associated convex cone has a vertex) polyhedral integral monoid has a unique smallest set of generators known as a Hilbert basis (see eg Schrijver [7]). The set of generators in example 2 is such a basis.

In section 2 we show that even if an integral monoid with n generators is not polyhedral it can be "lifted" into a space of dimension n (if $n > m$) so as to become the projection of lattice points within a polyhedral cone of dimension n. Example 1 illustrates this result, for the 1 dimensional monoid defined there is the projection of the integer lattice within the cone defined in example 2, onto the x-coordinate.

Ryan and Trotter [6], following Blair and Jeroslow have shown that

Weyl's and Minkowski's theorems have analogues for finitely generated integral monoids if one extends the linear functions of the inequalities to include (non-linear) Chvátal [3] functions.

Definition. The set of Chvátal functions C is the smallest class of functions $\mathcal{F}$ such that

(i) f linear $\Rightarrow f \in \mathcal{F}$

(ii) $f, g \in \mathcal{F}$ and $\alpha, \beta \in \mathbb{Q}_+ \Rightarrow (\alpha f + \beta g) \in \mathcal{F}$

(iii) $f \in \mathcal{F} \Rightarrow \lfloor f \rfloor \in \mathcal{F}$

where $\lfloor f \rfloor (x) = \lfloor f(x) \rfloor$, "$\lfloor\rfloor$" being the "integer round-down" operation.

We adopt the definition of Chvátal functions given by Blair and Jeroslow to include linear functions with negative coefficients but only allow the combination of functions by non-negative linear combinations. (Any negative coefficients therefore apply to variables at the "deepest" level within a rounding) eg $\lfloor -2x \rfloor + 3y$ is a Chvátal function. This is in contrast to the treatment of Schrijver. Chvátal functions are also used to derive cuts for Integer Programmes. When all the constraints are of the "$\leq$" form the Chvátal functions will contain no negative coefficients.

This result of Ryan and Trotter also follows from our result by eliminating the extra variables from the inequalities and congruences to produce inequalities involving Chvátal functions. The explanation is given in section 3. We illustrate the result by giving such a representation for the integral monoid of example 1.

Example 3. The non-polyhedral integer monoid generated by $\{3, 7\}$ is the solution set of $\{\lfloor \frac{1}{3}x \rfloor + \lfloor -\frac{2}{7}x \rfloor \geq 0\}$.

A third way of representing a non-polyhedral monoid by inequalities is also possible. This follows from applying an integer generalisation of Fourier-Motzkin Elimination due to Williams [9] (and further explained in Williams [10]) to eliminating the extra variables from the linear inequality representation. The resulting representation can be viewed as either the solution set of a finite disjunction of inequalities together with linear congruences in the variables or, equivalently the solution set of a finite set of linear inequalities and congruences involving extra bounded integer variables. Example 4 illustrates this representation for the non-polyhedral monoid used in example 1.

Example 4. The non-polyhedral monoid generated by $\{3, 7\}$ is the integer solution set of the finite disjunction

$$V_{s \in \{0,1,2\}}(x \geq 7s . x \equiv s \pmod 3)$$

or, equivalently, the integer solution set of

$$\{x - 7s \geq 0, 0 \leq s \leq 2, x - s \equiv 0 \pmod 3\}.$$

2 Lifting Integral Monoids to Create Polyhedral Monoids

We consider the integral monoid M generated by the columns of the $m \times n$ integral matrix $A = \|a_{ij}\|$ of rank r, ie $M = \{A\underline{x} \mid \underline{x} \in \mathbb{Z}_+^n\}$. For convenience we assume the rows of A have been permuted, so as to make the first r rows linearly independent (over $\mathbb{Q}$).

If A is not of full row rank we append any non-singular $(m-r) \times (m-r)$ matrix U to the last $(m-r)$ rows of A by appending $(m-r)$ extra columns to create

$$\tilde{A} = \left[A \mid \frac{o}{u}\right].$$

Clearly $\tilde{A}$ is of full row rank. Also we can express M by

$$M = \left\{\tilde{A}\begin{pmatrix}\underline{z}\\ \underline{z}'\end{pmatrix} \mid \underline{z} \in \mathbb{Z}_+^n, \underline{z}' = \underline{0}^{(m-r)}\right\}$$

ie $\overline{\underline{z}} = \begin{pmatrix}\underline{z}\\ \underline{z}'\end{pmatrix}$ is an $(n+m-r)$ nonnegative integer vector whose last $(m-r)$ components are set to zero.

While any non-singular $(m-r) \times (m-r)$ matrix U will suffice it is convenient to take $U = I_{m-r}$.

Lemma. *$\exists$ an $(n+m-r) \times (n+m-r)$ non-singular rational matrix B such that $\tilde{A}B = [I_m \mid 0]$.*

Proof: Since $\tilde{A}$ is of full row rank we can apply elementary column operations to $\tilde{A}$ to create an identity matrix in the first m columns. Applying these elementary column operations to the identity matrix I_{n+m-r} gives B.

Theorem. *Let $N = \left\{\underline{x} \mid B\begin{pmatrix}\underline{x}\\ \underline{y}\end{pmatrix} \geq \underline{0}, B\begin{pmatrix}\underline{x}\\ \underline{y}\end{pmatrix} \equiv \underline{0} \pmod{\frac{\underline{1}^{(n)}}{\underline{0}^{(m-r)}}}, \underline{x} \in \mathbb{Q}^m,\right.$ $\left.\underline{y} \in \mathbb{Q}^{n-r}\right\}$ then $M = N$, where $\underline{1}^{(n)}$ is an n-vector of 1s and $\underline{0}^{(m-r)}$ an $(m-r)$-vector of 0s. " mod $\underline{v}$" signifies equivalence of the corresponding expressions to the appropriate component of $\underline{v}$. If a component is 0 then the corresponding expression is constrained to be 0.*

Proof: Suppose $\underline{x} \in M$. Then $\exists \underline{z} \in \mathbb{Z}_+^n$, $\underline{z}' \in \underline{0}^{(m-r)}$ such that $\underline{x} = \tilde{A}\overline{\underline{z}}$ when $\overline{\underline{z}} = \begin{pmatrix}\underline{z}\\ \underline{z}'\end{pmatrix}$.

Define $\begin{pmatrix}\underline{s}\\ \underline{t}\\ \underline{u}\end{pmatrix} = B^{-1}\overline{\underline{z}}$ where $\underline{s}$ is an m-vector, $\underline{t}$ an $(n-m)$-vector and $\underline{u}$ an $(m-r)$-vector.

$B\begin{pmatrix}\underline{s}\\ \underline{t}\\ \underline{u}\end{pmatrix} = z$. Since $\underline{z} \in \mathbb{Z}_+^n$ and $\underline{z}' = \underline{0}^{(m-r)}$, $\underline{s}$ satisfies the conditions in the definition of N. Therefore $\underline{s} \in N$.

But $\underline{x} = \tilde{A}\underline{z} = \tilde{A}B\begin{pmatrix}\underline{s}\\ \underline{t}\\ \underline{u}\end{pmatrix} = [I_m \mid 0]\begin{pmatrix}\underline{s}\\ \underline{t}\\ \underline{u}\end{pmatrix} = \underline{s}$.

Hence $\underline{x} \in N$.

Conversely if $\underline{x} \in N$, $\exists \underline{y} \in \mathbb{Q}^{n-r}$ such that $\begin{pmatrix}\underline{s}\\ \underline{t}\\ \underline{u}\end{pmatrix} = B\begin{pmatrix}\underline{x}\\ \underline{y}\end{pmatrix}$ and $\underline{s} \in \mathbb{Z}_+^m$, $\underline{t} \in \mathbb{Z}_+^{n-m}$, $\underline{u} = \underline{0}^{(m-r)}$

$$\tilde{A}\begin{pmatrix}\underline{s}\\ \underline{t}\\ \underline{u}\end{pmatrix} = \tilde{A}B\begin{pmatrix}\underline{x}\\ \underline{y}\end{pmatrix} = [I_m \mid 0]\begin{pmatrix}\underline{x}\\ \underline{y}\end{pmatrix} = \underline{x}.$$

Hence $\underline{x} \in M$. Therefore $M = N$. □

The import of the theorem is that an integer monoid with n generators can be viewed as the projection of the intersection of an n dimensional lattice and cone. We illustrate this firstly for the monoid of example 1 and then for a more complicated example.

Example 1A. Let M be the non-polyhedral integer monoid generated by $\{3, 7\}$. It can be verified that, using the definitions of the lemma $\tilde{A} = A = [3, 7]$, $B = \begin{bmatrix}-2 & 7\\ 1 & -3\end{bmatrix}$.

Hence the monoid is the projection of the solution set of $\{-2x + 7y \geq 0, x - 3y \geq 0, x, y \in \mathbb{Z}\}$ (the polyhedral monoid of example 2) onto the x axis. The associated lattice is the integer lattice. Figure 1 illustrates this.

In figure 1 the "x"s represent the lattice points lying within the cone defined by the inequalities and the "x"' the projection of these points giving the monoid.

Example 5. Let M be the non-polyhedral integer monoid generated by $\left\{\begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}4\\12\end{pmatrix}, \begin{pmatrix}4\\8\end{pmatrix}\right\}$. It can be verified that

$$\tilde{A} = A = \begin{bmatrix}2 & 4 & 4\\ 3 & 12 & 8\end{bmatrix}, B = \begin{bmatrix}2 & -1 & 4\\ 0 & 0 & 1\\ -\frac{3}{4} & \frac{1}{2} & -3\end{bmatrix}.$$

After simplifying the resultant inequalities and congruences M can be seen to be the solution set of

$$\begin{aligned} 2x_1 - x_2 + 4x_3 &\geq 0 \\ x_3 &\geq 0 \qquad x_1 \equiv 0 \pmod{2}, x_2, x_3 \equiv 0 \pmod{1}. \\ -3x_1 + 2x_2 - 12x_3 &\geq 0 \end{aligned}$$

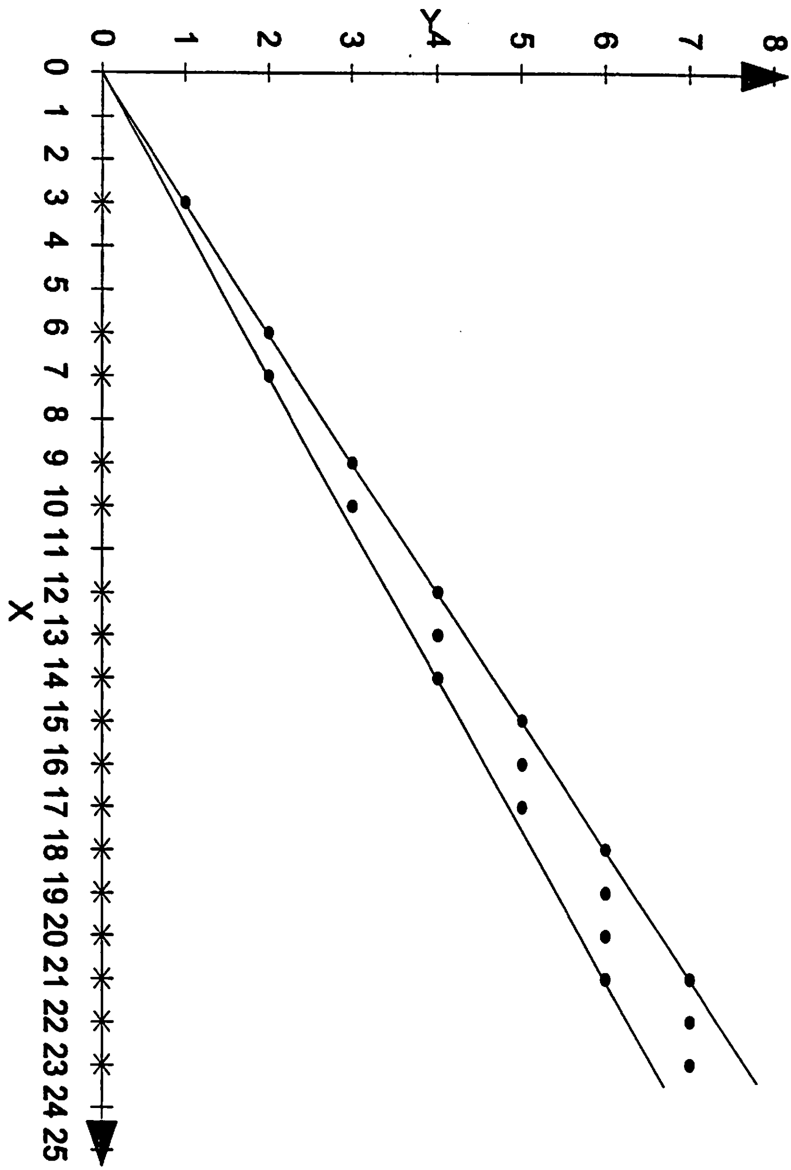

Figure 1

Example 6. Let M be the integer monoid generated by $\left\{ \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \right\}$.

It can be verified that A is of rank 2 and that therefore

$$\tilde{A} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 1 & 3 & 1 \end{bmatrix}, B = \begin{bmatrix} \frac{5}{6} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{6} & -\frac{2}{3} & 1 \end{bmatrix}.$$

After some simplification of the resultant inequalities and congruences M can be seen to be the solution set of

$$\begin{array}{ll} 5x_1 - 2x_2 \geq 0 & x_1 \equiv 0 \pmod 2), x_2 \equiv \ (\text{mod } 1) \\ -x_1 + x_1 \geq 0 & x_1 - 4x_2 + 6x_3 = 0 \end{array}$$

3 Representing Internal Monoids by Chvátal Inequalities

It has already been shown by Blair and Jeroslow and Ryan and Trotter that a finitely generated integral monoid can be represented as the solution set of a series of inequalities involving Chvátal functions, in the same dimension as the monoid, (and conversely). We use the result of section 2 to give an alternative proof of the theorem.

Theorem. *Let m be an integer monoid generated by the columns of the $m \times n$ integer matrix A. Then there exist m-dimensional Chvátal functions $f_1, f_2, \ldots, f_p$ such that*

$$M = \{\underline{x} \in \mathbb{Z}^m \mid f_1(\underline{x}) \geq \underline{0}, i = 1, 2, \ldots, p\}$$

and conversely.

Proof: Define $\tilde{A}$ and B as in section 2. Then introducing an integer vector $\begin{pmatrix} \underline{s} \\ \underline{t} \end{pmatrix}$, M is the solution set $\underline{x}$ of the integer programming polytope

$$B\begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix} - \begin{pmatrix} \underline{s} \\ \underline{t} \\ \underline{0}^{(m-r)} \end{pmatrix} = \underline{0}$$

$$\underline{s} \geq \underline{0}$$

$$\underline{t} \geq \underline{0}$$

$$\underline{x} \in \mathbb{Q}^m,\ \underline{y} \in \mathbb{Q}^{n-r},\ \underline{s} \in \mathbb{Z}^m,\ \underline{t} \in \mathbb{Z}^{n-m}.$$

For a given $\underline{x}$ the above inequalities define a mixed integer programme in variables $\underline{y}$, $\underline{s}$ and $\underline{t}$. From results in [1] the Consistency Tester is a set of Chvátal inequalities in $\underline{x}$. M is then the solution set of those Chvátal inequalities. □

In general it is difficult to calculate specific Consistency Testers except when $n - r = 1$ or when $m = n - r$ by a method described in Williams [11].

This result is illustrated by further considering example 2 to give the result in example 3 and also further considering example 4.

Example 2A. It is shown in example 2 that the monoid generated by $\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \end{pmatrix}, \begin{pmatrix} 9 \\ 3 \end{pmatrix} \right\}$ is the projection, onto x-space, of the integer solutions to

$$-2x + 7y \geq 0 \tag{1}$$
$$x - 3y \geq 0 \tag{2}$$

From (1)

$$-y \leq -\frac{2}{7}x \rightarrow -y \leq \left\lfloor -\frac{2}{7}x \right\rfloor \tag{3}$$

From (2)

$$y \leq \frac{1}{3}x \rightarrow y \leq \left\lfloor \frac{1}{3}x \right\rfloor \tag{4}$$

Adding (3) and (4) gives the feasibility condition on y

$$\left\lfloor \frac{1}{3}x \right\rfloor + \left\lfloor -\frac{2}{7}x \right\rfloor \geq 0 \tag{5}$$

The congruence relation (integrality) on x can be represented by

$$\lfloor x \rfloor - x = 0 \tag{6}$$

(5) is the Chvátal inequality given in example 3.

Example 4A. For the monoid, defined in example 5, we have

$$2x_1 - x_2 + 4x_3 \geq 0 \tag{7}$$
$$x_3 \geq 0 \tag{8}$$
$$-3x_1 + 2x_2 - 12x_3 \geq 0 \tag{9}$$

$$x_1 \equiv 0 \pmod 2), x_2, x_3 \equiv 0 \pmod{(1)} \tag{10}$$

Hence

$$-x_3 \leq \left\lfloor \frac{1}{2}x_1 - \frac{1}{4}x_2 \right\rfloor \tag{11}$$
$$-x_3 \leq 0 \tag{12}$$
$$x_3 \leq \left\lfloor -\frac{1}{4}x_1 + \frac{1}{6}x_2 \right\rfloor \tag{13}$$

Adding (11) to (13) and (12) to (13) gives

$$\left\lfloor \frac{1}{2}x_1 - \frac{1}{4}x_2 \right\rfloor + \left\lfloor -\frac{1}{4}x_1 + \frac{1}{6}x_2 \right\rfloor \geq 0 \qquad (14)$$

$$\left\lfloor -\frac{1}{4}x_1 + \frac{1}{6}x_2 \right\rfloor \geq 0 \qquad (15)$$

The congruence relations involving x_1 and x_2 can be represented by

$$\left\lfloor \frac{x_1}{2} \right\rfloor - \frac{x_1}{2} = o \qquad (16)$$

$$\lfloor x_2 \rfloor - x_2 = 0 \qquad (17)$$

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