Two Classes of Super-magic Quartic Graphs

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ABSTRACT. We deal with finite graphs which admit a labeling of edges by pairwise different positive integers from the set $\{1, 2, \ldots, |E(G)|\}$ in such a way that the sum of the labels of the edges incident to a particular vertex is the same for all vertices. We construct edge labelings for two families of quartic graphs, i.e., regular graphs of degree $d = 4$.

1 Introduction and definitions

In this paper all graphs are finite, simple, undirected and connected. Let G be such a graph with the vertex set $V(G)$ and the edge set $E(G)$, where $|V(G)|$ and $|E(G)|$ are the number of vertices and edges of G.

A graph G is *magic* [3] if and only if there exists a mapping f from $E(G)$ into the set of positive integers such that

(i) $f(e_i) \neq f(e_j)$ for all distinct $e_i, e_j \in E(G)$,

(ii) $\sum_{e \in E(G)} \alpha(v, e) f(e) = \lambda$ for all $v \in V(G)$,

where $\alpha(v, e)$ is 1 when the vertex v and the edge e are incident and 0 in the opposite case.

Magic graphs were first introduced by J. Sedláček [5]. A characterization of regular magic graphs was given by M. Dood [1]. Necessary and sufficient conditions for the existence of a magic graph can be found in [2,3].

We say that G is super-magic if and only if there exists a mapping f from $E(G)$ into the set $\{1, 2, \ldots, |E(G)|\}$ which satisfies the conditions (i), (ii). The mapping f is called a super-magic labeling of G and the value λ is the super-magic index of the super-magic labeling f.

B.M. Stewart [8] has proved (by construction of triangle labelings) that for $n > 5$, $n \not\equiv 0 \pmod 4$, the complete graph K_n is super-magic. It is easy to see that the classic concept of a magic square of n^2 boxes corresponds to the fact that the complete bipartite graph $K(n, n)$ is super-magic for $n \geq 3$ (see also [7]).

J. Sedláček in [6] considered the graph M_{2m} (also called the Möbius ladder) and constructed its super-magic labeling for m odd, $m \geq 3$.

The present notion of super-magic labelings is different from that of magic edge labelings defined in [4]. However, a super-magic labeling of a plane graph G, in our sense, is reminiscent of a magic edge labeling of the plane dual graph G^* of G as defined in [4].

This paper describes super-magic labelings for two classes of regular graphs of degree $d = 4$, which were obtained modifying of labeling L_{10} of the prism R_n [4].

2 Construction of graphs $\mathbb{R}_n$ and $\mathbb{Z}_n$

Let $J = \{1, 2, 3, \ldots, n\}$ and $I = \{1, 2, 3, \ldots, k\}$ be index sets. We make the convention that $x_{n+1} = x_1$ to simplify the notation. Suppose that n is even, $n \geq 4$. Let the prism R_n be the Cartesian product $P_2 \times C_n$ of a path on 2 vertices with a cycle on n vertices, embedded in the plane. We insert exactly one vertex x_i, $i \in J$, into each 4-sided face of R_n.

Case 1. If $n = 4k$ and $k \geq 1$, then into the internal (external) n-sided face of R_n we insert the vertices $y_1, y_2, \ldots, y_k$ $(z_1, z_2, \ldots, z_k)$ and consider the graph $\mathbb{R}_n$ with the vertex set $V(\mathbb{R}_n) = V_1 \cup V_2 \cup V_3$ and the edge set $E(\mathbb{R}_n) = E_1 \cup E_2 \cup E_3$, where $V_1 = \{x_i : i \in J\}$, $V_2 = \{y_i : i \in I\}$, $V_3 = \{z_i : i \in I\}$, $E_1 = \{x_i x_{i+1} : i \in J\}$, $E_2 = \bigcup_{j=0}^{3} \{x_{jk+i} y_i : i \in I\}$ and $E_3 = \bigcup_{j=0}^{3} \{x_{jk+i} z_i : i \in I\}$.

Case 2. If $n = 4k + 2$ and $k \geq 1$, then into the internal (external) n-sided face of R_n we insert the vertices $y, y_1, y_2, \ldots, y_k$ $(z_1, z_2, \ldots, z_k)$ and consider the graph $\mathbb{Z}_n$ whose vertex set is $V(\mathbb{Z}_n) = V_1 \cup V_2 \cup V_3 \cup \{y\}$ and the edge set is $E(\mathbb{Z}_n) = E_1 \cup \{x_i y_i : i \in I\} \cup \{x_{k+i} y_i : i \in I\} \cup \{x_{2k+1+i} y_i : i \in I\} \cup \{x_{3k+1+i} y_i : i \in I\} \cup \{x_{i+1} z_i : i \in I\} \cup \{x_{k+1+i} z_i : i \in I\} \cup \{x_{2k+2+i} z_i : i \in I\} \cup \{x_{3k+2+i} z_i : i \in I\} \cup \{x_1 y, x_n y, x_{2k+1} y, x_{2k+2} y\}$.

The $\mathbb{R}_n$ and $\mathbb{Z}_n$ are regular graphs of degree $d = 4$; let their vertices be labeled as in Figure 1 (if $n = 4k$) and Figure 2 (if $n = 4k + 2$).

3 Super-magic labelings of graphs $\mathbb{R}_n$ and $\mathbb{Z}_n$

If $n = 4k \geq 8$, we construct an edge labeling f_1 of the regular graph $\mathbb{R}_n$ in the following way.

$$f_1(x_i x_{i+1}) = \begin{cases} 3i+2 & \text{if } 1 \leq i \leq 2k-2, \\ 6k & \text{if } i = 2k-1, \\ 4 & \text{if } i = 2k, \\ 18k-1-3i & \text{if } 2k+1 \leq i \leq 4k-2, \\ 6k+1 & \text{if } i = 4k-1, \\ 12k-3 & \text{if } i = 4k, \end{cases}$$

$$f_1(x_i y_i) = \begin{cases} 2 & \text{if } i = 1, \\ 12k-3i & \text{if } 2 \leq i \leq k, \end{cases}$$

$$f_1(x_{k+i} y_i) = \begin{cases} 9k-3i & \text{if } 1 \leq i \leq k-2, \\ 6k+2 & \text{if } i = k-1, \\ 12k & \text{if } i = k, \end{cases}$$

$$f_1(x_{2k+i} y_i) = \begin{cases} 12k-1 & \text{if } i = 1, \\ 3i+1 & \text{if } 2 \leq i \leq k, \end{cases}$$

$$f_1(x_{3k+i} y_i) = \begin{cases} 3k+1+3i & \text{if } 1 \leq i \leq k-2, \\ 6k-1 & \text{if } i = k-1, \\ 1 & \text{if } i = k, \end{cases}$$

$$f_1(x_i z_i) = \begin{cases} 12k-3i+1 & \text{if } i \in I, \end{cases}$$

$$f_1(x_{2k+i} z_i) = \begin{cases} 3i & \text{if } i \in I, \end{cases}$$

$$f_1(x_{k+i} z_i) = \begin{cases} 9k+1-3i & \text{if } 1 \leq i \leq k-2, \\ 6k+4 & \text{if } i = k-1, \\ 6k-2 & \text{if } i = k, \end{cases}$$

$$f_1(x_{3k+i} z_i) = \begin{cases} 3k+3i & \text{if } 1 \leq i \leq k-2, \\ 6k-3 & \text{if } i = k-1, \\ 6k+3 & \text{if } i = k. \end{cases}$$

Lemma 1. *f_1 is a bijection from the set $\{1, 2, \ldots, |E(\mathbb{R}_n)|\}$ onto the edges of $\mathbb{R}_n$ if $n = 4k$ and $k \geq 2$.*

It is not difficult to check that the labeling f_1 uses each integer $1, 2, \ldots, |E(\mathbb{R}_n)|$ exactly once.

We denote by $\lambda(v)$ the sum of the labels of the edges incident to the vertex v.

Lemma 2. *If $n = 4k \geq 8$ then the labeling f_1 of $\mathbb{R}_n$ has the same value $\lambda(v)$ for all vertices of $\mathbb{R}_n$.*

Proof: For $i \in I$ we have

$$\lambda(y_i) = f_1(x_i y_i) + f_1(x_{k+i} y_i) + f_1(x_{2k+i} y_i) + f_1(x_{3k+i} y_i) = 24k + 2,$$
$$\lambda(z_i) = f_1(x_i z_i) + f_1(x_{k+i} z_i) + f_1(x_{2k+i} z_i) + f_1(x_{3k+i} z_i) = 24k + 2,$$

and analogously for $i \in J$ the $\lambda(x_i)$ is equal to $24k + 2$.

Our previous results lead to the following theorem.

Theorem 3. *If $n = 4k$ and $k \geq 1$ then $\mathbb{R}_n$ is super-magic.*

Proof: If $n = 4$, then $\mathbb{R}_n$ is the octahedron from the platonic family. The octahedron is super-magic and Figure 3 shows one of its super-magic labelings.

If $n = 4k \geq 8$ then from Lemma 1 and Lemma 2 it follows that the edge labeling f_1 is a super-magic labeling of $\mathbb{R}_n$ with the super-magic index $\lambda = 24k + 2$. Hence the proof is complete.

In the sequel we shall deal with the regular graph $\mathbb{Z}_n$ if $n = 4k + 2$ and $k \geq 1$. Define the edge labeling f_2 of $\mathbb{Z}_n$ as follows.

$$f_2(x_i x_{i+1}) = \begin{cases} 3i + 2 & \text{if } 1 \leq i \leq 2k - 1 \text{ except } i = 2, \\ 3n - 7 & \text{if } i = 2 \text{ and } k \geq 2, \\ n + 2k + 2 & \text{if } i = 2k, \\ 4 & \text{if } i = 2k + 1, \\ 4n + 2k - 3i & \text{if } 2k + 2 \leq i \leq 4k \text{ except } i = 2k + 3, \\ 8 & \text{if } i = 2k + 3 \text{ and } k \geq 2, \\ n + 2k + 1 & \text{if } i = 4k + 1, \\ 3n - 3 & \text{if } i = 4k + 2, \end{cases}$$

$$f_2(x_i y_i) = \begin{cases} 2 & \text{if } i = 1 \text{ and } k \geq 1, \\ n & \text{if } i = 2 \text{ and } k = 2, \\ 15 - 3i & \text{if } 2 \leq i \leq 3 \text{ and } k \geq 3, \\ 3n - 3i & \text{if } 4 \leq i \leq k \text{ and } k \geq 4, \end{cases}$$

$$f_2(x_{k+i}y_i) = \begin{cases} 2n-1 & \text{if } i=1 \text{ and } k=1, \\ (5k+1)i-2k-1 & \text{if } 1 \le i \le 2 \text{ and } k=2, \\ 2n+k-1 & \text{if } i=1 \text{ and } k \ge 3, \\ 2n+k+3-3i & \text{if } 2 \le i \le k-1 \text{ and } k \ge 3, \\ 6k+6 & \text{if } i=k \text{ and } k \ge 3, \end{cases}$$

$$f_2(x_{2k+1+i}y_i) = \begin{cases} 3n-1 & \text{if } i=1 \text{ and } k \ge 1, \\ 3n-15+3i & \text{if } 2 \le i \le \min\{3,k\} \text{ and } k \ge 2, \\ 3i & \text{if } 4 \le i \le k \text{ and } k \ge 4, \end{cases}$$

$$f_2(x_{3k+1+i}y_i) = \begin{cases} 3n-5 & \text{if } i=1 \text{ and } k=2, \\ 3k+1+3i & \text{if } 1 \le i \le k-1 \text{ and } k \ge 3, \\ 6k+2 & \text{if } i=k \text{ and } k \ge 1, \end{cases}$$

$$f_2(x_1y) = \begin{cases} 3n-2, \end{cases}$$

$$f_2(x_{2k+2}y) = \begin{cases} 3, \end{cases}$$

$$f_2(x_{2k+1}y) = \begin{cases} 3n, \end{cases}$$

$$f_2(x_ny) = \begin{cases} 1, \end{cases}$$

$$f_2(x_{i+1}z_i) = \begin{cases} 2n & \text{if } i=1 \text{ and } k=1, \\ 3n-2i-4 & \text{if } 1 \le i \le k \text{ and } k=2, \\ 3n-3i-2 & \text{if } 1 \le i \le k \text{ and } k \ge 3, \end{cases}$$

$$f_2(x_{k+i+1}z_i) = \begin{cases} 2n+k-1-3i & \text{if } 1 \le i \le k-2 \text{ and } k \ge 3, \\ 2n-2k+1 & \text{if } i=k-1 \text{ and } k \ge 3, \\ 2n-2k+2 & \text{if } i=k-1 \text{ and } k=2, \\ n+2k-2 & \text{if } i=k \text{ and } k \ge 1, \end{cases}$$

$$f_2(x_{2k+2+i}z_i) = \begin{cases} 2i+5 & \text{if } 1 \le i \le k \text{ and } k \le 2, \\ 3i+4 & \text{if } 1 \le i \le k-1 \text{ and } k \ge 3, \\ 3k+3 & \text{if } i=k \text{ and } k \ge 3, \end{cases}$$

$$f_2(x_{3k+2+i}z_i) = \begin{cases} 3k+3+3i & \text{if } 1 \le i \le k-2 \text{ and } k \ge 3, \\ 6k+1 & \text{if } i=k-1 \text{ and } k \ge 2, \\ n+2k+5 & \text{if } i=k \text{ and } k \ge 1. \end{cases}$$

Theorem 4. *If $n = 4k+2$ and $k \ge 1$ then $\mathbb{Z}_n$ is super-magic.*

Proof: It is enough to show the existence of a super-magic labeling of $\mathbb{Z}_n$.

It is simple to verify that the labeling f_2 is a bijection from the set $\{1, 2, \ldots, |E(\mathbb{Z}_n)|\}$ onto the edges of $\mathbb{Z}_n$. Furthermore, by direct computation, we obtain that for every vertex $v \in V(\mathbb{Z}_n)$ (under the labeling f_2) the sum of the labels of the edges incident to the vertex v is equal to $6n + 2$.

This implies that the labeling f_2 is super-magic and the proof is complete.

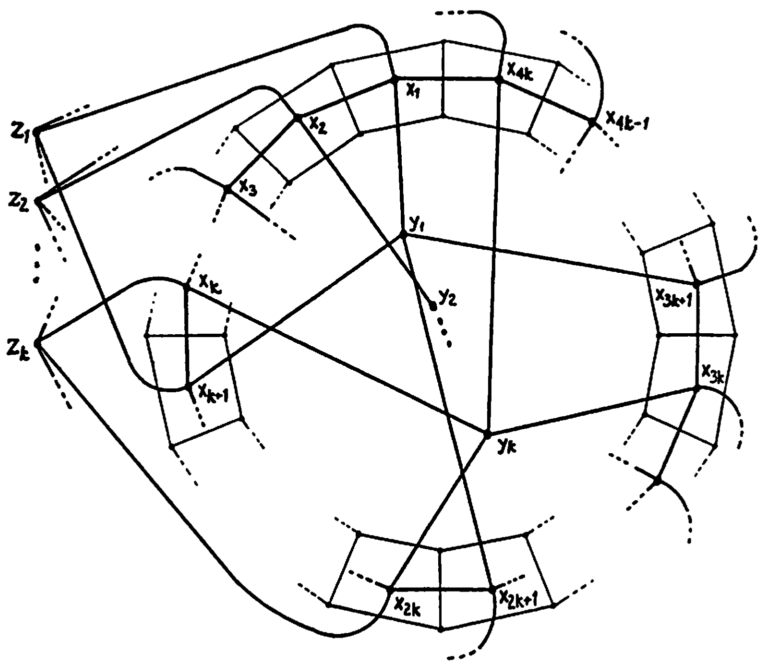

Figure 1. The labeled graph $\mathbb{R}_n$ when $n = 4k$

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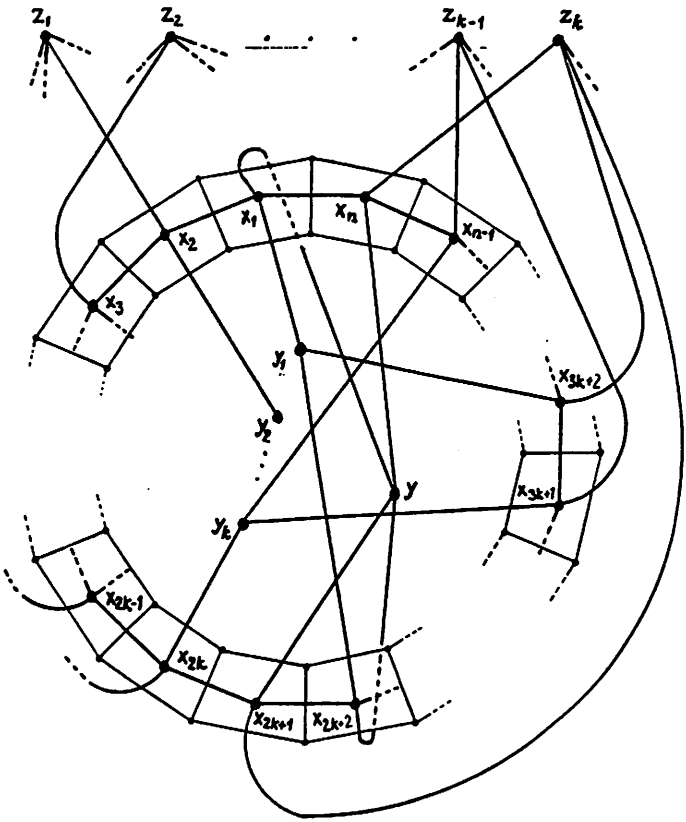

Figure 2. The labeled graph $\mathbb{Z}_n$ when $n = 4k + 2$

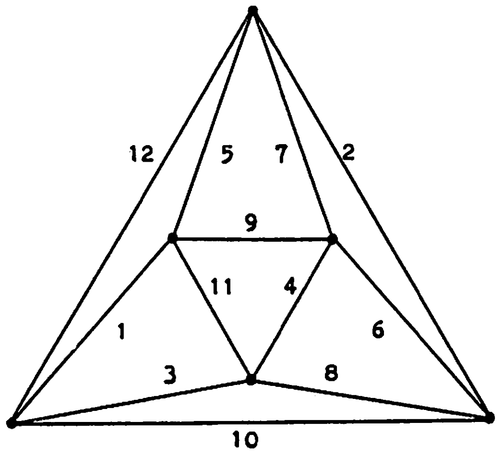

Figure 3. Super-magic labeling of octahedron