# Characteristic Parameters, Chordal Graphs, and Common Neighborhoods

Terry A. McKee*
Department of Mathematics & Statistics
Wright State University, Dayton, Ohio 45435

**Abstract**

The "characteristic" of a graph—the number of vertices, minus the number of edges, plus the number of triangles, etc.—is a little-studied, overtly combinatorial graph parameter intrinsically related to chordal graphs and common neighborhoods of subgraphs. I also introduce a sequence of related "higher characteristic" parameters.

## 1 The characteristic of a graph

For any graph $G$, let char $G$ denote the *(Euler) characteristic* of $G$, defined by

$$\text{char } G = k_1(G) - k_2(G) + k_3(G) - k_4(G) + \cdots,$$

where each $k_i(G)$ denotes the number of subgraphs of $G$ that are isomorphic to $K_i$; thus $k_1(G)$ is the order of $G$, $k_2(G)$ is the size, and so on. Previous work with this parameter occurs in [3, 4, 5, 6, 8]. See [2] for any undefined notation or terminology.

Let comp $G$ denote the number of components of $G$. Recall that a graph is *chordal* whenever when it contains no induced cycle of length greater than three. Reference [1] contains a thorough survey of the theory and applications of chordal graphs (called "trianglulated graphs" there); [5] is a more recent survey, from a different point of view. As observed in [3, 5, 8] and as is easily proved by induction, every chordal graph $G$ satisfies char $G$ = comp $G$. Many nonchordal graphs do too, including all wheels, but it is observed in [5] that $G$ is chordal if and only if char $H$ = comp $H$ for all induced subgraphs $H$ of $G$.

Agree always to use the symbol $Q$ to denote a complete subgraph of $G$ and $N(Q)$ to denote the *common neighborhood* of $G$, meaning the subgraph

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induced by those $v \in V(G)$ that are adjacent to every vertex in $Q$. Notice that this makes $N(Q) \cap Q = \emptyset$. The following lemma is proved in [6], primarily by manipulation of binomial coefficients.

**Lemma 1** *For every graph $G$,*

$$\sum_Q [1 - \text{char } N(Q)] = \text{char } G. \qquad \square$$

Realize that this formula,

$$\text{char } G = \sum_Q [1 - k_1(N(Q)) + k_2(N(Q)) - k_3(N(Q)) + k_4(N(Q)) - \cdots],$$

is not meant to be used to compute the characteristic, but rather is a step toward understanding what it means—how it relates to other parameters. Contrast this formula with

$$\text{char } G = \sum_v [1 - \frac{1}{2}k_1(N(v)) + \frac{1}{3}k_2(N(v)) - \frac{1}{4}k_3(N(v)) + \frac{1}{5}k_4(N(v)) - \cdots]$$

from [4].

The following provides another view of char $G$ in a special, but non-chordal context. Recall that a set of cycles is *dependent* if one is, when viewed as a set of edges, the symmetric difference of some of the others. A *cycle basis* is a maximal independent set of cycles and always consists of $k_2(G) - k_1(G) + \text{comp } G$ (the *cycle rank* of $G$) cycles.

**Theorem 1** *Suppose $G$ has no dependent set of triangles. Then* comp $G -$ char $G$ *is the number of induced cycles of length greater than three needed to use with the triangles to make a cycle basis.*

**Proof.** Suppose $G$ has no dependent set of triangles. Thus $G$ is $K_4$-free and so $k_i(G) = 0$ for all $i \geq 4$. The cycle rank equals $k_2(G) - k_1(G) + \text{comp } G + [k_1(G) - k_2(G) + k_3(G)] - \text{char } G = k_3(G) + [\text{comp } G - \text{char } G]$. The theorem then follows. $\square$

The following lemma is proved in [7], primarily by manipulating summations.

**Lemma 2** *For every graph $G$,*

$$\sum_Q [1 - \text{comp } N(Q)] \leq \text{comp } G,$$

*with equality holding if and only if $G$ is chordal.* $\square$

It is sometimes useful to allow the *null subgraph*—the $K_0$ subgraph—as a complete subgraph of $G$. Agree always to use the symbol $R$ to denote a complete or null subgraph of $G$, noting that $N(R) = G$ when $R$ is null. Any summation $\sum_R$ is then over all complete or null subgraphs of $G$. This allows the equality in Lemma 1 to be rewritten as $\sum_R[1 - \text{char } N(R)] = 1$ and the inequality in Lemma 2 to be rewritten as $\sum_R[1 - \text{comp } N(R)] \le 1$.

**Theorem 2** *For every graph $G$,*

$$\sum_R \text{char } N(R) \le \sum_R \text{comp } N(R),$$

*with equality if and only if $G$ is chordal.*

**Proof.** This follows directly from the $\sum_R$ reformulations of Lemma 1 and Lemma 2. □

## 2 Higher characteristics

Define $\text{char}_1 G = \text{char } G$, $\text{char}_2 G = k_2(G) - 2k_3(G) + 3k_4(G) - \cdots$, $\text{char}_3 G = k_3(G) - 3k_4(G) + 6k_5(G) - \cdots$, and so on. In general,

$$\text{char}_i G = \sum_{j \ge i} (-1)^{i+j} \binom{j-1}{i-1} k_j(G).$$

The following observation is a handy check when finding the various $\text{char}_i$ values.

**Theorem 3** *For every graph $G$, $\sum_i \text{char}_i G = k_1(G)$.*

**Proof.**

$$\begin{aligned} \sum_i \text{char}_i G &= \sum_{i\ge 1}\sum_{j \ge i} (-1)^{i+j} \binom{j-1}{i-1} k_j(G) \\ &= \sum_{1 \le j}\sum_{1 \le i \le j} (-1)^{i+j} \binom{j-1}{i-1} k_j(G), \end{aligned}$$

which, by the Binomial Theorem, equals $\sum_j (1-1)^{j-1} k_j(G) = k_1(G)$. □

**Lemma 3** *For every graph $G$ and every $i \ge 1$,*

$$\text{char}_i G = \text{char}_i (G - v) + \text{char}_{(i-1)} N(v) - \text{char}_i N(v).$$

**Proof.** Define $k_0$ of any graph always to be one. Then for every $v \in V(G)$ and $j \geq 1$, $k_j(G) = k_j(G-v) + k_{(j-1)}(N(v))$. Therefore $\operatorname{char}_i G$

$$\begin{aligned}
&= \sum_{j\geq i}(-1)^{i+j}\binom{j-1}{i-1}k_j(G-v) + \sum_{j\geq i}(-1)^{i+j}\binom{j-1}{i-1}k_{(j-1)}(N(v)) \\
&= \operatorname{char}_i(G-v) + \sum_{j\geq i-1}(-1)^{i-1+j}\binom{j}{i-1}k_j(N(v)) \\
&= \operatorname{char}_i(G-v) + \sum_{j\geq i-1}(-1)^{i-1+j}\binom{j-1}{i-2}k_j(N(v)) \\
&\qquad + \sum_{j\geq i-1}(-1)^{i-1+j}\binom{j-1}{i-1}k_j(N(v)) \\
&= \operatorname{char}_i(G-v) + \operatorname{char}_{(i-1)} N(v) - \sum_{j\geq i}(-1)^{i+j}\binom{j-1}{i-1}k_j(N(v)),
\end{aligned}$$

which equals $\operatorname{char}_i(G-v) + \operatorname{char}_{(i-1)} N(v) - \operatorname{char}_i N(v)$. □

Define the *1-blocks of G* to be the components of $G$, the *2-blocks of G* to be the nontrivial blocks (i.e., blocks—maximal nonseparable subgraphs—that are not isolated vertices) of $G$, and in general the *i-blocks of G* to be the maximal subgraphs of $G$ that are either $i$-connected or isomorphic to $K_i$. Let $b_i(G)$ denote the number of $i$-blocks in $G$.

**Theorem 4** *For every chordal graph G and every* $i \geq 1$, $\operatorname{char}_i G = b_i(G)$.

**Proof.** Suppose $G$ is chordal. By a standard result in chordal graph theory, Theorem 4.1 in [1], $V(G)$ can be ordered $v_1, \ldots, v_n$ such that each open neighborhood $N(v_j)$ is complete or null in the induced subgraph $G_j$ of $G$ induced by $v_j, \ldots, v_n$. Let $N_j(v_j)$ denote this open neighborhood in $G_j$. By Lemma 3, $\operatorname{char}_i G_j = \operatorname{char}_i(G_j - v_j) + \operatorname{char}_{(i-1)} N_j(v_j) - \operatorname{char}_i N_j(v_j)$. Set $d = |N_j(v_j)| = \deg v_j$ in $G_j$. Then

$$\operatorname{char}_i(N_j(v_j)) = \operatorname{char}_i(K_d) = \begin{cases} 0 & \text{if } d < i \\ 1 & \text{if } d \geq i \end{cases},$$

and so

$$\operatorname{char}_i G_j = \begin{cases} \operatorname{char}_i(G_j - v_j) & \text{if } d \neq i-1 \\ \operatorname{char}_i(G_j - v_j) + 1 & \text{if } d = i-1 \end{cases}.$$

Notice that if $d < i-1$, then $v_j$ is in no $i$-block and $\operatorname{char}_i G_j = \operatorname{char}_i G_{(j+1)}$. If $d > i-1$, then $N_j(v_j) \cup \{v_i\}$ is in every $i$-block that contains $N_j(v_j)$ and $\operatorname{char}_i G_j = \operatorname{char}_i G_{(j+1)}$. If $d = i-1$, then $N_j(v_j) \cup \{v_j\} \cong K_d$ is an $i$-block. Hence the count of $i$-blocks increases by one exactly when $\operatorname{char}_i G_j$ increases by one as $j$ runs from 1 to $n$. Thus $\operatorname{char}_i G = b_i(G)$. □

Indeed, $G$ is chordal if and only if $\text{char}_i H = b_i(H)$ for every induced subgraph $H$ of $G$, since $G$ is not chordal if and only if it contains an induced $C_k$ with $k \geq 4$, and $\text{char}_i C_k = b_i(C_k)$ only when $n = 3$ (and $i \leq 3$).

The following are open questions, part conjecture and part ignorance.

**Query 1.** *Is $G$ chordal if and only if* $\text{char}_i G = b_i(G)$ *for* every *$i$?*

The next extends (reversing) Theorem 2, which says that $\sum_R \text{char}_1 N(R) \leq \sum_R b_1(N(R))$.

**Query 2.** *When $i > 1$, must $\sum_R \text{char}_i N(R) \geq \sum_R b_i(N(R))$? (And if so, for which graphs are they equal?)*

**Query 3.** *Must, for every $G$, $\sum_i \text{char}_i G \geq \sum_i b_i(G)$?*

Theorem 3 shows that this can be phrased without mentioning characteristics: *Must the order of $G$ always be at least $\sum_i b_i(G)$?*

# References


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