Large digraphs with small diameter: A voltage assignment approach*

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Abstract

The theory of lifting voltage digraphs provides a useful tool for constructing large digraphs with given properties from suitable small base digraphs endowed with an assignment of voltages (=elements of a finite group) on arcs. We revisit the degree/diameter problem for digraphs from this new perspective and prove a general upper bound on diameter of a lifted digraph in terms of properties of the base digraph and voltage assignment. In addition, we show that all currently known largest vertex-transitive Cayley digraphs for semidirect products of groups can be described by means of a voltage assignment construction using simpler groups.

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1 Introduction

One fruitful application of graph theory to communication problems is in the design of interconnection networks, such as parallel computers, switching system architecture in LSI technology, design of local area networks, problem of data alignment, etc. Any of these applications imposes two basic constraints on the underlying network (=directed graph): The number of connections that can be attached to a node is limited, and a short communication route between any two nodes is required. This naturally leads to the following well known (directed version of)

Degree/Diameter Problem. Construct digraphs with the largest possible number of vertices for given maximum (in-and out-) degree d and diameter k.

A straightforward general upper bound on the order (=number of vertices) n of a digraph of maximum degree d and diameter k is the *Moore bound* $M_{d,k}$ for directed graphs

$$n \leq M_{d,k} = 1 + d + d^2 + \ldots + d^k \ . \qquad (1)$$

The equality $n = M_{d,k}$ holds only when $d = 1$ or $k = 1$ [6][16], and hence in all other cases the upper bound can be lowered by 1.

Case $k = 2$. It is well known that the bound $M_{d,2} - 1$ can be achieved for all $d \geq 2$ by line digraphs of complete digraphs of order $d + 1$. The existence of digraphs of $M_{d,2} - 1$ vertices other than the line digraphs has been studied in [2][3][5].

Case $k \geq 3$. In general, it is not known whether or not $M_{d,k} - 1$ is attainable. The existence of digraphs of degree d, diameter $k \geq 3$ and order $M_{d,k} - 1$ has been studied and several necessary conditions have been given in [3]–[5]. For degree $d = 2$, it was shown in [15] that $M_{2,k} - 1$ is not attainable.

Moreover, it was shown in [14] that $M_{2,k} - 2$ cannot be attained for *many* values of $k \geq 3$ (for example, if $3 \leq k \leq 10^7$ then $M_{2,k} - 2$ cannot be achieved for all $k \neq 274485, 5035921$).

Hence in many cases (depending on k and d) the upper bound on n is actually 2 or 3 less than the Moore bound. But apart from that, no other upper bounds on n are known.

A general *lower* bound on the *largest order* of $n = n(d, k)$ for the degree/diameter problem is given by Kautz digraphs $K(d, k)$ [13] of order $d^k + d^{k-1}$; these digraphs can be obtained by $(k-1)$-fold iteration of the line digraph construction applied to the complete digraph of order $d + 1$. It is also known that $n(2, 4) = 25$ (which implies that $n(2, j) \geq 25.2^{j-4}$ for

$j \geq 4$ by the iterated line digraph construction); the corresponding digraph of order 25 was found by Alegre [1].

Much effort has been spent on the vertex-transitive version of the degree/diameter problem, which asks for the largest possible *vertex-transitive* digraph of given diameter k and maximum degree d (see e.g., [8, 7]). The most current list of orders of largest known vertex-transitive digraphs of degree d and diameter k for $d, k \leq 10$ can be found in [12] where most of the values for $k \geq d$ were found (by computer search) as Cayley digraphs of semidirect products of (mostly cyclic) groups.

The purpose of this paper is to draw attention to a construction well known in topological graph theory (and in algebraic topology), derived from the theory of covering spaces. It enables us to "blow up" a given base digraph G in order to obtain a larger digraph (called a "lift") whose incidence structure depends on both G and a mapping ("voltage assignment") from the arc set of G into a finite group. Precise definitions will be given in the next section; at this point we only note two facts. First, all Cayley digraphs as well as some other vertex-transitive digraphs can be obtained by this kind of voltage construction. Second, the way lifts are defined allows us to reduce the computing time for checking the diameter to a reasonable minimum. Therefore, constructions involving voltage assignments appear to be good candidates for both the vertex-transitive as well as the original version of the degree/diameter problem.

The paper is organised as follows. In Section 2 we introduce new concepts and give the corresponding background. Section 3 is devoted to proving a general lower bound on the diameter of a lifted digraph. A way of controlling transitivity of the lift is presented in Section 4. Finally, Section 5 focuses on the known Cayley digraph constructions for the vertex-transitive degree/diameter problem from the voltage assignment perspective.

2 Voltage assignments on directed graphs

Let G be a digraph and let $D(G)$ be the set of arcs (= directed edges) of G (we allow loops as well as parallel arcs). Let Γ be an arbitrary group. Any mapping $\alpha: D(G) \to \Gamma$ is called a *voltage assignment*. The *lift* of G by α, denoted by G^α, is the digraph defined as follows: $V(G^\alpha) = V(G) \times \Gamma$, $D(G^\alpha) = D(G) \times \Gamma$, and there is an arc (x, f) in G^α from (u, g) to (v, h) if and only if $f = g$, x is an arc from u to v, and $h = g\alpha(x)$. For example, Figure 1 shows digraph G and the lift of G with $\Gamma = Z_5$ and the voltage assignment α in Z_5 given by $\alpha(a) = \alpha(b) = 0$, $\alpha(c) = 1$ and $\alpha(d) = 4$.

Intuitively, voltage assignments are a tool for "blowing up" small digraphs in order to obtain large ones. Since the lift is completely determined in terms of the original *base digraph* G and the voltage assignment α, this

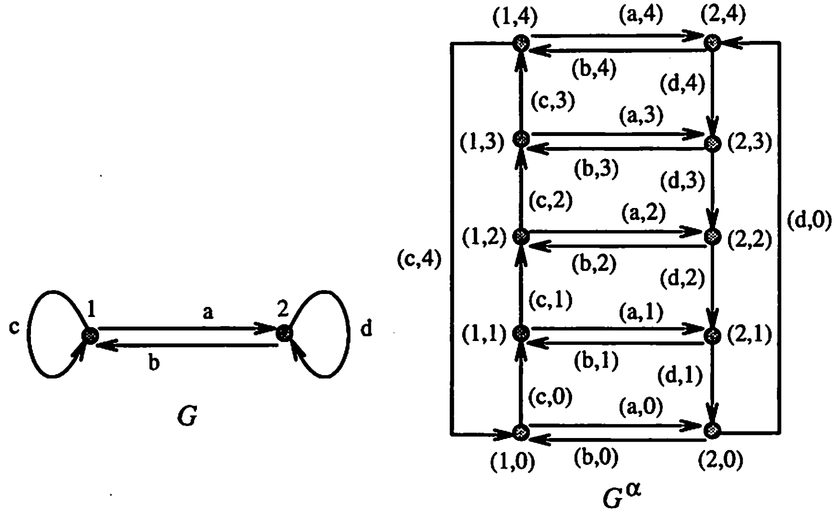


Figure 1: Digraph G and its lift G^α.

type of construction is suitable for handling large digraphs in terms of the properties of the base graph and the assignment.

The theory of voltage assignments on undirected graphs was introduced in the early 70's [9] as a dualisation of the theory of the so-called current graphs; the latter played a key role in proving the famous Map Color Theorem [17]. Most of this theory (summarised in [10]) can be immediately transferred to digraphs, and in what follows we outline the basic facts concerning voltage assignments on digraphs and properties of lifts.

Let G be a digraph and let G^α be a lift of G obtained by means of a voltage assignment α in a group Γ. Let $\pi : G^\alpha \to G$ be the *natural projection* which erases the second coordinates, that is, $\pi(u, g) = u$ and $\pi(x, g) = x$ for each $u \in V(G)$, $x \in D(G)$ and $g \in \Gamma$. Clearly, π is a digraph homomorphism; the sets $\pi^{-1}(u)$ and $\pi^{-1}(x)$ are called *fibres* above the vertex u or above the arc x, respectively.

The most important tool for understanding properties of a lift is the examination of walks in the base digraph. Let α be a voltage assignment on a digraph G in a group Γ. Let $W = e_1 e_2 \ldots e_m$ be a walk in G, that is, for each i, $2 \le i \le m$, the terminal vertex of the arc e_{i-1} coincides with the initial vertex of the arc e_i (we allow an arc to be used repeatedly). The number m is the *length* of the walk W. The walk W is *closed* if the initial vertex of e_1 and the terminal vertex of e_m are the same. The *net voltage of* W is the product $\alpha(W) = \alpha(e_1)\alpha(e_2)\ldots\alpha(e_m)$. For convenience, at each vertex we also admit a *trivial* closed walk of length 0 and of unit net voltage.

It is easy to see that for each walk W in G with initial vertex u and for each $g \in \Gamma$ there exists a *unique* walk $\tilde{W}$ (also often denoted by W^α) in the lift G^α which starts at the vertex (u, g) and such that $\pi(\tilde{W}) = W$. Indeed, if $W = e_1 e_2 \ldots e_m$ is a walk in G emanating from u and if $\alpha(e_i) = x_i$, $1 \leq i \leq m$, then the walk $\tilde{W} = (e_1, g)(e_2, gx_1) \ldots (e_m, gx_1 x_2 \ldots x_{m-1})$ emanates in the lift G^α from the vertex (u, g) and has the property that $\pi(\tilde{W}) = W$; its uniqueness is obvious. Note also that if the walk W ends at the vertex v of G, then $\tilde{W}$ terminates in G^α at the vertex $(v, g\alpha(W))$. The walk $\tilde{W}$ is often called a *lift* of W. Thus, each walk W in the base graph, emanating from a fixed vertex u, has $|\Gamma|$ different lifts - for each $g \in \Gamma$ there is a lift of W in G^α which starts at the vertex $(u, g) \in \pi^{-1}(u)$.

Some properties of the lift G^α may be deduced immediately from the base digraph G and the voltage assignment $\alpha : D(G) \rightarrow \Gamma$. For example, let G be a strongly connected digraph. Then the lift G^α is strongly connected if and only if there is a vertex u in G such that for each $g \in \Gamma$ there is a closed walk W emanating from u with $\alpha(W) = g$. In fact, if the latter holds, then it holds for an *arbitrary* vertex u of G; such a voltage assignment on a strongly connected digraph will be called *proper*.

An another easy example is consider the cycle lengths in the lift. Let C be a (directed) cycle of length k in G. Let u be a vertex of C and let C_u be the closed walk of length k starting at u whose arcs are precisely the arcs of C. Let $g = \alpha(C_u)$ be the net voltage of C_u and let n be the order of the element g in the group Γ. Then C_u lifts to $|\Gamma|/n$ cycles of G^α, each of length kn. Again, it is easy to see that these considerations do not depend on the chosen vertex u in C.

When trying to construct voltage assignments with specified properties, one may prescribe a certain amount of voltages without loss of generality. To be more precise, we say that two voltage assignments α and β of a digraph G in the same group Γ are *equivalent* if the lifts G^α and G^β are isomorphic. Now, if α is a voltage assignment on a connected digraph G in a group Γ and if T is any spanning tree of G, then there is an equivalent voltage assignment β on G such that $\beta(x) = id$ for every arc x in the tree T [10].

3 The diameter of lifted digraphs

A frequent goal in constructions of large digraphs with given properties is to keep the diameter as small as possible. We now present an upper bound on the diameter of lifted digraphs, applicable to a fairly general class of base digraphs and general groups. We start with a quick observation.

Lemma 1 *Let α be a voltage assignment on a digraph G in a group Γ. Then $diam(G^\alpha) \leq k$ if and only if for each ordered pair of vertices u, v of*

G (possibly $u = v$) and for each $g \in \Gamma$ there exists a walk of length $\leq k$ from u to v whose net voltage is g.

Proof. This is obvious since for any two distinct vertices $(u,g),(v,h)$ in $V(G^\alpha)$ there exists a path $\tilde{W}$ of length at most k from (u,g) to (v,h) if and only if the projection $W = \pi(\tilde{W})$ is a walk in the base digraph G of length at most k from u to v with $\alpha(W) = g^{-1}h$. □

Next, we introduce some more concepts and notation. Let Γ be a (finite) group and let X be a generating set for Γ. The *Cayley digraph* $C(\Gamma, X)$ has vertex set Γ, and for any ordered pair of vertices $g, h \in \Gamma$ there is an arc emanating from g and terminating at h whenever $gx = h$ for some $x \in X$. We observe that $C(\Gamma, X)$ is a vertex-transitive digraph of degree $|X|$.

Let G be a digraph and let $w \in V(G)$. Let r_w^+ be the largest distance *from* w to a vertex in G; similarly, let r_w^- be the largest distance *to* w from a vertex in G. Let $r(G) = min\{r_w^+ + r_w^-\}$ where the minimum is taken over all vertices $w \in V(G)$; any vertex w for which the minimum is attained will be called *central.* Also, let $\delta(G)$ be the largest t such that the outdegree and the indegree of each vertex of G is at least t.

Theorem 1 *Let $H = C(\Gamma, X)$ be an arbitrary connected Cayley digraph and let G be a strongly connected digraph such that $\delta(G) \geq |X| + 1$. Then there exists a voltage assignment $\alpha: D(G) \to \Gamma$ such that*

$$diam(G^\alpha) \leq r(G) + diam(H) \ .$$

Proof. Let w be a central vertex of G and let T_w^+ (T_w^-) be a directed spanning tree of G rooted at w such that $d_{T_w^+}(w,u) = d_G(w,u)$ (and, respectively, $d_{T_w^-}(u,w) = d_G(u,w)$) for each vertex u of G. (Loosely speaking, T_w^+ (T_w^-) is a spanning tree rooted at w and directed outward from (inward to) w, of depth at most $r(G)$.) Define now a voltage assignment $\alpha: D(G) \to \Gamma$ as follows. Set $\alpha(e) = id$ for each arc e in the spanning tree T_w^-. For each $u \in V(G)$ let u^+ denote the set of all arcs emanating from u which have not yet been assigned a voltage; since $\delta(G) \geq |X| + 1$, we have $|u^+| \geq |X|$. Now, we may define α on the remaining arcs of G (that is to say, on $D(G) \setminus D(T_w^-)$) in such a way that, for each $u \in V(G)$, the restriction of α to u^+ is an arbitrarily chosen surjection from u^+ onto X.

Note that in the lift G^α we have $|\Gamma|$ vertex-disjoint copies T_g of the tree T_w^-, with $V(T_g) = \{(v,g);\ v \in V(G)\}$ for each $g \in \Gamma$.

We shall now provide an upper bound for the distances d_{G^α} in the lift G^α. In order to simplify the notation, we write u_g instead of (u,g) when referring to vertices of the lift. Thus, let u_g and v_h be an ordered pair of distinct vertices of G^α.

We start with taking the (unique) $w \to v$ path R in the "w-outward" spanning tree T_w^+; let the net voltage of R be $\alpha(R) = q$. Let R^α denote the lift of R in G^α which emanates from the vertex $w_{hq^{-1}}$ and terminates at the vertex v_h (note that $h = hq^{-1}\alpha(R)$).

Now we invoke our Cayley digraph H in which there exists a path from the vertex $id \in V(H)$ to the vertex $g^{-1}hq^{-1} \in V(H)$ of length at most $diam(H)$. That is, there exist $x_1, \ldots, x_l \in X$ such that $g^{-1}hq^{-1} = x_1x_2\ldots x_l$ and $l \le diam(H)$.

Since our voltage assignment α restricted to arcs emanating from an arbitrary vertex s of G is a surjection from s^+ onto X there exists a walk P in G emanating from u of the form $P = e_1e_2\ldots e_l$ such that for the arcs e_i we have $\alpha(e_i) = x_i$, $1 \le i \le l$. Let u' be the terminal vertex of the walk P. The lift P^α of P which starts at our $u_g \in V(G^\alpha)$ terminates at the vertex $u'_{g'}$, where $g' = g\alpha(P) = g(g^{-1}hq^{-1}) = hq^{-1}$. Consequently, P^α is a $u_g \to u'_{hq^{-1}}$ walk in G^α of length l.

Finally, let Q be the (unique) $u' \to w$ path in the "w-inward" spanning tree T_w^-. Let Q^α be the lift of Q in G^α emanating from the vertex $u'_{hq^{-1}}$; observe that Q^α terminates at $w_{hq^{-1}}$ because $\alpha(Q) = id$. It remains to put the pieces together: The path $P^\alpha Q^\alpha R^\alpha$ in the lifted digraph G^α starts at the vertex u_g, terminates at the vertex v_h and therefore

$$d_{G^\alpha}(u_g, v_h) \le l + r_w^- + r_w^+ \le diam(H) + r(G) .$$

This completes the proof. □

Corollary 1 *Let Γ be a group and let G be a strongly connected digraph such that $\delta(G) \ge |\Gamma|$. Then there exists a voltage assignment $\alpha :\ D(G) \to \Gamma$ such that*

$$diam(G^\alpha) \le r(G) + 1 .$$

Proof. Let $X = \Gamma \setminus \{id\}$ and let $H = C(\Gamma, X)$. Clearly, H is a complete digraph and so $diam(H) = 1$. The assertion now follows directly from Theorem 1. □

The preceding two results are in general the best possible, as can be seen from the following infinite family of digraphs.

Example. Let T be the unique undirected tree of radius r, maximum degree $d \ge 2$ and order $1 + d + d(d-1) + d(d-1)^2 + \ldots + d(d-1)^{r-1}$. The tree T has a unique central vertex, which we denote w. Let G be the digraph which arises from T by replacing every edge by a pair of oppositely directed arcs and attaching $d-1$ directed loops at each pendant vertex of T. Then G is a digraph of degree d such that $r_w^+ = r_w^- = r = r(G)/2$. Let Γ be an arbitrary group of order $\leq d$ and let $X = \Gamma \setminus \{id\}$. We claim that $diam(G^\alpha) \geq r(G)+1$ for *each* voltage assignment $\alpha: D(G) \to \Gamma$. Indeed, if u, v are distinct vertices of G both incident to a loop and $d_G(u,v) = r(G)$ then there is exactly one directed path from u to v in G of length $r(G)$. Invoking Lemma 1, this shows that the diameter of the lift G^α must be at least $r(G)+1$, as claimed. □

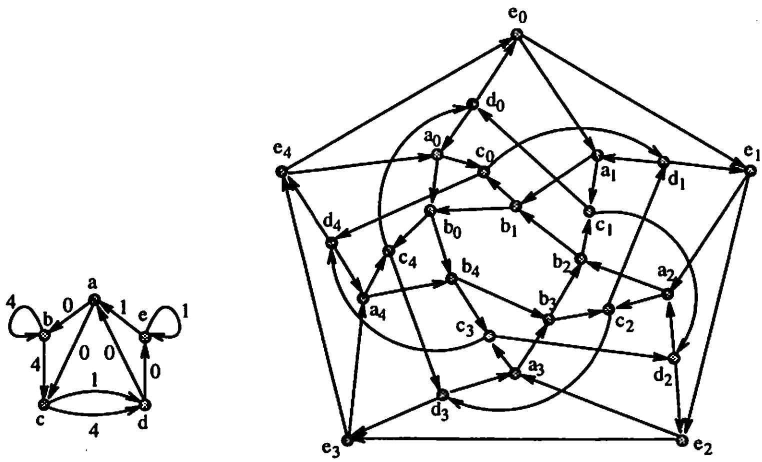
On the other hand, improvements to Theorem 1 (and Corollary 1) are certainly possible for some special base digraphs. For example, let G be the digraph obtained from the complete digraph K_n of degree $n-1$ by attaching a (directed) loop to each of its vertices. Clearly, G has degree n and $r(G) = 1$. By Corollary 1, there exists a lift G^α of order n^2 vertices, degree $d = n$ and diameter $k = 2$; moreover, the voltage assignment α can be taken in *any* group of order n. However, we may obtain a lift with more vertices by taking voltages in an *arbitrary* group Γ of order $n+1$. To begin with, we may wlog assume that $V(G) = \Gamma \setminus \{id\}$. The voltage assignment $\beta: D(G) \to \Gamma$ is described by the following simple rule: If x is an arc of G emanating from the vertex $g \in \Gamma \setminus \{id\}$, then $\beta(x) = g$. It is a matter of routine to check that the lift G^β is, in fact, isomorphic to the line digraph of a complete digraph K_{n+1} (note that the latter is true *independently* of what group of order $n+1$ has been used). This line digraph serves as an example of a largest possible digraph of degree $d = n$ and diameter $k = 2$ (its order is $d^2 + d$). The fact that this extremal graph can be obtained as a lift indicates the usefulness of voltage assignment constructions in the degree/diameter problem.

Improvements to Theorem 1 are possible also by choosing assignments in special groups; this will be the case with assignments considered in the last section of this paper. At this point we just note that, for example, the largest known digraph of degree 2 and diameter 4 (Alegre's digraph of order 25) can be obtained as a lift of a digraph of order 5 with voltage assignment in the group $\mathcal{Z}_5$, as indicated in Figure 2.

4 Symmetries of lifts

In many cases one would like to guarantee the existence of some automorphisms in the lift; this is particularly interesting if a lift which is vertex-transitive (or even arc-transitive) is required.

Observe first that the voltage construction itself introduces some special automorphisms for free. Let G be a digraph and let G^α be a lift of G



(a) (b)

Figure 2: The Alegre's digraph and its base digraph.

obtained by means of a voltage assignment α in a group Γ. As before, let $\pi : G^\alpha \to G$ be the natural projection erasing the second coordinates, that is, $\pi(u,g) = u$ and $\pi(x,g) = x$ for each $u \in V(G)$, $x \in D(G)$ and $g \in \Gamma$. Now, for any two vertices *in the same fibre* $\pi^{-1}(u)$ there exists an automorphism of the lift which sends the first vertex to the second. Indeed, without loss of generality, let $(u, id), (u, g) \in \pi^{-1}(u)$ be a pair of such vertices. Then it can be easily checked that the mapping $\tilde{B}_g : G^\alpha \to G^\alpha$, given by $\tilde{B}_g(v,h) = (v, gh)$ for each $(v,h) \in V(G^\alpha)$, is an automorphism of the lifted digraph G^α such that $\tilde{B}_g(u, id) = (u, g)$. (Note also that $\pi\tilde{B}_g = \pi$.) Thus $Aut(G^\alpha)$, the group of all automorphisms of G^α, acts transitively on each fibre and hence we always have $|Aut(G^\alpha)| \geq |\Gamma|$.

More automorphisms of the lift may in some cases be constructed by lifting automorphisms of the base digraph. To be specific, we say that an automorphism A of G *lifts* to an automorphism $\tilde{A}$ of G^α if $\pi\tilde{A} = A\pi$, that is, if $\pi(\tilde{A}(v,h)) = A(\pi(v,h))$ for each vertex $(v,h) \in G^\alpha$. Observe that since $A(\pi(v,h)) = A(v)$, the lifted automorphism $\tilde{A}$ maps vertices from the fibre $\pi^{-1}(v)$ onto vertices in the fibre $\pi^{-1}(A(v))$, that is, $\tilde{A}$ is necessarily *fibre-preserving*.

It is also important to note that if an automorphism $A \in Aut(G)$ lifts to some $\tilde{A} \in Aut(G^\alpha)$ then A has at least $|\Gamma|$ distinct lifts. Indeed, for each $g \in \Gamma$, the composition $\tilde{B}_g\tilde{A}$ is also a lift of A because $\pi\tilde{B}_g\tilde{A} = \pi\tilde{A}$.

We now prove a necessary and sufficient condition for an automorphism to have a lift. For notational convenience, if W is a walk in the base digraph G then $W*$ will stand for the terminal vertex of W. If A is an automorphism of G, we denote by AW the image of W under A. Two walks W and W' in G will be said to be *correlated* (in symbols, $W \sim_\alpha W'$) if $W* = W'*$ and, at the same time, $\alpha W = \alpha W'$.

Theorem 2 *Let G be a digraph, let α be a proper voltage assignment on G in a finite group Γ and let A be an automorphism of G. Then A lifts to an automorphism of G^α if and only if for any two walks W and W' emanating from a fixed vertex of G,*

$$W \sim_\alpha W' \Leftrightarrow AW \sim_\alpha AW' .$$

Proof. *Necessity:* Fix an arbitrary vertex w in G and let W and W' be two correlated walks emanating from w. Let W^α and W'^α be the lifts of W and W', both emanating from the vertex (w, id) of G^α. The fact that the net voltages of W and W' are equal implies that $W^\alpha* = W'^\alpha*$. Now, if an automorphism A of G lifts to an automorphism $\tilde{A}$ of G^α then $\tilde{A}(W^\alpha)* = \tilde{A}(W'^\alpha)*$. Hence the net voltages of $AW = \pi\tilde{A}(W^\alpha)$ and $AW' = \pi\tilde{A}(W'^\alpha)$ in the base digraph G must be equal to each other and their terminal vertices must coincide. Thus $W \sim_\alpha W'$ implies $AW \sim_\alpha AW'$. The same considerations applied to the inverse A^{-1} at the vertex $A(w)$ yield the reverse implication.

Sufficiency: Again, fix $w \in V(G)$ and let $W \sim_\alpha W' \Leftrightarrow AW \sim_\alpha AW'$ for any two walks W and W' in G, both emanating from w. We define a mapping $\tilde{A} : V(G^\alpha) \to V(G^\alpha)$ in the following way. Let (u, g) be an arbitrary vertex of G^α. Since the voltage assignment α is proper, there exists a walk W in G which emanates from w, terminates at u, and has net voltage g. Thus instead of (u, g) we may simply write $(W*, \alpha(W))$ without any loss of generality; this method of encoding all vertices of G^α will facilitate subsequent considerations. Let us set

$$\tilde{A}(W*, \alpha(W)) = (AW*, \alpha(AW))$$

where W is any walk in G emanating from w. We show that $\tilde{A}$ is (a) well defined, (b) a bijection, and finally (c) an automorphism of the lift G^α.

(a) Let W' be another walk in the base digraph G emanating from w such that $(W*, \alpha(W)) = (W'*, \alpha(W'))$. This means that $W \sim_\alpha W'$ which (by our assumption) implies $AW \sim_\alpha AW'$. The latter implies that $(AW*, \alpha(AW)) = (AW'*, \alpha(AW'))$ and so $\tilde{A}$ is well defined.

(b) Assume that $(AW*, \alpha(AW)) = (AW'*, \alpha(AW'))$ for some walks W, W' in G emanating from w. Similarly to (a), this translates to $AW \sim_\alpha AW'$, which implies $W \sim_\alpha W'$, and hence $(W*, \alpha(W)) = (W'*, \alpha(W'))$. Since G^α is finite, this shows that $\tilde{A}$ is bijective.

(c) Let an arc $(x.\alpha(W))$ of G^α emanate from the vertex $(W*,\alpha(W))$ and terminate at the vertex $((Wx)*,\alpha(Wx))$ where $\alpha(Wx)=\alpha(W)\alpha(x)$. Then the arc $\tilde{A}(x,\alpha(W))=(A(x),\alpha(AW))$ emanates in the lift G^α from the vertex $\tilde{A}(W*,\alpha(W))=(AW*,\alpha(AW))$ and terminates at the vertex $\tilde{A}((Wx)*,\alpha(Wx))=(A(Wx)*,\alpha(A(Wx)))$ because for the voltages we have $\alpha(A(Wx))=\alpha(AW)\alpha(A(x))$. This proves that $\tilde{A}$ is an automorphism of G^α. □

An analogous result for lifts of undirected graphs (in terms of closed walks) can be found in [11]. In fact, Theorem 2 could also have been stated in terms of "closed walks" – however, one would have to allow "walking" in the opposite direction as well, which seems unnatural in digraphs.

At a first glance, it is not obvious how to construct voltage assignments with the property as stated in Theorem 2. We now propose a fairly general method based on considering automorphisms of the voltage group.

Corollary 2 *Let G be a digraph and let $\mathcal{A}$ be a group of automorphisms of G. Let Γ be a voltage group and let $\phi:\mathcal{A}\to Aut(\Gamma)$ be an arbitrary group homomorphism which sends each digraph automorphism $A\in\mathcal{A}$ to an automorphism ϕ_A of the group Γ. Let α be a proper voltage assignment on G in the group Γ such that*

$$\alpha(A(x))=\phi_A(\alpha(x))$$

for each arc $x\in D(G)$. Then each digraph automorphism $A\in\mathcal{A}$ lifts to an automorphism of the lifted digraph G^α.

Proof. Let $W=e_1e_2\dots e_k$ be a walk in the digraph G and let $AW=A(e_1)A(e_2)\dots A(e_k)$ be its image under a digraph automorphism $A\in\mathcal{A}$. Then, since ϕ_A is an automorphism of the group Γ, we have

$$\alpha(AW)=\prod_{i=1}^{k}\alpha(A(e_i))=\prod_{i=1}^{k}\phi_A(\alpha(e_i))=\phi_A(\prod_{i=1}^{k}\alpha(e_i))=\phi_A(\alpha(W))\ .$$

Invoking the fact that $\phi_A\in Aut(\Gamma)$ again, we see that $\alpha(W)=\alpha(W')$ if and only if $\phi_A(\alpha(W))=\phi_A(\alpha(W'))$. By the above chain of equalities this is equivalent to $\alpha(AW)=\alpha(AW')$. It follows immediately that $W\sim_\alpha W'\Leftrightarrow A(W)\sim_\alpha A(W')$ for any two walks W, W' based at a common vertex. The rest follows from Theorem 2. □

The following special case of Corollary 2 is often useful in applications and we will discuss it in detail in the last section.

Corollary 3 *Let A be an order k automorphism of a digraph G. Let α be a proper voltage assignment on G in the additive group $\mathcal{Z}_n$. Assume that there is an element b in $\mathcal{Z}_n$ of multiplicative order k, which has a multiplicative inverse in the ring $(\mathcal{Z}_n, +, .)$ and such that $\alpha(A(x)) = b\alpha(x)$ for each arc $x \in D(G)$ (the multiplication is again in the ring $(\mathcal{Z}_n, +, .)$). Then A lifts to an automorphism $\tilde{A}$ of the digraph G^α.*

Proof. Let $\mathcal{A}$ be the cyclic group of order k generated by the automorphism A. Then we have an obvious homomorphism $\phi : \mathcal{A} \rightarrow Aut(\mathcal{Z}_n, +)$, given by $\phi_A(r) = br$ for each $r \in (\mathcal{Z}_n, +)$. The result now follows from Corollary 2. □

5 Semidirect products and voltage assignments

The theory developed in the preceding section can prove useful in the search for large *vertex-transitive* digraphs of given degree and diameter by looking for suitable voltage assignments on small digraphs. As an illustration, in this section we shall now focus on lift constructions based on the result of Corollary 2 in the special case when the underlying digraph G is a Cayley digraph $G = C(\Lambda, X)$ and the group $\mathcal{A} \simeq \Lambda$ of automorphisms of G is determined by the left action of Λ on G.

For the sake of generality, we will now work with a slightly more general concept of a Cayley digraph – namely, we shall allow loops and multiple arcs. Formally, let Λ be a group and let $X = (x_1, x_2, \ldots, x_k)$ be a *sequence* of generators of Λ, that is, we allow repeated use of the same elements in X. The Cayley digraph $G = C(\Lambda, X)$ is now defined in much the same way as before: Vertices of G are elements of Λ, and for each vertex $b \in \Lambda$ and for each i, $1 \leq i \leq k$, there is an arc (b, x_i) emanating from b and terminating at the vertex bx_i. Clearly, for each $a \in \Lambda$, the left multiplication $A_a : b \mapsto ab$ is an automorphism of the Cayley digraph G.

In order to come back to the connection with Corollary 2, let $G = C(\Lambda, X)$ be a Cayley digraph for a group Λ and a sequence of generators X. Let $\mathcal{A} = \{A_a;\ a \in \Lambda\}$; obviously $\mathcal{A} \simeq \Lambda$. Assume that, for our voltage group Γ, there is a group homomorphism $\phi : \Lambda \rightarrow Aut(\Gamma)$ (which sends $a \in \Lambda$ to an automorphism ϕ_a of Γ) and a proper voltage assignment on G in Γ such that $\alpha(ab, x_i) = \phi_a(\alpha(b, x_i))$ for each arc (b, x_i) of G. Setting $b = id$, it follows that then, for each $a \in \Lambda$, we have

$$\alpha(a, x_i) = \phi_a(\alpha(id, x_i)) \ . \qquad (2)$$

In what follows we shall refer to the equation (2) as *compatibility condition.*

Summing up the discussion under the above assumptions, the compatibility condition implies that the voltage assignment α is completely determined by the distribution of voltages on the arcs emanating from the vertex $id \in \Lambda$. Clearly, combining Corollary 2 with the well known (directed version of) theorem of Sabidussi [18], we can conclude that the lift G^α is necessarily a Cayley graph as well. However, we shall be interested in the structure of the underlying group of the lift. For that reason we recall the concept of *semidirect product* $\Lambda \times_\phi \Gamma$ of the groups Λ and Γ (which depends on the above homomorphism $\phi: \Lambda \to Aut(\Gamma)$) where the multiplication of elements $(a,g),(b,h) \in \Lambda \times \Gamma$ is given by $(a,g)(b,h) = (ab, g\phi_a(h))$.

Theorem 3 *Let $G = C(\Lambda, X)$ be a Cayley digraph for a group Λ and a generating sequence X and let Γ be a voltage group. Let $\phi: \Lambda \to Aut(\Gamma)$ be a group homomorphism and let α be a proper voltage assignment such that $\alpha(a,x_i) = \phi_a(\alpha(id,x_i))$ for each arc (a,x_i) of G. Then the lift G^α is isomorphic to the Cayley digraph $C(\Lambda \times_\phi \Gamma, X^\alpha)$ with generating sequence $X^\alpha = (x_1, \alpha(id,x_1)), (x_2, \alpha(id,x_2)), \ldots, (x_k, (\alpha(id,x_k))$.*

Proof. According to the definition of a lift, there is an arc in G^α from (a,g) to (b,h) if and only if $ax_i = b$ for some x_i in X and, at the same time, $h = g\alpha(a,x_i) = g\phi_a(\alpha(id,x_i))$. But this adjacency condition is equivalent to the following multiplicative property in the semidirect product $\Lambda \times_\phi \Gamma$:

$$(a,g)(x_i, \alpha(id,x_i)) = (ax_i, g\phi_a(\alpha(id,x_i))) = (b,h)$$

which actually defines the Cayley graph $C(\Lambda \times_\phi \Gamma, X^\alpha)$, as claimed. □

The preceding theorem tells us that if a voltage assignment on a Cayley graph satisfies the compatibility condition (2), then the resulting lift is a Cayley graph of a semidirect product of the original group and the voltage group. Interestingly enough, the converse is true as well.

Theorem 4 *Let $\phi: \Lambda \to Aut(\Gamma)$ be a homomorphism which sends an element $a \in \Lambda$ to an automorphism ϕ_a of the group Γ. Let $C(\Lambda \times_\phi \Gamma, Y)$ be a Cayley digraph for the semidirect product $\Lambda \times_\phi \Gamma$ with a generating sequence Y. Then there exists a Cayley digraph $G = C(\Lambda, X)$ and a voltage assignment α on G which satisfies the compatibility condition (2), such that $G^\alpha \simeq C(\Lambda \times_\phi \Gamma, Y)$.*

Proof. Let $Y = (x_1,y_1),(x_2,y_2),\ldots,(x_k,y_k)$ be the generating sequence for the semidirect product. Then $X = x_1, x_2, \ldots, x_k$ is a generating sequence for the group Λ. For each arc (a,x_i) of the Cayley digraph $C(\Lambda, X)$ let us define the voltage assignment α by $\alpha(a,x_i) = \phi_a(y_i) \in \Gamma$.

The rest is now almost identical to the proof of the preceding theorem and therefore omitted. □

The 1–1 correspondence between Cayley digraphs of semidirect products of groups on one side and lifts of Cayley digraphs (via voltage assignments satisfying the compatibility condition) on the other side, established in the preceding two results, opens up a new perspective to the search for large vertex-transitive digraphs of given degree and diameter. Namely, as the literature on computer search for such digraphs reveals [12], a large number of the currently known record examples were found among Cayley graphs of semidirect products of cyclic groups. Our Theorem 4 shows how to reconstruct each such Cayley digraph in terms of a lift of a smaller Cayley digraph of a *cyclic* group, with voltages taken in some smaller *cyclic* group as well. This strongly suggests that a computer search over lifts of Cayley digraphs may lead to some new examples of large digraphs of given diameter and degree.

Before discussing examples, let us remind the reader that we defined the semidirect product $\Lambda \times_\phi \Gamma$ by setting $(a,g)(b,h) = (ab, g\phi_a(h))$ for a given homomorphism $\phi : \Lambda \to Aut(\Gamma)$. Occasionally, however, a different definition of semidirect product (which we denote here by $\Lambda \times_\phi^* \Gamma$) is considered in the literature: For the same Λ, Γ and ϕ as above, the multiplication $*$ in this $\Lambda \times_\phi^* \Gamma$ is given by $(a,g) * (b,h) = (ba, h\phi_b(g))$. Clearly, these two semidirect products are algebraically equivalent under the isomorphism $\Phi : (a,g) \mapsto (a,g)^{-1} = (a^{-1}, \phi_{a^{-1}}(g^{-1}))$; we note that this notation is not ambiguous because the inverse of an element (a,g) is the *same* in both $\Lambda \times_\phi \Gamma$ and $\Lambda \times_\phi^* \Gamma$. It follows that Cayley digraphs of these semidirect products are isomorphic under taking inverse generating sequences, i.e., $C(\Lambda \times_\phi^* \Gamma, X) \cong C(\Lambda \times_\phi \Gamma, X^{-1})$. Moreover, although in general $C(\Lambda \times_\phi \Gamma, X^{-1}) \not\cong C(\Lambda \times_\phi \Gamma, X)$, the latter two Cayley digraphs obviously have the same diameter (which is the smallest k such that each element of the group can be expressed as a product of at most k elements in X - and we have the same k for X^{-1}). Consequently, $diam\ C(\Lambda \times_\phi^* \Gamma, X) = diam\ C(\Lambda \times_\phi \Gamma, X)$, which means that when concentrating on the diameter only, it does not matter which of the two definitions of semidirect product we are actually using.

In order to illustrate the above facts, let us take, say, the largest known vertex-transitive digraph of degree 3 and diameter 5, which has 165 vertices and was found [12] as the Cayley digraph $H = C(\mathcal{Z}_5 \times_\phi^* \mathcal{Z}_{33}, Y)$; the homomorphism $\phi : \mathcal{Z}_5 \to Aut(\mathcal{Z}_{33})$ is given by $\phi_a(j) = 4^a j$, $a = 0,1,2,3,4$ (the multiplication takes place in the *ring* $(\mathcal{Z}_{33}, +, .)$), and $Y = ((2,21),(4,4),(4,17))$. (We note that for the sake of brevity, in [12] this semidirect product is denoted by the symbol $5 \times_4 33$.) According to the

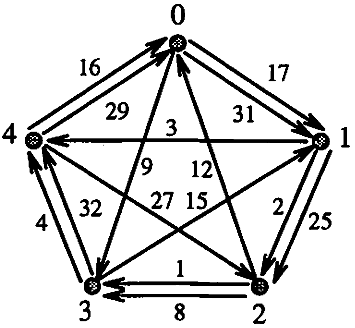


Figure 3: The base digraph G with the voltage assignment α.

comments in the preceding paragraph, $H \cong H' = C(\mathcal{Z}_5 \times_\phi \mathcal{Z}_{33}, Y^{-1})$ where $Y^{-1} = ((1,17),(1,31),(3,9))$. By Theorem 4, the digraph H' is isomorphic to the lift of the Cayley digraph $G = C(\mathcal{Z}_5, X)$ where $X = (x_1, x_2, x_3)$, $x_1 = x_2 = 1$, $x_3 = 3$. The voltage assignment α in the group $\mathcal{Z}_{33}$ is given by $\alpha(0,x_1) = 17$, $\alpha(0,x_2) = 31$, $\alpha(0,x_3) = 9$, and (in accordance with the compatibility condition (2)) it extends to the remaining arcs of G by setting $\alpha(a,x_i) = \phi_a(\alpha(0,x_i))$. The digraph G with the voltage assignment α is depicted in Figure 3, and it well illustrates the idea of "blowing up" a small digraph by a suitable voltage assignment in order to obtain the desired large lifted digraph.

A computer search run on the digraph G revealed another suitable voltage assignment β in $\mathcal{Z}_{33}$ such that $diam(G^\beta) = 5$. It is given by $\beta(0,x_1) = 1$, $\beta(0,x_2) = 32$, $\beta(0,x_3) = 0$, and extends to all other arcs in the same way as α above.

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