On the Number of Continua Having a Finite Set of Non-cut Points

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ABSTRACT. A continuum with finitely many non-cut points is an irreducible tree. A two variable power series is obtained for the number of (unlabelled) irreducible trees with p pendant and q interior vertices. The result is then specialized to get Harary's series for the number of irreducible trees with n vertices and to another series for the number of irreducible trees with p pendant vertices, a result of interest in continuum theory.

Introduction

An irreducible tree is a connected graph having no cycles and no vertices of degree two. A continuum is a nonempty compact connected metric space. S.B. Nadler [N 154] has shown that any continuum having p non-cut points ($p < \infty$) is a tree. Each pendant vertex in a tree is a non-cut point whilst interior vertices are cut points. Hence, the irreducible trees with p ($1 < p < \infty$) pendant vertices and the continua with p non-cut points are the same. Figure 1 exhibits all non-isomorphic (non-homeomorphic) irreducible trees (continua) having four or five pendant vertices (non-cut points).

The aim of this paper is to construct a generating function for irreducible trees with p pendant vertices. However, the generating function that is obtained is somewhat more general: it is the function $t(x,y) = \sum_{p,q\geq 0} t_{pq}x^py^q$, where t_{pq} is the number of irreducible trees having p pendant vertices and q interior vertices that are distinct under graph isomorphism ($t_{41} = t_{42} = t_{51} = t_{52} = t_{53} = 1$). A tree having p pendant vertices and q interior vertices is called a (p,q) tree.

Since there are no degree two vertices in an irreducible tree, the notions of isomorphism and homeomorphism coincide. Thus, letting $y = 1$ in $t(x,y)$, we obtain the topologically interesting result of enumerating irreducible

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trees with p pendant vertices; the result answers a question that is implicit in 9.43 of [N]. By letting $y = x$ in $t(x, y)$, we obtain a result of Harary and Prins enumerating irreducible trees with n vertices [HP 150, cor. 1]. Since many interested readers may not be familiar with the method employed here, some preliminaries are included.

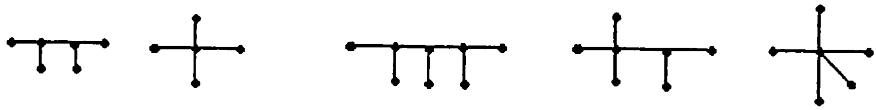

Figure 1

Preliminaries

For our purpose, it is necessary to introduce Polya's Enumeration Theorem, following the method of Harary and Prins [HP 142–144]. Let *figure* be an undefined term. To each figure there is assigned an ordered pair of nonnegative integers called its content. Let a_{mn} denote the number of different figures of content (m, n). Then the figure counting series $a(x, y)$ is defined by $a(x, y) = \sum_{m,n\geq 0}^{\infty} a_{mn}x^m y^n$.

Let A be a permutation group of degree d and order h. A *configuration* of length d is a sequence of d figures. The content of a configuration is the vector sum of the contents of its figures. Two configurations are A-equivalent if there is a permutation in A sending one into the other, so we call A the configuration group. Let F_{mn} denote the number of A-inequivalent configurations with content (m, n). Then the configuration counting series is defined to be $F(x, y) = \sum_{m,n\geq 0} F_{mn}x^m y^n$.

The aim is to express $F(x, y)$ in terms of $a(x, y)$ and A. This is accomplished using the cycle index for the group A which is denoted $Z(A)$ and defined as follows. $Z(A)$ is a polynomial in the variables $f_1, f_2, \ldots, f_d$ with

$$Z(A) = \frac{1}{|A|} \sum_{\alpha \in A} \prod_{k=1}^{d} f_k^{j_k(\alpha)}$$

where $j_k(\alpha)$ is the number of cycles of length k in the disjoint cycle decomposition of α. For any power series $f(x, y)$ let $Z(A, f(x, y))$ denote the function $Z(A)$ replacing each f_k by $f(x^k, y^k)$. Now, it is possible to state Polya's Theorem.

Polya's Enumeration Theorem. *The configuration counting series $F(x, y)$ is determined by substituting the figure counting series into the cycle index of the configuration group. Symbolically, $F(x, y) = Z(A, a(x, y))$.*

The *degree* of a vertex, v, in a graph, G, is the number of edges to which it is incident. Let $\diamond(G)$ be the automorphism group of G. Two points a and

b in $V(G)$ are similar if there is a $\diamond$ in $\diamond(G)$ such that $\diamond a = b$. The notion of similar edges is defined analogously. A *symmetry line* is an edge whose endpoints are similar. A *tree* of type (p,q) is a tree that has p pendant and q interior vertices (sometimes called a (p,q) tree). A *rooted tree* is a tree with a distinguished vertex, called its root. An *edge rooted tree* is a tree with a distinguished edge. A *symmetric tree* is a tree which contains a symmetry line. The following theorem of R. Otter gives a relationship key to the enumeration of many species of trees.

Otter's dissimilarity characteristic. *Let p^*, q^*, and s represent the number of dissimilar points, the number of dissimilar edges and the number of symmetry lines respectively in a tree T. Then $p^* - q^* + s = 1$. Furthermore, $s = 0$ or $s = 1$.* [O 588]

Results

A *branch* of a rooted tree determined by an edge adjacent to the root is the subtree consisting of the root and all points reachable by a path through that particular edge. A *plantable tree* is a rooted tree whose branches are irreducible and whose root degree is not one.

Lemma 1. *If $\bar{L}_{pq}$ is the number of plantable irreducible (p,q) trees ($\bar{L}_{00} = 0$) and $\bar{L}(x,y) = \sum_{p,q\geq 0} \bar{L}_{pq}x^p y^p$ then $\bar{L}(x,y) = x + y\sum_{n=2}^{\infty} Z(S_n, \bar{L}(x,y))$.*

Proof: Each plantable (p,q) tree has a root with degree r ($r = 0$ or $2 \leq r \leq p$). Let $\bar{L}^{(r)}(x,y)$ be the counting series for plantable (p,q) trees with root degree r and let T be such a tree. Consider the graph obtained by deleting this root and all incident edges. Rooting each of the r resulting plantable (p,q) trees at the point which was adjacent to the root gives a set of r plantable trees. Clearly, then each plantable tree with root degree r corresponds to a combination with repetition of r plantable trees with pendant vertices totalling p and interiors totalling $q-1$. Applying the Polya's Enumeration Theorem with configuration group S_r and counting series $\bar{L}(x,y)$, $\bar{L}^{(r)}(x,y)$ is obtained to be $yZ(S_r, \bar{L}(x,y))$. Summing over all possible root degrees completes the proof. □

For some explicit coefficient values of $\bar{L}(x,y)$, see Table 1.

Now the counting series for plantable trees is used to obtain counting series for symmetric irreducible, rooted irreducible and edge-rooted irreducible trees. Clearly a symmetric (p,q) tree can exist only if both p and q even.

Lemma 2. *The counting series for symmetric irreducible trees of type (p,q) is $\bar{L}(x^2,y^2)$.*

Proof: There is a one-to-one correspondence between symmetric trees with type (p,q) and identical pairs of plantable trees with $p/2$ pendant vertices and $q/2$ interior vertices, each rooted at a point on the symmetry line. □

Lemma 3. *The counting series for rooted irreducible (p,q) trees is*

$$L(x,y) = (1+x)\bar{L}(x,y) - yZ(S_2, \bar{L}(x,y)).$$

Proof: Let L_{pq} be the number of rooted irreducible trees with type (p,q). Then L_{pq} can be obtained in the following manner. All rooted (p,q) trees having a root with degree one can be formed by attaching an edge to the root of all plantable trees with type $(p-1,q)$ and rooting the resulting tree at its new vertex. Thus there are $\bar{L}_{(p-1)q}$ rooted (p,q) trees with root degree one. There is also a one to one correspondence between rooted (p,q) trees ($r \neq 1$) and plantable (p,q) trees and $r \geq 3$. Therefore, $L_{pq} = \bar{L}_{(p-1)q} + \bar{L}_{pq} - \bar{L}^{(2)}_{pq}$, where $\bar{L}^{(2)}_{pq}$ equals the number of plantable (p,q) trees with root degree two. In words, L_{pq} equals the number of plantable $(p-1,q)$ trees plus the number with type (p,q) minus the number with type (p,q) and $r = 2$. The result then follows by Polya's Theorem. □

Some coefficients of $L(x,y)$ are listed in Table 2.

Lemma 4. *The counting series for edge rooted irreducible (p,q) trees is* $Z(S_2, \bar{L}(x,y))$.

Proof: There is a one-to-one correspondence between edge rooted irreducible (p,q) trees and unordered pairs of plantable trees whose pendancies total p and interiors total q, each rooted at a vertex of the distinguished edge. By applying Polya's enumeration theorem with configuration group S_2 and figure counting series $\bar{L}(x,y)$ we obtain the series for edge rooted irreducible trees with type (p,q). □

We are now prepared to state the main result.

Theorem 5. *The counting series for irreducible (p,q) trees is*

$$t(x,y) = (1+x)\bar{L}(x,y) - (1+y)Z(S_2, \bar{L}(x,y)) + \bar{L}(x^2,y^2)$$

Proof: First summing Otter's dissimilarity characteristic over all irreducible (p,q) trees, then inserting the respective counting series for rooted, edge rooted and symmetric irreducible (p,q) trees, the counting series for the number of such trees is obtained. In other words, the sum of the number of dissimilar points minus the sum of the number of dissimilar edges plus the sum of the number of symmetry lines over all irreducible (p,q) trees is the same as the number of rooted minus the number of edge rooted plus the number of symmetric irreducible (p,q) trees. Symbolically, by lemmas 2 and 4,

$$t(x,y) = L(x,y) - Z(S_2, \bar{L}(x,y)) + \bar{L}(x^2,y^2)$$

and the result follows by lemma 3. □

For several explicit coefficient values of $t(x,y)$, see Table 3.

If we set $h(x) = t(x,x)$ and $\bar{L}(x) = \bar{L}(x,x)$ we see that $h(x)$ and $\bar{L}(x)$ are the counting series for irreducible and plantable trees respectively with $n = p + q$ vertices. Letting $y = x$ in Theorem 5 then gives the result of Harary and Prins.

Corollary 1. $h(x) = (1+x)[\bar{L}(x) - Z(S_2, \bar{L}(x))] + \bar{L}(x^2)$.
Explicitly, $h(x) = x + x^2 + x^4 + x^5 + 2x^6 + 2x^7 + 4x^8 + 5x^9 + 10x^{10} + 14x^{11} + 26x^{12} + 42x^{13} + 78x^{14} + 132x^{15} + \dots$

Similarly, if we let $\lambda(x) = t(x,1)$ and $\bar{L}_0(x) = \bar{L}(x,1)$ then $\lambda(x)$ and $\bar{L}_0(x)$ are the counting series for irreducible and plantable trees respectively with p pendant vertices. A concise formula for $\lambda(x)$ now follows.

Corollary 2. $\lambda(x) = (1+x)\bar{L}_0(x) - \bar{L}_0^2(x)$.

Proof: Let $y = 1$ in Theorem 5 and use the fact that

$$Z(S_2, \bar{L}_0(x)) = (1/2)[\bar{L}_0^{-2}(x) + \bar{L}_0(x^2)]$$

□

Explicitly, $\lambda(x) = x + x^2 + x^3 + 2x^4 + 3x^5 + 7x^6 + 13x^7 + 32x^8 + 73x^9 + 190x^{10} + 488x^{11} + 1350x^{12} + 3741x^{13} + 10765x^{14} + 31309x^{15} + \dots$

The coefficients given in the corollaries are readily obtained once the series $\bar{L}(x,y)$ is obtained. However, the coefficients of $\bar{L}(x,y)$ are obtained (with some difficulty) using *Mathematica* and the formula given in Lemma 1.

Acknowledgments

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References

[HP] Frank Harary and Geert Prins, The Number of Homeomorphically Irreducible Trees, and Other Species. *Acta. Math.* **101** (1959), 141–162.

[HP1] Frank Harary and Ed Palmer, *Graphical Enumeration*, Academic Press, New York and London , 1973.

[N] Sam B. Nadler, Jr., *Continuum Theory, An Introduction*, Marcel Dekker, Inc. (Pure and appl. Math. series, Vol. 158), New York, NY, 1992.

[O] Richard Otter, The number of Trees. *Ann. of Math.* **49** (1948), 583–599.

Appendix I

Tables of Coefficients

$x\backslash y$	0	1	2	3	4	5	6	7	8	9	10
1	1										
2	0	1									
3	0	1	1								
4	0	1	2	2							
5	0	1	3	5	3						
6	0	1	4	10	12	6					
7	0	1	5	16	29	28	11				
8	0	1	6	24	57	84	66	23			
9	0	1	7	33	99	192	231	157	46		
10	0	1	8	44	157	382	615	634	373	98	
11	0	1	9	56	234	682	1380	1905	1704	890	207

Plantable irreducible (p, q) trees: $\bar{L}(x, y)$

$x\backslash y$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1									
3	0	2								
4	0	2	2							
5	0	2	4	4						
6	0	2	6	10	7					
7	0	2	8	20	25	13				
8	0	2	10	32	60	60	25			
9	0	2	12	48	117	176	143	52		
10	0	2	14	66	202	399	494	345	106	
11	0	2	16	88	319	789	1297	1369	829	225

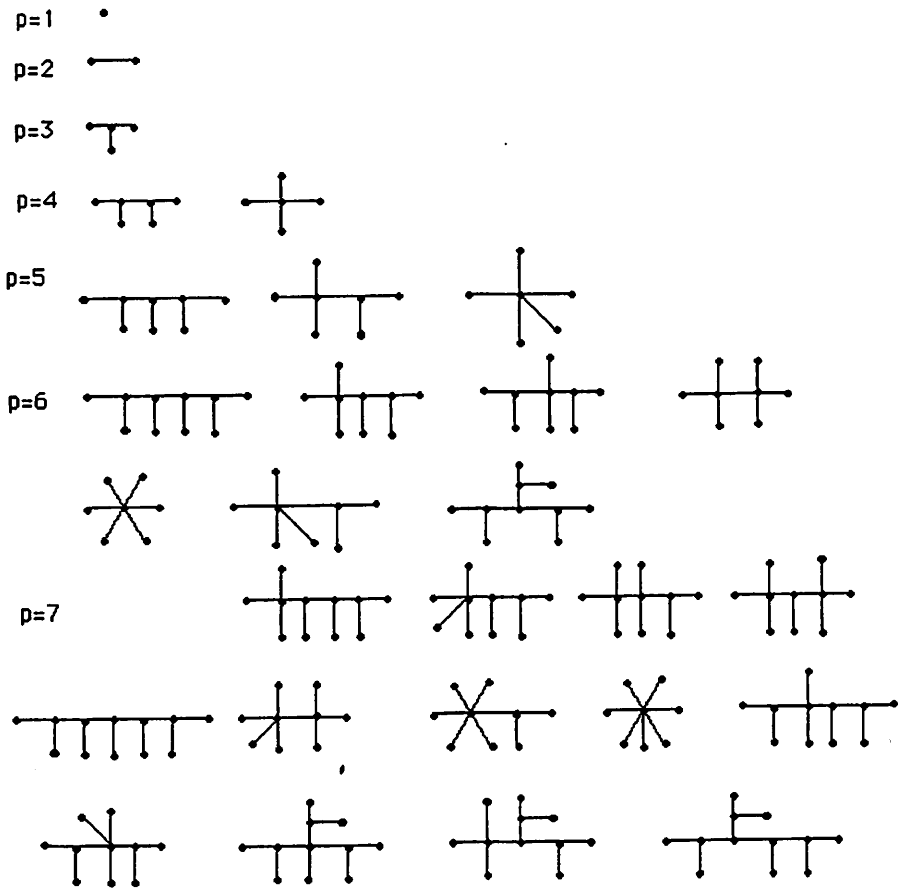
Rooted irreducible (p, q) trees: $L(x, y)$

$x\backslash y$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1									
3	0	1								
4	0	1	1							
5	0	1	1	1						
6	0	1	2	2	2					
7	0	1	2	4	4	2				
8	0	1	3	6	10	8	4			
9	0	1	3	9	17	22	15	6		
10	0	1	4	12	30	47	53	32	11	
11	0	1	4	16	44	91	127	121	66	18

Irreducible (p, q) trees: $t(x, y)$

Appendix II

Diagrams of all irreducible trees with p ($p \leq 8$)



p=8