On Minimal Connected Dominating Sets

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ABSTRACT. A connected dominating set is a dominating set S with the additional property that the subgraph induced by S is connected. We are interested in the collection **C** of graphs in which every minimal connected dominating set is of one size. Trees, for instance, clearly belong to this collection. A partial characterization will be discussed; in particular, we determine those graphs which have the property that all spanning trees have the same number of leaves. It is noted that membership in this sub-collection of **C** can be determined in polynomial time.

1 Introduction

We begin with some terminology used in this paper.

Let G be a graph. A *dominating set* of G is a set, D, of vertices of G such that every vertex in G is either in, or adjacent to, a member of D. D will be a *connected dominating set* of G if the subgraph induced by D is connected. A *minimal connected dominating set* of G is a connected dominating set of G such that the removal of any member of the set leaves G no longer dominated or the set no longer connected. Let **C** be the collection of graphs in which every minimal connected dominating set is of one size. A *cut vertex* of G is a vertex of G such that its removal leaves G no longer connected. A *non-cut vertex* of G is a vertex of G such that its removal does not disconnect G.

It is easily seen that every minimal connected dominating set of a tree with n vertices is of constant size, $n-L$, where L is the number of leaves [2]. Similarly, the leaves of a spanning tree of a graph G can be removed to find

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a connected dominating set of G. In fact, every minimal connected dominating set corresponds to some spanning tree with its leaves removed, an observation first made in [4]. Therefore, the spanning tree with the maximum number of leaves can be used to find a minimum connected dominating set of a graph G. Unfortunately, finding this tree for an arbitrary graph is NP-complete [3]! As pointed out in [4], this can be used to show that determining a minimum connected dominating set for an arbitrary graph is also NP-complete (the reader interested in more detailed complexity results is referred to [5] and [6]).

If all spanning trees of a graph G have the same number of leaves, then any spanning tree will give a minimal connected dominating set and, in fact, G will be in the collection **C**. Let **D** be the class of graphs such that G belongs to **D** if and only if all spanning trees of G have the same number of leaves. Observe that a 4-cycle with a tree rooted at one vertex on the 4-cycle but the other vertices on the 4-cycle being of degree two is in the collection **C** but fails to be in **D**. In this paper we will focus our attention on **D**.

Our main result is the following:

Theorem. *A graph G has the property that each pair of spanning trees have the same number of leaves if and only if both of the following conditions hold.*

(1) *About each cycle in the graph G, the vertices are either all cut vertices, all non-cut vertices, or alternating cut and non-cut vertices.*

(2) *Every vertex of degree 3 or more is a cut vertex.*

2 Proof of the Theorem

Lemma 1. *Let G be a graph in which all spanning trees have the same number of leaves. If a vertex v is of degree three or more, then v must be a cut vertex.*

Proof: Assume not. That is, let G be a graph in which all spanning trees have the same number of leaves but with a vertex v of degree at least 3 where v is not a cut vertex. Let the neighbors of v be $x_1, x_2, \ldots, x_k$, $k \geq 3$ (see Figure 1).

Since v is not a cut vertex, we can find a spanning tree T of $G - \{v\}$. Let m be the number of leaves in T. Note that in T the x_i's can be leaves or non-leaves. We consider various possibilities.

Case 1: There exists at least one vertex, x_r say, that is a leaf, and one vertex, x_s say, that is a non-leaf in T.

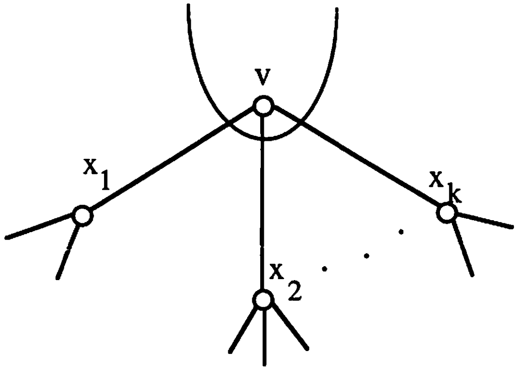

Figure 1

One way to form a spanning tree of G is to to extend T to include the edge between v and x_r. This spanning tree of G will have m leaves since v takes the place of x_r as a leaf. On the other hand, a spanning tree of G can be formed by including the edge between v and x_s. This spanning tree of G will have a different number of leaves, $m + 1$, since v becomes a new leaf.

Hence Case 1 cannot occur as all spanning trees of G must have the same number of leaves.

Case 2: All the x_i's are leaves in T.

One way to form a spanning tree of G is to extend T to include the edge between v and x_i. This spanning tree of G will have m leaves since v takes the place of x_i as a leaf. Another way to form a spanning tree of G is to first include two edges between v and any two x_i's, say x_r and x_s. Exactly one cycle has been formed, since there are now two paths between x_r and x_s; one in T and one through v. Along this cycle, there must be at least one vertex of degree 3 or more to connect it to vertices outside the cycle. Let one such vertex be called y. At least one vertex outside of the cycle must exist since $\deg(v) \geq 3$. Now remove an edge (not incident with v) along the cycle to form a spanning. tree of G. By removing the edge between y and a neighbor in the cycle, a spanning tree with $m - 2$ or $m - 1$ leaves will be formed. If the degree of the neighbor is 2, and the neighbor is not x_r, nor x_s, then removing the edge will create a new leaf, but x_r and x_s are no longer leaves, thus giving $m - 1$ leaves. In the event the neighbor is one of x_r or x_s, x_r say, then removing the edge results in a spanning tree with $m - 1$ leaves (as x_s no longer a leaf). If the degree of the neighbor is greater than 2, then removing the edge will not create a new leaf, and x_r and x_s are no longer leaves, thus giving $m - 2$ leaves.

In either situation, a spanning tree with a different number of leaves than m is obtained. Thus case 2 cannot occur.

Case 3: None of the x_i's are leaves in T.

One way to form a spanning tree of G is to extend T to include an edge between v and any x_i. This spanning tree will have $m+1$ leaves as v becomes a new leaf.

Another way to form a spanning tree of G is to first include two edges between v and any two x_i's, say x_r and x_s. Similar to case 2, exactly one cycle is formed. If either x_r or x_s has a neighbor on the cycle that is of degree greater than two, then the edge between the neighbor and that x_i can be removed. This spanning tree of G will have m leaves since no new leaves were created. As this argument will hold for any cycle formed by including two edges between v and any two x_i's, it can be assumed that the neighbors on the cycle of all x_i's will be of degree 2. By including three edges between v and any three x_i's we note that two edges must be removed to form a spanning tree of G. If each edge removed is between an x_i and its degree 2 neighbor, both on a cycle, then the spanning tree of G will have $m+2$ leaves.

In either situation, a spanning tree with a number of leaves different than $m+1$ is obtained. Hence case 3 is impossible.

This completes the proof of Lemma 1. □

Lemma 2. *Let G be a graph in which all spanning trees have the same number of leaves. If C is a cycle of G, then the vertices of C must either be all cut vertices or all non-cut vertices or the cycle must be of even length where the vertices alternate between cut and non-cut.*

Proof: Let G be a graph in which all spanning trees have the same number of leaves. Assume G has some cycle in which the vertices are not all cut vertices, nor all non-cut vertices, nor alternating cut and non-cut vertices. Then on that cycle, one of two possible sequences of vertices will be found.

Case 1: There is a cycle in G with three consecutive vertices x, y and z where x and y are non-cut vertices and z is a cut vertex. By Lemma 1, non-cut vertices cannot be of degree three or more. Therefore, we may assume that all the non-cut vertices on the cycle are of degree 2 (see Figure 2).

Consider any spanning tree, say T, of $G - \{y\}$ and let m be the number of leaves. To form a spanning tree of G, the edge xy or the edge yz must be added to T. By including the edge xy, the spanning tree of G will have m leaves since y takes the place of x as a leaf. On the other hand, by including the edge yz, the spanning tree of G will have a different number of leaves, $m+1$, since y becomes a new leaf.

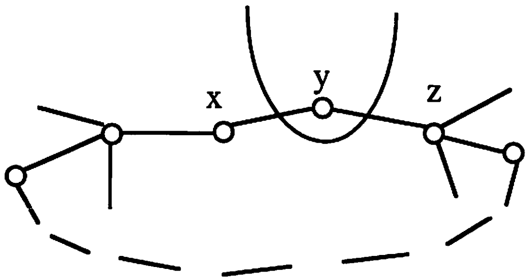

Figure 2

Case 2: There is a cycle in G with three consecutive vertices x, y and z where x and y are cut vertices and z is a non-cut vertex (see Figure 3).

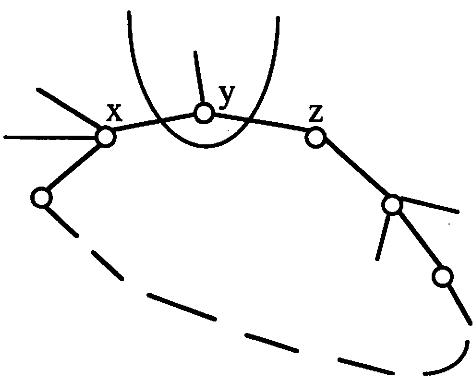

Figure 3

Let G_1 be the component of the graph $G - \{y\}$ containing the vertices x and z. Since x is a cut vertex in G, it must also be a cut vertex in G_1. Hence, it will not be a leaf in any spanning tree of G_1. To form a spanning tree of G, we first take a spanning tree of each component of $G - \{y\}$ and then add an edge from y to each component (other than G_1) of $G - \{y\}$. Finally, we include either the edge xy or the edge yz. The former case results in a spanning tree of G with exactly one more leaf (namely, z).

Hence, neither case 1 nor case 2 can occur.

This completes the proof of Lemma 2. □

Lemma 3. *If a graph G has both the following properties, then any two spanning trees of G have the same number of leaves.*

(1) *Around each cycle, either cut and non-cut vertices alternate, or all vertices are cut vertices, or all vertices are non-cut vertices.*

(2) *Every vertex of degree 3 or more is a cut vertex.*

Proof: Assume a graph G which satisfies conditions 1 and 2 has two spanning trees with different numbers of leaves. Call the spanning trees A and B, where B has fewer leaves than A, but let B be chosen in such a way that it has as many leaves as possible in common with A. Then there will be at least one vertex v that is a leaf in A but not in B. Since v is a leaf in a spanning tree, it must be a non-cut vertex in G, and by condition 2, it will be of degree 2. Let vw be the edge in G that is not included in the spanning tree A.

The vertices v and w lie on a cycle in the graph G because there exists two paths between these vertices: the edge vw and the unique path in the spanning tree A. This cycle must satisfy condition 1, so there are two possibilities for the vertices in the cycle. All vertices may be non-cut, and hence of degree 2, in which case G itself consists only of one cycle. Since all spanning trees of cycles have the same number of leaves, a contradiction has been reached. A second possibility is that the vertices alternate around the cycle between cut and non-cut, in which case w is a cut vertex.

Consider the spanning tree B and the two components of $B - vw$. Let B_v be the component consisting of those vertices in the same component as v and B_w consisting of those in the same component as w. Observe that there must be at least one edge, say ab, on the unique path between v and w in the spanning tree A, where a belongs to B_v and b belongs to B_w. Now form a new spanning tree B' from B by deleting the edge vw and adding the edge ab. This edge is between a cut and a non-cut vertex which means one leaf will be destroyed. Therefore, the number of leaves in B' is the same as the number of leaves in B. But, this is a contradiction since B' now has more leaves in common with A than B (namely, v). Therefore, our original assumption was incorrect and all spanning trees must have the same number of leaves.

This completes the proof of Lemma 3. □

3 Conclusions

We conclude by observing that one can determine membership in the family characterized by the Theorem in polynomial time. For instance, one could employ a depth-first search to a graph G to determine the biconnected components (see [1]). Then the conditions for either all cut or all non-cut or alternating cut and non-cut on cycles can be readily verified.

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