On the Characterizing Properties of the Circuit Polynomial

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ABSTRACT. It is shown that the circuit polynomial characterizes many of the well-known families of graphs . These include chains, stars, cycles, complete graphs, regular complete bipartite graphs and wheels. Some analogous results are deduced for the characteristic polynomial and the μ-polynomial.

1 introduction

The graphs considered here are finite and contain neither loops nor multiple edges. Let G be such a graph. We define the circuits (or cycles) with one and two nodes to be a node and an edge respectively. Cycles with more than two nodes are called *proper*. A *cycle (circuit) cover* in G is a spanning subgraph of G, in which every component is a cycle. With every cycle C_n with n nodes, let us associate an indeterminate or *weight* $w_n = w(C_n)$ and with every cycle cover Z in G, the weight $w(Z) = \prod(C_n)$, where the product is taken over all the components C of Z. Then the circuit polynomial of G is

$$C(G; \underline{w}) = \sum w(Z),$$

where the summation is taken over all the cycle covers of G , and $w = (w_1, w_2, w_3, \dots)$ is the vector of indeterminates. The basic results on circuit polynomials have been given in the introductory paper [3].

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Let G be a graph and $F(G;\underline{w})$ an F-polynomial (see [2]) associated with G. We say that $F(G;\underline{w})$ *characterizes* G, if and only if for any graph H, whenever $F(G;\underline{w}) = F(H;\underline{w})$, then $G \cong H$. It is of interest to determine the families of graphs that are characterized by their circuit polynomials. In this paper, we identify several of these families. They include cycles, *chains* (trees with nodes of valencies 1 and 2 only), stars, complete graphs, regular complete bipartite graphs and wheels. Analogous results can also be deduced for characteristic polynomials and μ-polynomials.

Suppose that we restrict the elements of a circuit cover to improper cycles (i.e. nodes and edges) only, then each circuit cover becomes a *matching*. The resulting polynomial is then called a *matching polynomial*. This polynomial was introduced in [4]. Let us denote the matching polynomial of G, by $M(G;\underline{w})$. Then clearly

$$M(G;\underline{w}) = C(G;(w_1, w_2, 0, 0, \ldots, 0)).$$

It is not difficult to see that if a graph is characterized by its matching polynomial, then it must be also characterized by its circuit polynomial. In [7], it was shown that cycles, stars (except the 4-star), chains of even lengths and complete graphs, are all characterized by their matching polynomials. It follows that all these graphs are also characterized by their circuit polynomials.

The circuit polynomial is related to the matching polynomial, as shown above. It is also related to the μ-polynomial (see Gutman and Polansky [9]). The connection between these polynomials has been given in Farrell and Gutman [8]. Both the matching polynomial and the μ-polynomial have been applied to various problems in Chemistry ([9]). It follows that developments in the theory of circuit polynomials might be of interest to researchers who apply graph polynomials in Chemistry.

In the material which follows, we denote by C_p, P_p, K_p and W_p the cycle, chain, complete graph and wheel respectively with p nodes. The m by n complete bipartite graph is denoted by $K_{m,n}$.

2 Preliminaries

The following lemma gives graph-theoretical interpretations for the coefficients of the polynomial $C(G;\underline{w})$. It can be easily proved.

Lemma 1. *Let G be a graph with p nodes and q edges. Then*

(i) *$C(G;\underline{w})$ has the term w_1^{p-2}; and this occurs with the coefficient 1.*

(ii) *The coefficient of $w_1^{p-2}w_2$ is q.*

(iii) *The coefficient of $w_1^{p-4}w_2^2$ is $\binom{q}{2} - \sum_{i=1}^{p} \binom{d_i}{2}$, where d_o is the valency of node i in G.*

(iv) *The coefficient of w_p is the number of hamiltonian cycles in G.*

The following result (see [3]) gives a relationship between the circuit polynomial of G and the characteristic polynomial of G, denoted by $\phi(G;x)$.

Lemma 2.

$$\phi(G;x) = C(G;(x,-1,-2,-2,\dots)).$$

Lemmas 1 and 2 yield the following result.

Lemma 3. *Let G be a graph with p nodes and q edges. Then*

(i) *$\phi(G;x)$ has the term x^p; and this occurs with coefficient 1.*

(ii) *The coefficient of $-x^{p-2}$ is q.*

The following lemmas give the relations between the circuit polynomial, μ-polynomial and the characteristic polynomial of a graph. The μ-polynomial of G will be denoted by $\mu(G;\underline{t},x)$.

Lemma 4. *$\mu(G;\underline{t},x) = C(G;(x,-1,-2t_1,-2t_2,\dots))$, where $\underline{t} = (t_1,t_2,t_3,\dots)$ and more general weights are assigned to the cycles as defined in [8].*

Lemma 5. *$\phi(G;x) = \mu(G;\underline{1},x)$, where $\underline{1} = (1,1,1,\dots,1)$.*

The following result can be easily deduced from above.

Lemma 6. *If a graph G is characterized by its characteristic polynomial, then it is also characterized by its circuit polynomial and its μ-polynomial. If G is characterized by its μ-polynomial, then it is also characterized by its circuit polynomial (with the implied general weights).*

3 Chains and complete bipartite graphs

The following theorem shows that chains are characterized by their circuit polynomials. It is well known that the chain is not characterized by its chromatic polynomial. Chains are characterized by their star polynomials ([6]). Also, only the even chains are characterized by their matching polynomials; as reported in [7] .

Theorem 1. *The circuit polynomial characterizes chains.*

Proof: This is straightforward. □

The non-characterization of $K_{m,n}$ $(m \neq n)$ for matching polynomials, was established in [7]. For $m = n$, the problem is still open. The following theorem can be easily proved. It shows that $K_{n,n}$ is characterized by its circuit polynomial.

Theorem 2. *The circuit polynomial characterizes regular complete bipartite graphs.*

The characterization of $K_{n,n}$ by its characteristic polynomial is shown in Cvetkovic et al [1].

4 Wheels

Definition. The *wheel* W_p $(p > 2)$ with p nodes, is the graph formed by joining a new node to all the nodes of C_{p-1}. W_1 is a node and W_2 is an edge.

The following lemma can be easily established from Lemma 1 and by observing the form of the subscripts of w in $C(Wp; \underline{w})$, given in [5].

Lemma 7. *Let G be a graph such that $C(G;\underline{w}) = C(W_p;\underline{w})$. Then*

(i) *G has p nodes*

(ii) *G has $2p-2$ edges*

(iii) $\sum_{1=1}^{p} \binom{d_i}{2} = \frac{(p-1)(p-2)}{2} + 3(p-1)$

(iv) *G is hamiltonian*

(v) *G has cycles of all lengths, up to p*

(v) *G does not contain any pair of node-disjoint proper cycles.*

Lemma 8. *Let G be a graph such that $C(G;\underline{w}) = C(W_p;\underline{w})$, where $p > 5$. Then G contains a node v of valency $d_v \geq 4$.*

Proof: Assume the contrary. Then for all i, $d_i \leq 3$. Therefore the sum of the valencies is $\leq 3p$. But the sum of the valencies is $2q = 2(2p-2) = 4p-4$. It follows that $4p-4 \leq 3p$. Thus $p \leq 4$. This is a contradiction. Hence the result follows. □

Lemma 9. *Let G be a graph such that $C(G;\underline{w}) = C(W_p;\underline{w})$, where $p \geq 9$. Then G contains a node v such that $d_v = p-1$.*

Proof: By (iv) of Lemma 7, G is hamiltonian. $\Rightarrow$ all the nodes are located on a cycle H. By lemma 8, there is a node $v \in V(G)$ such that $d_v \geq 4$. Let us label the nodes of H as shown in Figure 1(i), with $d_i \geq 4$.

We will prove the result by considering the different possible values of d_i. Since there are $q = 2p-2$ edges in G, and H has p edges, another

$p-2$ "diagonal" edges must be drawn between the p nodes of H. We will consider four cases:

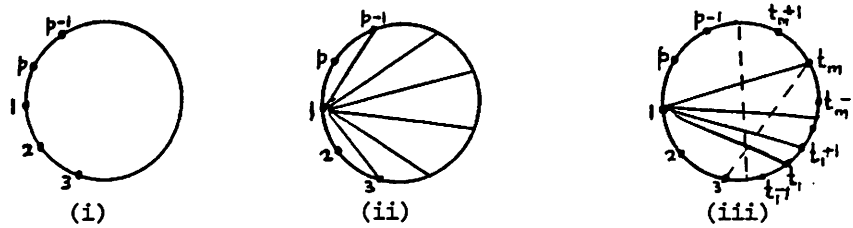

Figure 1

Case (1). $p-3$ diagonal edges are drawn from node 1.
In this case , we get that $d_1 = p-1$ and the result follows. This situation is shown above in Figure 1(ii).

Case (2). $m[(4 \leq m \leq (p-4)]$ diagonal edges are drawn from node 1.
Let the diagonal edges be $(1,t_1),(1,t_2),\ldots,(1,t_m)$, where $3 \leq t_1 < t_2 < \cdots < t_m \leq p-1$. Since there are $p-2$ diagonal edges to be added, at least 2 edges must be added between nodes other than node 1.

We show that it is impossible to add two or more edges between the nodes, other than node 1 without violating the property that G has no node-disjoint proper circuit subgraphs.

First of all, we note that no two nodes in the set $\{2,3,\ldots,t_m-1\}$ can be joined by an edge; otherwise a proper cycle, disjoint from the cycle $1 \to t_m \to p \to 1$, will be formed. Similarly, no edges can be drawn between the nodes in the set $\{t_1+1, t_2+2, \ldots, p-1, p\}$. Therefore the required edges can only be added between nodes of the sets $A_1 = \{t_m, t_m+1, \ldots, p\}$ and $B_1 = \{2,3,\ldots,t_1\}$ (N.B. A_1 and B_1 are non-empty, since $p \in A_1$ and $2 \in B_1$). If A_1 and B_1 are singletons then it is impossible to add two or more edges. Otherwise, since $m \geq 4$, $\exists$ nodes i and j such that $t_1 < i < j < t_m$ with $1 \to i \to i+1 \to \cdots \to j \to 1$ as one cycle and a (disjoint) cycle consisting of two diagonals joining a nodes in A_1 to nodes in B_1 and edges of the hamiltonian cycle H. This contradicts (vi) of Lemma 7. Therefore $4 \leq m \leq p-4$ is impossible.

Figure 2

Case (3). $m = 3$ diagonal edges are drawn from node 1.
This situation is shown in Figure 2(i).

Let the edges be $(1, t_1)$, $(1, t_2)$ and $(1, t_3)$, where $3 \le t_1 < t_2 < t_3 \le p-1$. In this case, there are $(p-2) - 3 = p - 5 \ge 4$ edges that must be drawn between the nodes other than node 1. These edges must be drawn between nodes of the sets $A_2 = \{t_3, t_3 + 1, \ldots, p\}$ and $B_2 = \{2, 3, \ldots, t_1 - 1, t_1\}$. Furthermore, we may restrict the number of edges starting from t_3 (or t_1) to the set B_2 (or A_2) to be at most 2; otherwise there will be 4 or more diagonal edges starting from t_3 (or t_1) and so by Case(2), the result follows.

Now every cycle must use both t_1 and t_3 in order not to be disjoint from each other . Therefore one of the sets of edges $\{t_1t_3, t_1x\}$, $\{t_1t_3, t_3x\}$ or $\{t_1t_3, t_1x, t_3x\}$ must be used. It follows that 2 or 3 edges must join elements of the sets A_2 and B_2. Any other edges joining a pair of nodes other than t_1 and t_3 will form independent cycles.

Case (4). $m = 2$ diagonal edges are joined from node 1.
This case can be analyzed in the same manner as Case(3) above (see Figure 2 (ii)). Therefore the proof is completed. □

Lemma 10. *Let $C(G; \underline{w}) = C(W_p; \underline{w})$. Suppose that there is a node $v \in V(G)$ such that $d_v = p - 1$. Then $G \cong W_p$.*

Proof:

$$\sum_{u \in (V(G) - \{v\})} d_u = 2(2p-2) - (p-1) = 3(p-1)$$

$$\sum_{u \in (V(G) - \{v\})} \binom{d_u}{2} = [(p-1)(\frac{p-2}{2}) + 3(p-1) - \binom{p-1}{2} = 3(p-1)$$

by (iii) of Lemma 7.

It can be shown that $(3, 3, 3, \ldots)$ ($p-1$ times) is the only valency sequence satisfying the two conditions above. Therefore all the d_u's must be equal to 3. Hence the valency sequence of G is $(3, 3, 3, \ldots, p-1)$. Thus G is isomorphic either to the wheel W_p or some web (a graph consisting of concentric circles with radial edges connecting them). But no web can satisfy the condition (vi) of Lemma 7. Therefore $G \cong W_p$. □

The above lemmas lead to the following result.

Lemma 11. *The circuit polynomial characterizes all wheels with more than 8 nodes.*

We now establish the results for wheels with no more than 8 nodes.

Lemma 12. *The circuit polynomial characterizes W_p, for $1 \le p \le 7$.*

Proof: The result follows from direct calculation of the circuit polynomials of the graphs on 6 and 7 nodes.

The following lemma was recently established by Maharaj [10], using a computer program to investigate the circuit polynomials of all the graphs on 8 nodes.

Lemma 13. *The circuit polynomial characterizes W_8. (N.B. It is well-known (Xu and Li [11]) that the W_8 is not characterized by its chromatic polynomial.)*

The following theorem summarizes the above lemmas.

Theorem 3. *The circuit polynomial characterizes wheels.*

5 Discussion

The circuit polynomial has strong characterizing properties. However, it is not a characterizing polynomial for all graphs. For example, the following non-isomorphic graphs are cocircuit; i.e. they have the same circuit polynomial.

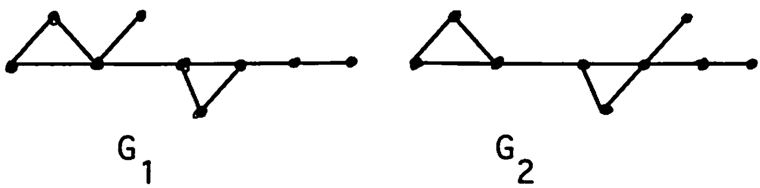

Figure 3

It can be verified that $C(G_1;\underline{w}) = C(G_2;\underline{w}) = w_1^9 + 10w_1^7w_2 + 29w_1^5w_2^2 + 25w_1^3w_2^3 + 5w_1w_2^4 + 2w_1^6w_3 + 10w_1^4w_2w_3 + 9w_1^2w_2^2w_3 + w_2^3w_3 + w_1^3w_3^2 + w_1w_2w_3^2$.

We have presented results which can be used to further compare the characterizing abilities of the various F-polynomials. This is an interesting problem. There are many families of graphs that are characterized by their characteristic polynomials, and by Lemma 6, they must also be characterized by their circuit polynomials. These include some of the graphs which have been considered above. However, in such cases, we have presented a novel combinatorial approach, to establish results which until now have been exclusively in the domain of Matrix Algebra.

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