On a coloring problem of P. Erdös

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ABSTRACT. Let $f(n,k)$ be the maximum chromatic number among all graphs whose edge set can be covered by n copies of $K(n)$, the complete graph on n vertices, so that any two of those $K(n)$ share at most k vertices. It has been known that $f(n,k) = (1-o(1)).n^{3/2}$ for $k \geq n^{1/2}$. We show that $(1-o(1))n.k \leq f(n,k) \leq (k+1)(n-k)$ for $k < n^{1/2}$, hence, for $1/k = o(1), f(n,k) = (1+o(1)).n.k$.

In a private letter P.Erdös proposed to study the following generalization of the famous Erdös, Faber, and Lovász conjecture. Determine the value of $f(n,k)$, the maximum chromatic number in the class $E(n,k)$ consisting of all graphs whose edge set can be covered by n copies of $K(n)$ such that any two of those $K(n)$ have at most k vertices in common, i.e. determine $f(n,k) = max\{\chi(G) : G \in E(n,k)\}$. In this setting, the famous conjecture of Erdös, Faber, and Lovász (see, e.g. [1]) claims that $f(n,1) = n$. So far the best result along this line is due to J.Kahn [4] who proved that $f(n,1) = (1+o(1))n$. On the other hand, in [3] it has been shown that if the size of intersection of $K(n)$'s is not limited then the chromatic number of G is at most $n^{3/2}$ and this bound is asymptotically best possible (see also [5]). In fact, as mentioned in [3], the same result is valid if the copies of $K(n)$ are assumed to share at most $n^{1/2}$ vertices. Hence, in this notation, $f(n,k) = (1-o(1))n^{3/2}$ for all $k \geq n^{1/2}$. As to the other values of k, P. Guan and T.Huang [2] stated that $f(n,k) \leq (2n-4).k+1$. (It seems to us that the method used in [2] allows to claim only that $f(n,k) \leq (2n-3).k+1$). We show that

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Theorem 1. *For any $k \leq (n-1)^{1/2}$, $(1-o(1))n.k \leq f(n,k) \leq (k+1)(n-k)$.*

As an immediate consequence we get:

Corollary 1. *If k is a function of n, $k < n^{1/2}, 1/k = o(1)$, then*

$$\lim_{n\to\infty} \frac{f(n,k)}{n.k} = 1.$$

We conjecture that $f(n,k) = (1+o(1)).n.k$ for any $k < n^{1/2}$.

Proof of Theorem 1: Let $G \in E(n,k)$. We denote by A the set of n copies of $K(n)$ covering the edge set of G. Sometimes elements of A will be called cliques from A. Further, for any vertex v of G, by valency of v, $val(v)$, we mean the number of cliques from A covering the vertex v. Let $val(v) = m > 1$, and let T be the set of vertices of G which are adjacent to v and are of valency at least m. We show that for t, cardinality of T, it holds

$$t \leq k - 1 + k.(n-m).m/(m-1) \qquad (\star)$$

Let A' be the set of m cliques from A covering v. Put $A'' = A - A'$. Consider the sum $S = \sum_{w\in T} val(w)$. Clearly, $t \leq S/m$. The contribution of any clique from A'' to S is at most $k.m$. More precisely, if w is a vertex of T covered by i, $1 \leq i \leq m$, cliques from A' then w is covered by at least $m-i$ cliques from A'', and any of those $m-i$ cliques contributes to S by at most $m.k-(i-1)$. Therefore, $S \leq m.k(n-m) - \sum_{1\leq i\leq m} s_i(m-i)(i-1) + \sum_{1\leq i\leq m} i.s_i$, where s_i is the number of vertices in T covered by exactly i cliques from A'. Thus, $S \leq m.k.(n-m) + \sum_{1\leq i\leq m} s_i(i^2-im+m) \leq m.k.(n-m)+s_1+s_{m-1}+m.s_m$ as $i^2 - im + m \leq 0$ for any $m > 1$ and $2 \leq i \leq m-2$. Consequntly, $t.m \leq S \leq m.k.(n-m) + t + (m-1)s_m$, and $(\star)$ is obtained by a simple rearrangement and the fact that $s_m \leq k-1$.

Now we are ready to prove the upper bound. Arrange the vertices of G in nonincreasing order with respect to their valency. We color the vertices in this order, one at a time. For $m \geq 2$, the function $f(m) = k-1+k(n-m).m/(m-1)$ is a decreasing function. Let v be a vertex of valency at least $k+1$. Since $f(m) < (k+1)(n-k)$ for $m \geq k+1$, $(\star)$ implies that there is at least one color left for proper coloring of v. On the other hand, any vertex v with $val(v) = m$ is adjacent to at most $(n-1)m$ vertices in G. As $(k+1).(n-k) \geq (n-1).k+1$ for $k \leq (n-1)^{1/2}$ there is a color available for coloring also any vertex v of $val(v) \leq k$.

To prove the lower bound we will use a slight modification of a construction given in [3]. Let $k < n^{1/2}$. Suppose that p is the largest prime power not exceeding $n^{1/2} - 1$. Then $p = (1-o(1)).n^{1/2}$ and the projective plane

$PG(p)$ of order p has $(1 - o(1)).n$ points and the same number of lines. Replace each point of the plane by a set of vertices of cardinality k. For any line l of the plane take the union U_l of vertices on l and add to them a new set of vertices U_l' of cardinality $n - (p+1).k$. Thus, to any line l we have assigned a set of vertices $V_l = U_l \cup U_l'$ of cardinality n. Consider a graph G with the vertex set $V = \cup_l V_l$, where two vertices from V are joined by an edge if they belong to a V_l. Clearly, the edges of G can be covered by n copies of $K(n)$. In addition, by the incidence axioms of $PG(p)$ any two $V_l's$ intersect in exactly k vertices, i.e. $G \in E(n,k)$. Finally, a subgraph of G induced by the set $\cup_l U_l$ forms a complete subgraph of G, hence $\chi(G) \geq (1 - o(1))n.k$. The proof is complete.

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References

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