Edge-3-coloring of a family of cubic graphs

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ABSTRACT. Let G be a cubic graph containing no subdivision of the Petersen graph. If G has a 2-factor F consisting of two circuits C_1 and C_2 such that C_1 is chordless and C_2 has at most one chord, then G is edge-3-colorable. This result generalizes an early result by Ellingham and is a partial result of Tutte's edge-3-coloring conjecture.

1 Introduction

A *cubic graph* is a 3-regular simple graph. A *2-factor* of a graph G is a 2-regular spanning subgraph of G. The *underlying graph* of a graph G, denoted by $\overline{G}$, is the graph homeomorphic to G and containing no degree two vertex. A *chord* of a circuit C is an edge not in C with both endvertices in $V(C)$. A cubic graph G is called a *permutation graph* if G has a 2-factor F which is the union of two chordless circuits. All other graph-theoretic terms that are used in this paper can be found, for instance, in [6].

The following well-known conjecture due to Tutte is a generalization of the 4-color problem ([3, 4, 5, 10]).

Conjecture 1 *(The Edge-3-coloring Conjecture, Tutte [11]) Every 2-edge-connected cubic graph containing no subdivision of the Petersen graph is edge-3-colorable.*

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It is easy to see that a smallest counterexample to Conjecture 1 must have a 2-factor which is a union of a few even-circuits and precisely two odd circuits. It is natural to study the edge-3-colorability of a cubic graph which has a 2-factor consisting of precisely two (odd) circuits. This motivates the following work by Ellingham.

Theorem 1 (Ellingham [7]) *If G is a permutation graph containing no subdivision of the Petersen graph, then*
(i). *G contains a 4-circuit,*
(ii). *G contains a Hamilton circuit,*
(iii). *G is edge-3-colorable.*

The edge-3-colorability of permutation graphs containing no subdivision of the Petersen graph is useful in cycle cover problems and is a key lemma in showing that a minimal counterexample to the cycle double cover conjecture contains a subdivision of the Petersen graph ([1, 2]).

Definition 1 *Let G be a cubic graph with at least four vertices and F be a 2-factor of G which is the union of two chordless circuits C_1 and C_2. The set of edges joining C_1 and C_2 is denoted by M. A circuit of length four containing exactly two edges of M is called an M-C_4. A subdivision of the Petersen graph in G is called a P_{10}-*subgraph. *A P_{10}-subgraph which has a 2-factor consisting of all edges of F (so that it has a perfect matching consisting of five edges of M) is called an M-P_{10}-subgraph.*

In [7], Ellingham actually showed that if a permutation graph has no M-P_{10}-subgraph then it contains an M-C_4. From this it is easy to construct a Hamilton circuit in G, so G is edge-3-colorable (all edges not in the Hamilton circuit have the same color). V. Klee ([9]) showed that *for each odd integer $n \geq 9$, there is a non-Hamiltonian permutation graph (with a P_{10}-subgraph) of order* $2n$, but was unable to determine what happens for each even n.

The main result of this paper is the following generalization of Theorem 1.

Theorem 2 *Let G be a cubic graph containing no subdivision of the Petersen graph. If G has an edge e such that $\overline{G \setminus \{e\}}$ is a permutation graph with a 2-factor F which is the union of two chordless circuits and e subdivides two edges in F, then G is edge-3-colorable.*

If the graph G described in Theorem 2 is itself a permutation graph, then we have Theorem 1. However, if the edge e subdivides two edges in one of the chordless circuits in F, then G might not have an M-C_4-circuit. Goldwasser and Zhang ([8]) showed that every permutation graph containing no M-P_{10}-subgraph has at least two M-C_4's, and constructed an infinite family having precisely two M-C_4's. If G is obtained by adding the edge e to one of the chordless circuits in F so as to cut both of the M-C_4's, then G has no M-C_4 (see an example illustrated in Figure 1). So Ellingham's method certainly cannot be used to obtain a Hamilton circuit in G and we have to take a different approach in this case.

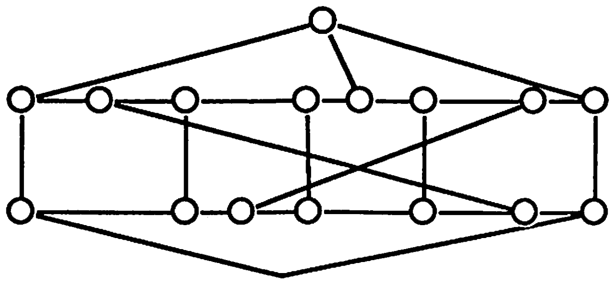

Figure 1. A graph with no $M - C_4$ circuit

In this paper, we characterize the permutation graph containing no M-P_{10} subgraph and precisely two M-C_4's. We use this characterization to get the desired edge-3-coloring to prove the main case of Theorem 2.

2 The easy cases

Theorem 2 is obviously true for the following cases.
(1). *G itself is a permutation graph;*
(2). *Both circuits of F are of even length* (because edges of F can be alternatively colored with two colors, with all other edges having the third color);
(3). *G has an M-C_4* (so G has a Hamilton circuit).

If $\overline{G \setminus \{e\}}$ has at least three M-C_4's then G has an M-C_4 and we are done. Goldwasser and Zhang proved the following lemma.

Lemma 3 ([8] Theorem 4) *A permutation graph G containing no M-P_{10}-subgraph contains at least two M-C_4's.*

Thus, the only remaining case is that the two chordless circuits in $\overline{G \setminus \{e\}}$ have odd length and $\overline{G \setminus \{e\}}$ has precisely two M-C_4's both of which are 5-circuits in G. So Theorem 2 will be proved if we can prove the following lemma.

Lemma 4 *Let G be a cubic graph of order $4k \geq 12$ containing no subdivision of the Petersen graph and F be a 2-factor of G such that F is the union of two circuits C_1 and C_2 where C_1 has a chord $e = xy$, and $\overline{G \setminus \{e\}}$ is a permutation graph which has precisely two M-C_4's each of which contains one of $\{x, y\}$ in a subdivided edge. Then G is edge-3-colorable.*

3 Nested permutation graphs

Define a bijection $f : Z \mapsto Z$ as follows:

$$f(i) = \begin{cases} i & \text{if } i \text{ is even} \\ -i & \text{if } i \text{ is odd.} \end{cases}$$

Let $k \geq 2$ be a positive integer and construct a permutation graph H_k as follows. Let $A = a_{-k} \cdots a_0 \cdots a_k a_{-k}$ and $b_{-k} \cdots b_0 \cdots b_k b_{-k}$ be two disjoint circuits, let $M = \{a_i b_{f(i)} : -k \leq i \leq k\}$, let $V(H_k) = V(A) \cup V(B)$ and let $E(H_k) = M \cup E(A) \cup E(B)$. Let $L_k = \overline{H_k \setminus \{a_0 b_0\}}$ and denote the family of permutation graphs L_k by $\mathcal{L}$ (see Figure 2). Obviously, $a_1 b_{-1} b_1 a_{-1} a_1$ and $a_k b_k b_{-k} a_{-k} a_k$ (when k is even) or $a_k b_{-k} b_k a_{-k} a_k$ (when k is odd) are the only 4-circuits of L_k (not just M-C_4's) if $k \geq 3$.

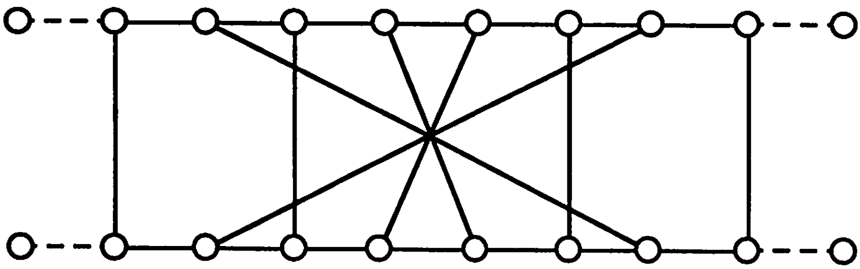

Figure 2. L_k

Goldwasser and Zhang proved the following lemma.

Lemma 5 (Goldwasser and Zhang [8]) *Each graph in $\mathcal{L}$ contains no M-P_{10} and precisely two M-C_4's.*

Let $\xi_i = 1$ or 2 for each $i = 2, \cdots, k-1$. A *nested permutation graph* is a graph which is the underlying graph obtained by deleting from L_k precisely $2 - \xi_i$ of the edges in $\{a_{-i}b_i, b_{-i}a_i\}$ if i is odd and in $\{a_{-i}b_{-i}, a_i b_i\}$ if i is even, for $i = 2, 3, \cdots, k-1$. We say such a nested permutation graph is of type $2, \xi_2, \xi_3, \cdots, \xi_{k-1}, 2$ and denote by $\mathcal{N}$ the set of nested permutation graphs.

Such a graph has $2[4 + \sum_{i=2}^{k-1} \xi_i]$ vertices, precisely two M-C_4's (since $\xi_i \neq 0$), and, by Lemma 5, no M-P_{10}. Two nested permutation graphs of the same type might not be isomorphic, while two of different types might be isomorphic. For example, Figures 3 (a) and (b) show one of type $2, 2, 1, 2$ and one of type $2, 1, 2, 2$ which are isomorphic (in fact there is one isomorphism class for all nested permutation graph of these two types), and Figure 3 (c) and (d) show two of type $2, 1, 1, 1, 2$ which are not.

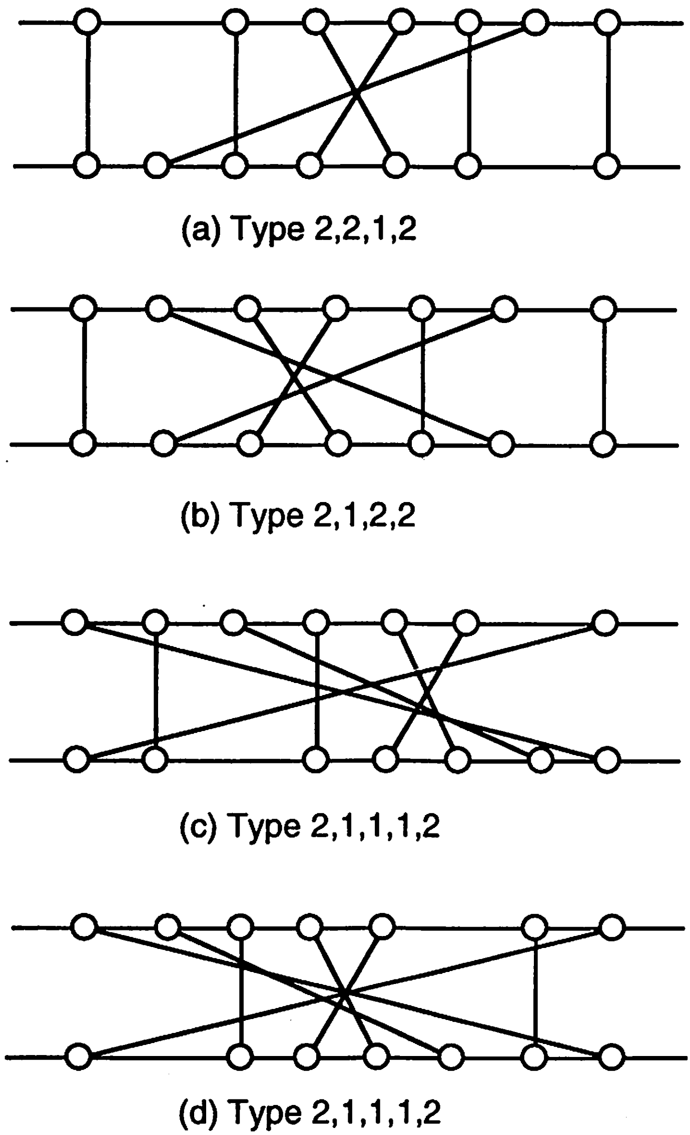

Figure 3. Nested permutation graphs of order 14

The statements in the following lemma follow easily from the definition of nested permutation graph.

Lemma 6 *Let $L_k \in \mathcal{L}$ and let G be a nested permutation graph of type $2, \xi_2, \xi_3, \cdots, \xi_{k-1}, 2$ (which is a subgraph of L_k).*

(a). *Let G' be the graph obtained by subdividing the edges $a_{-1}a_1$ and b_1b_2 of G to get $a_{-1}ra_1$ and b_1sb_2 and adding an edge rs (Figure 4 (a)). Then G' is a nested permutation graph (of type $2, 1, \xi_2, \xi_3, \cdots, \xi_{k-1}, 2$, with a new M-C_4 $a_{-1}rsb_1a_{-1}$).*

(b). *Let G'' be the graph obtained by subdividing the edges $a_{-1}a_1$ and $b_{-1}b_1$ of G to get $a_{-1}rr'a_1$ and $b_{-1}ss'b_1$ and adding edges rs and $r's'$ (Figure 4 (b)). Then G'' is a nested permutation graph (of type $2, 2, \xi_2, \cdots, \xi_{k-1}$, 2 with a new M-C_4 $rr's'sr$).*

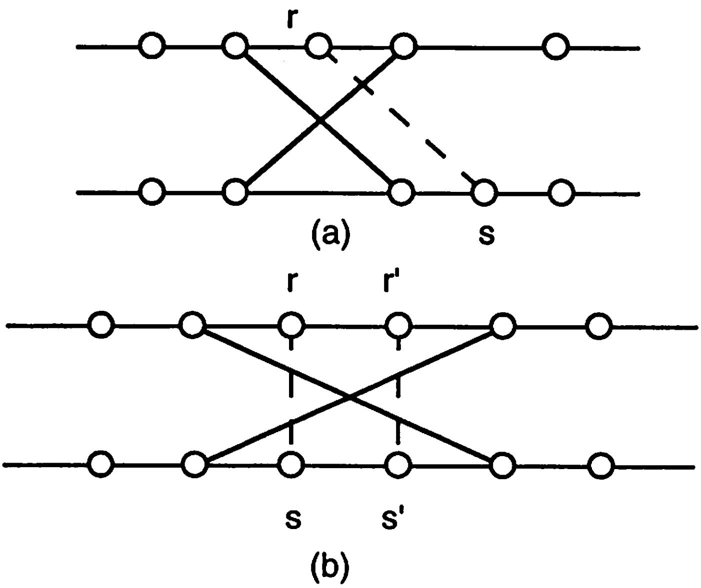

Figure 4. Adding edges to get nested permutation graphs

Theorem 7 *Let G be a permutation graph with no M-P_{10} subgraph. Then G has precisely two M-C_4's if and only if G is a nested permutation graph.*

Proof. Clearly each graph in $\mathcal{N}$ has precisely two M-C_4's and no M-P_{10}. Assume that G is a permutation graph with no M-P_{10} and precisely two M-C_4's. It is easy to check that such a graph G has at least 8 vertices and that if G has precisely 8 vertices then $G \in \mathcal{N}$ (of type $2, 2$). Assume that G is a permutation graph with minimum order $2n \geq 10$ such that G has no M-P_{10} and precisely two M-C_4's, but $G \notin \mathcal{N}$.

Let $A = c_1c_2\cdots c_nc_1$ and $B = d_1d_2\cdots d_nd_1$ be two chordless circuits whose union is a 2-factor of G and suppose $c_1d_2d_1c_2c_1$ is one of the M-C_4's. By Lemma 3, $\overline{G \setminus \{c_1d_2\}}$ contains at least two M-C_4's, so at least one of c_nd_n and c_3d_3 must be an edge of G. If only one, say c_nd_n, is an edge of G, then $\overline{G \setminus \{c_1 d_2\}}$ contains precisely two M-C_4's (one has edges $c_n c_1 c_2, c_2 d_1, d_1 d_n, d_n c_n$), so must be in $\mathcal{N}$ since G is a minimal counterexample. Then, by Lemma 6 (a) (with $c_2 c_1 c_n$ and $d_1 d_2 d_3$ in the role of $a_{-1} r a_1$, and $b_1 s b_2$ respectively), G is in $\mathcal{N}$. If both $c_n d_n$ and $c_3 d_3$ are in G, then $\overline{G \setminus \{c_1 d_2, d_1 c_2\}}$ is of smaller order and clearly has precisely two M-C_4's, so must be in $\mathcal{N}$. Hence, by Lemma 6 (b), $G \in \mathcal{N}$. □

4 Proof of the main theorem

The difficulty in trying to prove Lemma 4 by a straightforward induction is that arbitrary edge-3-coloring of a graph cannot always be readily modified to obtain an edge-3-coloring of a graph with an extra edge or two. So we will prove a stronger result than Lemma 4 (so that we can have a stronger inductive hypothesis)

Definition 2 *Let $\mathcal{N}_C$ be the set of all graphs G_C which can be obtained by adding an edge e (and two vertices) to one of the disjoint chordless n-circuits of a nested permutation graph G of order 2n ($n \geq 4$) so that e cuts both M-C_4's of G. Each M-C_4 of G has a subdivided edge in G_C and these two 5-circuits are called the* ears *of G_C. Let A and B be two disjoint chordless n-circuits of G and let A′ be the (n + 2)-circuit formed in G_C by adding the edge e to A in G. We say that an edge-3-coloring $\mathcal{T}$ of G_C is* simple *at the ear R if the two edges of R not contained in $A' \cup B$ have the same color in $\mathcal{T}$.*

We note that if $\mathcal{T}$ is simple at R then the seven edges of $A' \cup B$ incident to some vertex of R (two are in $A' \cap R$ and one is in $B \cap R$) must be 2-colored in $\mathcal{T}$.

Now we are ready to prove a stronger version of Lemma 4. It is easy to see that there is a unique graph which satisfies the hypothesis of Lemma 4 for $k = 3$, and Figure 6 shows that it has an edge-3-coloring.

Lemma 8 *Let G_C be a cubic graph of order $4k \geq 16$ containing no subdivision of the Petersen graph and F be a 2-factor of G such that F is the union of two circuits C_1 and C_2 where C_1 has a chord $e = xy$, and $\overline{G_C \setminus \{e\}}$ is a permutation graph which has precisely two M-C_4's each of which contains one of $\{x, y\}$ in a subdivided edge. Then, for each ear R of G_C, G_C has an edge-3-coloring which is simple at R.*

Proof. We use induction on k. The edge-3-colorings in Figure 5 show that the result holds for $k = 4$. There are four isomorphically distinct possibilities for G_C (from the three isomorphically distinct graphs of order 14 in $\mathcal{N}$, one of type $2, 2, 1, 2$ and two of type $2, 1, 1, 1, 2$), and a total of six isomorphically distinct ordered pairs (G_C, R) of a graph G_C of order 16 and an ear R in G_C (there are automorphisms in two of the four graphs G_C which transpose the ears). It is necessary to start the induction at $k = 4$ because while there is an edge-3-coloring of the unique graph of order 12 in $\mathcal{N}_C$ (Figure 6), there is no edge-3-coloring simple at an ear.

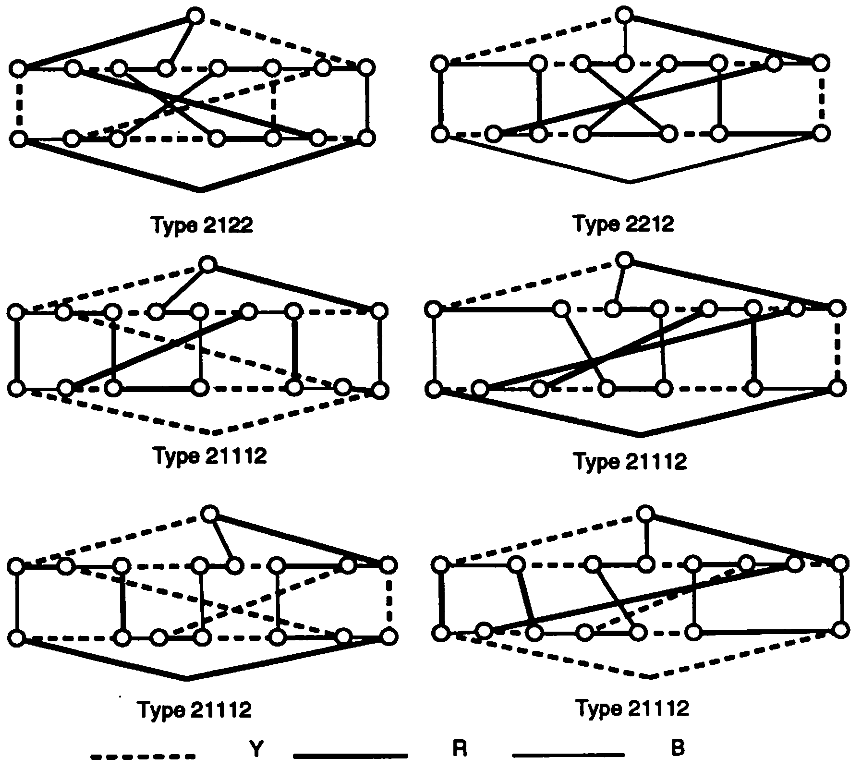

Figure 5. Edge-3-colorings simple at an ear of graphs of order 16

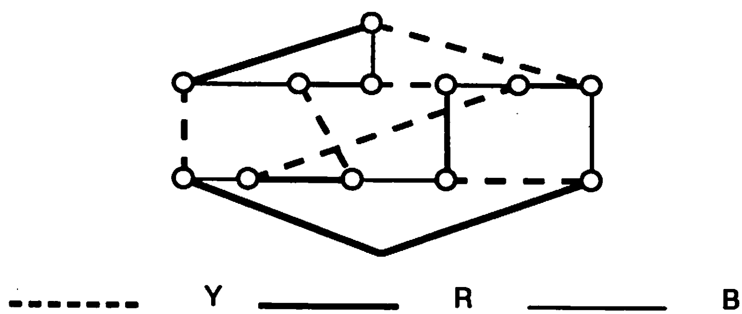

Figure 6. An edge-3-coloring of the graph in $\mathcal{N}_C$ of order 12

Now let k be at least 5 and we assume the result for all graphs in $\mathcal{N}_C$ of order less than $4k$. Let $G_C \in \mathcal{N}_C$ have order $4k$, and let e be an edge of G_C such that $G = G_C \setminus \{e\}$ is in $\mathcal{N}$ with type $2, \xi_2, \xi_3, \cdots, \xi_{h-1}, 2$, so G is the underlying graph of a subgraph of L_h, for some $h \geq k$, where L_h has disjoint chordless $2h$-circuits $C_1 = a_{-h} \cdots, a_{-1}a_1 \cdots a_h a_{-h}$ and $C_2 = b_{-h} \cdots, b_{-1}b_1 \cdots b_h b_{-h}$. Assume $a_{-1}b_1b_{-1}a_1a_{-1}$ is an M-C_4 of L_h (so also of G), that a_0a_{h+1} is the chord added to G to get G_C and let $R = a_{-1}a_0a_1b_{-1}b_1a_{-1}$ be an ear of G_C. There are four (similar) cases.

(a)

(b)

(c)

(d)

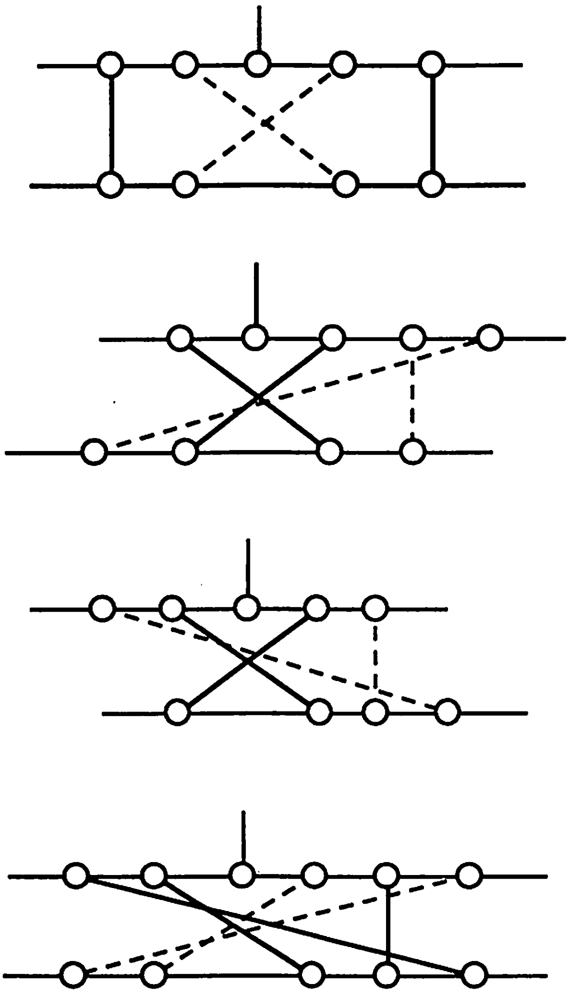

Figure 7. Subtracting edges to get graphs in $\mathcal{N}_C$

Case 1. $a_{-2}b_{-2}$ and a_2b_2 are edges of G_C (Figure 7 (a)).

Then $\overline{G\setminus\{a_{-1}b_1, b_{-1}a_1\}} \in \mathcal{N}$. Thus, $H' = \overline{G_C\setminus\{a_{-1}b_1, b_{-1}a_1\}}$ is in $\mathcal{N}_C$, has order $4(k-1)$, and has $R_{H'} = a_{-2}a_0a_2b_2b_{-2}a_{-2}$ as one of its ears. By the inductive hypothesis, H' has an edge-3-coloring $\mathcal{T}_{H'}$ which is simple at $R_{H'}$. Suppose the colors in $\mathcal{T}_{H'}$ of the edges $a_{-3}a_{-2}, a_{-2}a_0, a_0a_2, a_2a_3$ are R, Y, R, Y respectively, of the edges $b_{-3}b_{-2}, b_{-2}b_2, b_2b_3$ are R, Y, R respectively (they could be Y, R, Y instead), and that B is the third color in $\mathcal{T}_{H'}$. Define a coloring $\mathcal{T}$ of the edges of G_C by assigning the edges $a_{-2}a_{-1}, a_{-1}a_0, a_0a_1, a_1a_2$ Y, R, Y, R respectively, assigning the edges $b_{-2}b_{-1}, b_{-1}b_1, b_1b_2$ Y, R, Y respectively, letting $\mathcal{T}(a_{-1}b_1) = \mathcal{T}(b_{-1}a_1) = B$, and assigning all other colors in $\mathcal{T}$ as they are in $\mathcal{T}_{H'}$. It is easy to check that $\mathcal{T}$ is an edge-3-coloring of G_C which is simple at R.

Case 2. $a_{-2}b_{-2}, a_{-3}b_3 \notin E(G_C)$, $\{a_2b_2, a_3b_{-3}\} \subseteq E(G_C)$ (Figure 7 (b)).

Then, $H'' = \overline{G_C\setminus\{b_{-3}a_3, a_2b_2\}} \in \mathcal{N}_C$, so has an edge-3-coloring $\mathcal{T}_{H''}$ simple at the ear $R = a_{-1}a_0a_1b_{-1}b_1a_{-1}$. $\mathcal{T}_{H''}$ can be modified to get an edge-3-coloring $\mathcal{T}$ of G_C which is simple at R.

Case 3. $a_{-2}b_{-2}, b_{-3}a_3 \notin E(G_C)$, $\{a_2b_2, a_{-3}b_3\} \subseteq E(G_C)$ (Figure 7 (c)).

Then, $H''' = \overline{G_C\setminus\{a_{-3}b_3, a_2b_2\}} \in \mathcal{N}_C$, so has an edge-3-coloring simple at the ear $a_{-1}a_0a_1b_{-1}b_1a_{-1}$ and the argument is the same as before.

Case 4. $a_{-2}b_{-2} \notin E(G_C)$, $\{a_2b_2, a_{-3}b_3, b_{-3}a_3\} \subseteq E(G_C)$ (Figure 7 (d)).

Then, $H'''' = \overline{G_C\setminus\{b_{-3}a_3, b_{-1}a_1\}} \in \mathcal{N}_C$, so has an edge-3-coloring simple at the ear $a_{-1}a_0a_2b_2b_1a_{-1}$ and the argument is the same as before. □

5 A stronger theorem

A graph G satisfying the hypothesis of Theorem 2 must, in fact, have a Hamilton circuit (which implies that it is edge-3-colorable). This is obvious if G has an M-C_4. And, if not, it follows from the following lemma, a strengthening of Lemma 8.

Lemma 9 *Let G_C be a cubic graph of order $2n \geq 16$ containing no subdivision of the Petersen graph and F be a 2-factor of G_C such that F is the union of two circuits C_1 and C_2 where C_1 has a chord $e = xy$ and $\overline{G_C\setminus\{e\}}$ is a permutation graph which has precisely two M-C_4's, each of which contains one of $\{x, y\}$ in a subdivided edge. Let R be an ear of G_C and let f and g be the two edges in $R\setminus(C_1 \cup C_2)$. Then there is a Hamilton circuit in G_C which includes neither f nor g.*

The proof of Lemma 8 is actually a proof of Lemma 9 if G_C has order $4m \geq 16$ (Figure 5 shows the induction can start at $m = 4$, because the $R - Y$ subgraph is a Hamilton circuit in each graph). There are two isomorphically distinct graphs of order 12 in $\mathcal{N}$ (one of the type $2, 2, 2$, one of type $2, 1, 1, 2$) and three isomorphically distinct graphs of order 14 in $\mathcal{N}_C$, all of which have Hamilton circuits. But of the three isomorphically distinct ordered pairs (G_C, R) of a graph $G_C \in \mathcal{N}_C$ of order 14 and an ear R of G_C, only one has a Hamilton circuit with the special property required in Lemma 9. So to start the induction when G_C has order $4m+2$ we must check that for each ordered pair (G_C, R) of a graph $G_C \in \mathcal{N}_C$ of order 18 and an ear R of G_C there is a Hamilton circuit in G_C which misses $R \setminus (C_1 \cup C_2)$. There are 15 such ordered pairs to check, so we decided to omit the proof of Lemma 9 in this paper, and to be content to show edge-3-colorability instead (if G_C has order $4m+2$ then C_1 and C_2 are each even circuits, so G_C is obviously edge-3-colorable).

6 Problems

A minimal counterexample to the Tutte's Edge-3-coloring Conjecture has a 2-factor that consists of only two odd circuits and all other components are even circuits. Considering the edge-3-colorability of a graph with a 2-factor consisting of precisely two (odd) components was initially the motivation of the paper by Ellingham ([7]). Thus, it is natural to try to generalize Theorem 1 and Theorem 2 as follows.

Problem 1 *Let G be a bridgeless cubic graph and F be a 2-factor of G which consists of at most two components. If G does not contain a subdivision of the Petersen graph, is G edge-3-colorable?*

Apparently there is no other progress toward solving this problem. In the statements of Theorem 1 and Theorem 2, the assumption is made that G contains no P_{10}-subgraph. However, the proofs were actually done under the weaker assumption that G contains no M-P_{10}-subgraph. There is no hope for similar situation for Problem 1, as can be seen from the following example. Let C be a longest circuit of the Petersen graph P_{10} which is of length 9 and replace the vertex $v \in V(P_{10}) \setminus V(C)$ with a triangle C'. Then the new graph H has a 2-factor $C \cup C'$, does not have an M-P_{10}-subgraph (since $|M| = 3$), and is not edge-3-colorable (it does have a P_{10}-subgraph). Since the subgraph of H induced by $V(C)$ is $K_{3,3}$, could we relax the condition and consider the following problem?

Problem 2 *Let G be a bridgeless cubic graph and F be a 2-factor of G such that F is the union of two circuits C_1, C_2. If G has no M-P_{10} and each subgraph of G induced by $V(C_i)$ $(i = 1, 2)$ is planar, is G edge-3-colorable?*

Theorem 1 and Theorem 2 are special cases of Problem 2. Even if the whole graph G is assumed to be planar, we still do not have a proof without applying the 4-color theorem.

Problem 3 *Let G be a bridgeless cubic planar graph such that G has a 2-factor F consisting of two odd circuits. Can we prove that G is edge-3-colorable without applying the 4-color theorem?*

Adding extra chords to permutation graphs containing no P_{10}-subgraph makes edge-3-colorability harder to prove. Even without extra chords, the following problem may not be an easy one.

Problem 4 *Let G be a 3-connected, cyclically 5-edge-connected permutation graph. If $G \neq P_{10}$, is G edge-3-colorable?*

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