Hypohamiltonian/hypotraceable digraphs abound

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December 6, 1995

Abstract

The number of hypohamiltonian and that of hypotraceable n-vertex digraphs are both bounded below by a superexponential function of n for $n \geq 6$ and $n \geq 7$, respectively.

1 Introduction

All graphs considered are simple; digraphs loopless. A nonhamiltonian graph whose all vertex-deleted subgraphs are hamiltonian is called ***hypohamiltonian***. Similarly, a digraph is called ***hypohamiltonian*** [***hypotraceable***] if it has no hamiltonian dicycle [no hamiltonian. dipath] but each vertex-deleted subdigraph does have one. A digraph D is called ***homogeneously bilaterally traceable*** (i.e., ***bihomogeneously traceable***) if, for each vertex x of D, there are a hamiltonian dipath which starts at x and another one which ends at x. Hence, hypohamiltonian digraphs are nonhamiltonian and bihomogeneously traceable. Hypohamiltonian graphs are clearly homogeneously traceable. A digraph without any 2-dicycle is said to be an *oriented graph*.

There are known only exponentially many (cubic or so) hypohamiltonian graphs and minimum nonhamiltonian either homogeneously traceable graphs or bihomogeneously traceable oriented graphs all on n vertices (for all n large enough: $n \geq 18$, $n \geq 9$, $n \geq 7$, respectively, or for some smaller n's), cf. Skupień [7, 8] and [9, with erratum: add arrow $D \to b$ in Fig. 3], respectively. However, a superexponential number of n-vertex homogeneously traceable nonhamiltonian undirected graphs are constructed in Skupień [10] for all $n \geq 9$. Using constructions presented in Grötschel et

JCMCC **24** (1997), pp. 239-242

al. [3] and Thomassen [11], we prove the following result mentioned in [10, p. 29].

Theorem 1 *There are superexponentially many hypohamiltonian [resp., hypotraceable] digraphs on n vertices exactly for $n \geq 6$ [resp., $n \geq 7$].*

It is an open problem if the cardinality of n-vertex digraphs which are maximally hypohamiltonian (or maximally hypotraceable) is superexponential in n. In the hypohamiltonian case that cardinality, if nonzero, can be seen to be at least exponential. So is namely that of maximally hypohamiltonian undirected (triangle-free for all $n \geq 48$) graphs [6] and, furthermore, the property of being maximally hypohamiltonian is clearly invariant under replacing each edge by a 2-dicycle. A polyhedral approach to the Asymmetric Travelling Salesman Problem involves many of those maximal digraphs [5].

2 Preliminaries

Assume that a digraph D has a vertex y of in- and out-degree 2 and with three neighbours x, w and z such that D includes digraphs $\vec{w}yz$ and $\vec{x}yx$. Let D' be the digraph obtained from D by deleting the vertex y and adding two new vertices y' and z' together with eight arcs $x \to y' \to z' \to x$ and $w \to y' \to w \to z \to z' \to z$. Call the operation $D \mapsto D'$ the *expansion of D at y*. Then D' is hypohamiltonian provided that D is hypohamiltonian and satisfies the following conditions (i) and (ii).

(i) $D - \{y, x\}$ has no Hamiltonian w–z dipath;

(ii) $D - \{y, w\}$ has no Hamiltonian x–z dipath.

Moreover, the expansion can be iterated, i.e., the expansion of D' at y' gives a hypohamiltonian digraph because the counterparts of (i) and (ii) can be seen to be true. These are a construction and result due to Thomassen [11, Fig. 1].

Consider a hypohamiltonian digraph D on $n-1$ vertices. Let v be any vertex of D. Then *splitting v into a source, v', and a sink, v''*, such that v' dominates the same vertices as v and v'' is dominated by the same vertices as v, results in a hypotraceable digraph of order n. This construction is due to Grötschel et al. [3].

3 Proof and Comments

Proof. Bounds on n come from the fact that the cardinalities of the digraphs in question are proved to be nonzero exactly for $n \geq 6$ [1, 2, 11, 4] and

$n \geq 7$ [3], respectively. For each natural constant k, consider the digraph, M_{2k+1}, consisting of two disjoint odd dicycles $C^x := \vec{x}_1 x_2 \ldots x_{2k+1} x_1$ and $C^y := \vec{y}_1 y_2 \ldots y_{2k+1} y_1$ together with all 2-dicycles $\vec{x}_i y_i x_i$. In [11, 4] M_{2k+1} is proved to be hypohamiltonian; Thomassen [11] identifies M_{2k+1} with the Cartesian product $\vec{C}_2 \times \vec{C}_{2k+1}$. Fouquet and Jolivet prove in [1] that the digraph with $k = 1$ (the smallest one) is a factor of all hypohamiltonian digraphs on 6 vertices.

Let Y be any subset of vertices y in the dicycle C^y of the digraph M_{2k+1} such that Y contains no two consecutive vertices of C^y. Being hypohamiltonian is not spoiled—as noted in Grötschel et al. [3, proof of Thm 2]—if the complete digraph with vertex set Y is added to M_{2k+1}. Let D be a digraph obtained from M_{2k+1} by adding any set, say A', of Y–Y arcs. Then D is a hypohamiltonian digraph. Clearly, the order

$$n := 4k + 2,$$

of D is determined as is the cardinality $|A'|$ of A' and, if $A' \neq \emptyset$, then D determines the dicycle C^x, then C^y and, finally, the set A' up to the labels of the vertices. Furthermore, the labels of vertices in the dicycle C^y (and those in D) are determined up to the action of the group of rotations of C^y, the cyclic group of order $2k + 1$. Let $N(D)$ denote the number of labelled digraphs D. If the vertex set Y of the maximum possible cardinality k is fixed, there are 2^{k^2-k} arc sets A' possible. Hence $N(D) \geq 2^{k^2-k}$, the equality therein being true only if $k = 1$. Therefore, by Burnside's lemma, the number of isomorphism classes of digraphs D is at least $N(D)/(2k + 1) \geq 2^{(n^2-8n+28)/16}/n$ for our n,
$n \equiv 2 \pmod 4$.

Assume that $Y = \{y_2, y_4, \ldots, y_{2k}\}$. Note that each digraph D satisfies requirements (i) and (ii) of the preceding Section with $y = y_1$ and $(x, w, z) = (x_1, y_{2k+1}, y_2)$. Therefore the expansion of D at y_1 for each D gives hypohamiltonian digraphs of order $n + 1$; next, recursively, of order $n + 2$ and $n + 3$. Thus all residue classes of n modulo 4 can be covered. Moreover, cardinalities of resulting digraphs of any fixed order $n \geq 6$ clearly remain superexponential in n.

Finally, one can see that splitting the vertex y_{2k+1} into a source and a sink in all of the above digraphs D of any fixed order n, $6 \leq 4k + 2 \leq n \leq 4k + 5$, gives superexponentially many hypotraceable digraphs of order $n + 1$ (≥ 7). □

Our results do not show that hypohamiltonicity among digraphs is more frequent than among graphs, at least if the order n is large enough. To see this, recall that the number of n-vertex digraphs is asymptotic to $2^{2\binom{n}{2}}/n! = 2^{n^2(1+o(1))}$ and that of n-vertex graphs to $2^{\binom{n}{2}}/n! = 2^{\frac{1}{2}n^2(1+o(1))}$ only. It is an open problem to relate those frequencies (and also the frequencies of hypotraceability).

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