THE ASYMPTOTIC NUMBER OF EULERIAN ORIENTED GRAPHS WITH MULTIPLE EDGES

Xiaoji Wang

School of Mathematics, University of New South Wales,
PO Box 1, Kensington, NSW 2033, Australia

1. Abstract.

We estimate the number of labelled loop-free eulerian oriented graphs with multiple edges with n vertices by using an n-dimensional Cauchy integral. An asymptotic formula is obtained for the case where the multiplicity of each edge is $O(\log n)$.

2. Main result.

By an *eulerian* oriented graph we mean a digraph in which the in-degree is equal to the out-degree at each vertex and at most one of the edges (v, w) and (w, v) is permitted for any distinct v and w. Let $EOGME(n, t)$ be the number of labelled loop-free eulerian oriented graphs with n vertices in which the multiplicity of each edge is at most t. Allowing loops would multiply $EOGME(n, t)$ by exactly $(t+1)^n$, since loops do not affect the eulerian property. For the case where $t = 1$, McKay [1] obtained the asymptotic formula

$$EOGME(n, 1) = \left(\frac{3^{n+1}}{4\pi n}\right)^{(n-1)/2} n^{1/2} e^{-3/8} \left(1 + O(n^{-1/2+\epsilon})\right).$$

for any $\epsilon > 0$.

We will identify $EOGME(n, t)$ as a coefficient in an n-variable power series, and estimate it by applying the saddle-point method to the integral provided by Cauchy's Theorem. In particular, the choice of contour is trivial but substantial work is required to demonstrate that the parts of contour where the absolute value of the integrand is small contribute negligibly to the result.

The techniques used are similar to those to estimate the number of eulerian digraphs with multiple edges [2], but here we allow the bound of the multiplicity of each edge to be $O(\log n)$ where n is the order of the graph.

For $s \geq 0$ and $n \geq 1$, define $U_n(s) = \{(x_1, x_2, \ldots, x_n) \mid |x_i| \leq s \text{ for } 1 \leq i \leq n\}$. Let $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_n)$, $\boldsymbol{\theta}' = (\theta_1, \ldots, \theta_{n-1})$ and let $T : \mathbb{R}^{n-1} \mapsto \mathbb{R}^{n-1}$ be the linear transformation defined by $T : \boldsymbol{\theta}' \mapsto \boldsymbol{y} = (y_1, y_2, \ldots, y_{n-1})$, where

$$y_j = \theta_j - \sum_{k=1}^{n-1} \theta_k / (n + n^{1/2})$$

for $1 \leq j \leq n-1$. The following Theorem 2.1 was obtained by McKay [1] which is useful for our estimation.

Theorem 2.1. *Let a, b and c be real numbers with $a > 0$. Let $0 < \epsilon < 1/8$, and let $n \geq 2$ be an integer. Define*

$$J = J(a,b,c,n) = \int \exp\Big(-a \sum_{1\leq j<k\leq n} (\theta_j - \theta_k)^2 + b \sum_{1\leq j<k\leq n} (\theta_j - \theta_k)^4 + \frac{c}{n^2}\Big(\sum_{1\leq j<k\leq n} (\theta_j - \theta_k)^2\Big)^2\Big)\, d\theta',$$

where the integral is over $\theta' \in U_{n-1}(n^{-1/2+\epsilon})$ with $\theta_n = 0$. Then, as $n \to \infty$,

$$J = n^{1/2}\Big(\frac{\pi}{an}\Big)^{(n-1)/2} \exp\Big(\frac{6b+c}{4a^2} + O(n^{-1/2+4\epsilon})\Big). \quad \blacksquare$$

Lemma 2.2.

(i) For integer $t \geq 1$ and real x with $|x| < \pi/(2t+1)$,

$$\big|\, (1 + \exp(ix) + \exp(-ix) + \cdots + \exp(itx) + \exp(-itx) \,\big| \leq (2t+1)\exp\big(-\tfrac{1}{6}t(t+1)x^2\big).$$

(ii) For integer $t \geq 1$ and any real x,

$$\big|1 + \exp(ix) + \exp(-ix) + \cdots + \exp(itx) + \exp(-itx)\big| \leq 2t - 1 + \big(2 + 2\cos(x)\big)^{1/2}.$$

Proof. (ii) is true since $|1 + \exp(ix)| = \big(2 + 2\cos(x)\big)^{1/2}$ and each of the rest terms is bounded by 1. The proof for (i) is as follows. Clearly,

$$\begin{aligned}
&\big|\, (1 + \exp(ix) + \exp(-ix) + \cdots + \exp(itx) + \exp(-itx) \,\big| \\
&= \Big| \frac{1-\exp(i(t+1)x)}{1-\exp(ix)} + \frac{1-\exp(-i(t+1)x)}{1-\exp(-ix)} \Big| \\
&= \frac{\cos(tx) - \cos((t+1)x)}{1-\cos(x)} \\
&= \frac{\sin((2t+1)|x|/2)}{\sin(|x|/2)} \\
&= \exp\Big(\log(\sin((2t+1)|x|/2)) - \log\big(\sin(|x|/2)\big)\Big) \\
&\leq (2t+1)\exp\big(-\tfrac{1}{6}t(t+1)x^2\big),
\end{aligned}$$

since for $0 < x < \pi$,

$$\log(\sin(x)) = \log(x) + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k-1} B_{2k} x^{2k}}{k(2k)!},$$

where $\{B_n\}$ are the Bernoulli numbers, which satisfy $(-1)^k B_{2k} < 0$. ∎

Theorem 2.3. *For any $\epsilon > 0$, as $n \to \infty$,*

$$EOGME(n,t) = \left(\frac{3(2t+1)^n}{2t(t+1)\pi n}\right)^{(n-1)/2} n^{1/2} \exp\left(-\frac{3(2t^2+2t+1)}{20t(t+1)} + O(n^{-1/2+\epsilon})\right).$$

Proof. Since $\prod_{1\le j<k\le n}(1 + x_j^{-1}x_k + x_j x_k^{-1} + \cdots + x_j^{-t}x_k^t + x_j^t x_k^{-t})$ is the generating function for the digraphs in which the multiplicity of each edge is at most t, $EOGME(n,t)$ is the constant term. By Cauchy's Theorem,

$$EOGME(n,t) = \frac{1}{(2\pi i)^n} \times \oint \cdots \oint \frac{\prod_{j<k}(1 + x_j^{-1}x_k + x_j x_k^{-1} + \cdots + x_j^{-t}x_k^t + x_j^t x_k^{-t})}{x_1 x_2 \cdots x_n} dx_1 \cdots dx_n,$$

where each integration is around a simple closed contour encircling the origin once in the anticlockwise direction. We choose each contour to be the unit circle and substitute $x_j = e^{i\theta_j}$ for $1 \le j \le n$. We obtain

$$EOGME(n,t) = \frac{1}{(2\pi)^n} \int_{U_n(\pi)} \prod_{1\le j<k\le n} \left(1 + \sum_{m=1}^{t} \left(\exp(im(\theta_k - \theta_j)) + \exp(im(\theta_j - \theta_k))\right)\right) d\boldsymbol{\theta}.$$

Defining

$$T_{jk}(\boldsymbol{\theta}) = \frac{1 + \sum_{m=1}^{t}\left(\exp(im(\theta_k - \theta_j)) + \exp(im(\theta_j - \theta_k))\right)}{2t+1}$$

and

$$g(\boldsymbol{\theta}) = \prod_{1\le j<k\le n} T_{jk}(\boldsymbol{\theta}),$$

we have

$$EOGME(n,t) = \frac{(2t+1)^{n(n-1)/2} I}{(2\pi)^n},$$

where

$$I = \int_{U_n(\pi)} g(\theta)\, d\theta.$$

We will begin the evaluation of I with the part of the integrand which will turn out to give the major contribution. Let I_1 be the contribution to I of those θ such that $|\theta_j - \theta_n| \le n^{-1/2+\epsilon}$ for $1 \le j \le n-1$, where θ_j values are taken mod 2π. Since $g(\theta)$ is invariant under uniform translation of all θ_j, we see that the contributions to I_1 from different values of θ_n are the same. Hence,

$$I_1 = 2\pi \int_{U_{n-1}(n^{-1/2+\epsilon})} g(\theta)\, d\theta',$$

where $\theta' = (\theta_1, \ldots, \theta_{n-1})$ with $\theta_n = 0$.

By using Taylor's expansions for $\exp(ix)$ and $\log(1+z)$ for complex z, we obtain

$$\begin{aligned} g(\theta) &= \prod_{1 \le j < k \le n} T_{jk}(\theta) \\ &= \exp\Big(\sum_{1 \le j < k \le n} \log T_{jk}(\theta) \Big) \\ &= \exp\Big(-\frac{1}{6} t(t+1) \sum_{1 \le j < k \le n} (\theta_j - \theta_k)^2 \\ &\quad - \frac{1}{360} t(t+1)(2t^2+2t+1) \sum_{1 \le j < k \le n} (\theta_j - \theta_k)^4 \\ &\quad + O\big(\sum_{1 \le j < k \le n} |\theta_j - \theta_k|^5 \big) \Big). \end{aligned}$$

Applying Theorem 2.1, we have

$$\begin{aligned} I_1 = & 2\pi n^{1/2} \Big(\frac{6\pi}{t(t+1)n} \Big)^{(n-1)/2} \\ & \times \exp\Big(-\frac{3(2t^2+2t+1)}{20t(t+1)} + O(n^{-1/2+5\epsilon}) \Big). \end{aligned} \tag{2.1}$$

So our remaining work is to prove that the integral of $g(\theta)$ over the other parts of the region of integration is negligible compared to (2.1).

Let $\delta = \pi/6(2t+1)$. For $j = 0, 1, 2, 3, 24t+10, 24t+11$, define the interval $A_j = [(j-1)\delta, j\delta]$, and $B = [-\pi, -2\delta] \cup [2\delta, \pi]$. For any $\theta \in U_n(\pi)$, let us suppose that $A_0 \cup A_1$ contains $n/6(2t+1)$ or more of the θ_j. (If not, we can make this true by suitable translation). If $\theta_j \in B$ and $\theta_k \in A_0 \cup A_1$, then $\delta \le |\theta_j - \theta_k| \le \pi + \delta$. Define C to be the set consisting of such pairs (j,k) and I_2 to be the contribution to I of all the cases where n^ϵ or more of the θ_j lie in B. Since, if $(j,k) \in C$, $|T_{jk}(\theta)| \le (2t - 1 + (2 + 2\cos(\delta))^{1/2})/(2t+1)$, and for any other pair (j,k), $|T_{jk}(\theta)| \le 1$, we have that

$$|I_2| \le (2\pi)^n \big((2t - 1 + (2 + 2\cos(\delta))^{1/2})/(2t + 1)\big)^{n^{(1+\epsilon)/6(2t+1)}}.$$

From this it easily follows that $I_2 = O\big(\exp(-c_1 n^{1+\epsilon})\big) I_1$ for some $c_1 > 0$. Thus we can suppose that at least $n - n^\epsilon$ of the θ_j lie in $[-2\delta, 2\delta]$. Now define $I_3(r)$ to be the contribution to I of those θ such that

(i) $3\delta \le |\theta_j| \le \pi$ for r values of j,
(ii) $\theta_j \in [-2\delta, 2\delta]$ for at least $n - n^\epsilon$ values of j, and
(iii) $\theta_j \in A_3 \cup A_{24t+10}$ for any other values of j.

Clearly $I_3(r) = 0$ if $r > n^\epsilon$. If θ_j and θ_k are in classes (i) and (ii), respectively, then $\delta \le |\theta_j - \theta_k| \le \pi + 2\delta$, while if they are both in classes (ii) and (iii), by Lemma 2.2, $|T_{jk}(\theta)| \le \exp\big(-\frac{1}{6}t(t+1)(\theta_j - \theta_k)^2\big)$. Using $|T_{jk}(\theta)| \le 1$ for the other cases, we find

$$|I_3(r)| \le (2\pi)^r \binom{n}{r} \big((2t - 1 + (2 + 2\cos(\delta))^{1/2})/(2t+1)\big)^{r(n-n^\epsilon)} \times |I_3'(n - r)|, \tag{2.2}$$

where

$$I_3'(m) = \int_{U_m(\pi)} \prod_{1 \le j < k \le m} \exp\big(-\tfrac{1}{6}t(t+1)(\theta_j - \theta_k)^2\big) d\theta_1 \cdots d\theta_m.$$

We can apply the transformation T (using m in place of n) to easily obtain

$$I_3'(m) \le 2\pi m^{1/2} \Big(\frac{6\pi}{t(t+1)m}\Big)^{(m-1)/2}.$$

Substituting back into (2.2) we find that

$$\sum_{r=1}^{n^\epsilon} |I_3(r)| \le |I_1| \exp\Big(-c_2 n + o(n)\Big)$$

for some $c_2 > 0$. We conclude that the only substantial contribution must come from the case $r = 0$.

Next, define $I_4(h)$ to be the contribution to I of those θ such that

(i) $|\theta_n| \le 3\delta$,
(ii) $n^{-1/2+\epsilon} < |\theta_j - \theta_n| \le 6\delta$ for h values of j, and
(iii) $|\theta_j - \theta_n| \le n^{-1/2+\epsilon}$ for the remaining values of j.

Since $g(\theta)$ is invariant under uniform translation of all θ_j, we see that the contributions to $I_4(h)$ from different values of θ_n are the same. Hence, we have $|I_4(h)| \le 6\delta |I_4'(h)|$, where $|I_4'(h)|$ is the same integral over θ' with $\theta_n = 0$. Since we have $|T_{jk}(\theta)| \le \exp(-\frac{1}{6}t(t+1)(\theta_j - \theta_k)^2)$, apply the transformation T to transform the θ' to y and the values of θ' contributing to $I_4'(h)$ for $h \ge 1$ map to a subset of those y such that either $|\sum_{k=1}^{n-1} y_k| > n^\epsilon/2$ or $|y_k| > n^{-1/2+\epsilon}/2$ for some k. Since the contribution to

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp\Big(-\tfrac{1}{6}t(t+1)n\sum_{k=1}^{n-1} y_k^2\Big)\, dy$$

of those y is $O(n)(6\pi/t(t+1)n)^{(n-1)/2}\exp(-c_3 n^{2\epsilon})$ for some $c_3 > 0$, we conclude that

$$\sum_{h=1}^{n-1} |I_4(h)| \le O(n)\exp(-c_3 n^{2\epsilon})|I_1|.$$

The remaining case, $h = 0$, is covered by I_1. Therefore we have completed our proof. ∎

Acknowledgement.

I wish to thank Brendan McKay for his valuable comments.

References.

[1] B. D. McKay, The asymptotic numbers of regular tournaments, eulerian digraphs and eulerian oriented graphs, *Combinatorica,* 10 (4) (1990) 367–377.

[2] X. Wang, Asymptotic enumeration of eulerian digraphs with multiple edges, *Australas. J. Combin.*, 5 (1992) 293–298.