On Two Levels Balanced Arrays of Strength Six

D.V. Chopra
Wichita State University
Wichita, KS U.S.A.

R. Dio's
New Jersey Institute of Technology
Newark, New Jersey U.S.A.

Abstract. In this paper we derive some inequalities on the existence of balanced arrays (B-arrays) of strength six and with two symbols by using some results involving moments from Statistics. Besides providing illustrative examples, we will make brief comments on the use of these combinatorial arrays in Statistical Design of Experiments.

1. Introduction and Preliminaries

An array T with m constraints (rows), N runs (columns or treatment-combinations), and with two levels is merely a matrix T of size $(m \times N)$ and with two elements (say, 0 and 1). The weight of a column $\underline{\alpha}$ of T, denoted by $w(\underline{\alpha})$, is merely the number of 1's in $\underline{\alpha}$. Next we state the definition of a balanced array (B-array):

Definition 1.1. T is called a B-array of strength $t = 6$ if in every $(6 \times N)$ submatrix T^* of T, every vector $\underline{\alpha}$ of T^* with weight i $(i = 0, 1, \ldots, 6)$ appears a constant number (say) μ_i times, and $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_6)$ is called the index set of the array T. A B-array T is denoted by $(m, N, s = 2, t = 6, \underline{\mu}')$.

It is quite clear that $N = \sum_{c=0}^{6} \binom{6}{i} \mu_i$. Obviously orthogonal arrays (O-arrays) are special cases of B-arrays, and that the incidence matrices of incomplete block designs as well as those of BIB designs form special cases of B-arrays. B-arrays of strength six, under certain conditions, give rise to the construction of balanced fractional factorial designs which allow us to estimate all the effects up to and including three-factor interactions. In order to learn more about the applications of B-arrays to combinatorics and statistical design of experiments, the interested reader may consult the list of references (by no means an exhaustive list) at the end of this paper, and also further references given there in.

It is quite obvious that a B-arrays of strength six with $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_6)$ and $m = 6$ will always exist, but its existence for $m > 6$ is a difficult and non-trivial problem. The problem of the existence of B-arrays for a given $\underline{\mu}'$ and to obtain, for a given $\underline{\mu}'$, an upper bound on m is very important in combinatorics and statistical design of experiments. Such problems for O-arrays and B-arrays have been studied, among others, by Bose and Bush [1], Chopra and/or Dio's [3, 4], Rafter and Seiden [8], Saha et. al. [10], Yamamoto et. al. [14], etc. etc.

2. Main results on the existence of Balanced Arrays.

The following results are straight forward and easy to establish.

Lemma 2.1. A B-array T with $m = t = 6$ and with a given index set $\underline{\mu}'$ always exists.

Lemma 2.2. A B-array T with strength $t = 6$ and index set $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, ..., \mu_6)$ is also of strength t' where $0 < t' \le t = 6$.

Remark: Considered as an array of strength t', let its new index set $\underline{\mu}'^{*}$ be $(A_{0,t'}, A_{1,t'}, ..., A_{t',t'})$, where each $A_{j,t'}$ is a linear function of the μ_i's and is given by
$A_{j,t'} = \sum_{i=0}^{t-t'} \binom{t-t'}{i} \mu_{i+j}, \; j = 0, 1, 2, ..., t'$.

Lemma 2.3. Consider a B-array $T(m, N, s = 2, t = 6, \underline{\mu}')$, and let $x_j (0 \le j \le m)$ denote the number of columns of weight j in T. Then the following results are true:

$$\sum_{j=0}^{m} x_j = N \tag{2.1}$$

$$\sum j x_j = m_1 A_{1,1} \tag{2.2}$$

$$\sum j^2 x_j = \sum_{k=1}^{2} m_k A_{k,k} \tag{2.3}$$

$$\sum j^3 x_j = m_3 A_{3,3} + 3 m_2 A_{2,2} + m_1 A_{1,1} \tag{2.4}$$

$$\sum j^4 x_j = m_4 A_{4,4} + 6 m_3 A_{3,3} + 7 m_2 A_{2,2} + m_1 A_{1,1} \tag{2.5}$$

$$\sum j^5 x_j = m_5 A_{5,5} + 10 m_4 A_{4,4} + 25 m_3 A_{3,3} + 15 m_2 A_{2,2} + m_1 A_{1,1} \tag{2.6}$$

$$\sum j^6 x_j = m_6 A_{6,6} + 15 m_5 A_{5,5} + 65 m_4 A_{4,4} + 90 m_3 A_{3,3} + 31 m_2 A_{2,2} + m_1 A_{1,1} \tag{2.7}$$

where $m_r = m(m-1)(m-2)...(m-r+1)$, and each of $A_{k,k}$ is a linear function of the μ_i's. For a given $\underline{\mu}'$, it is not difficult to see that the R.H.S. in each of (2.1-2.7) is a polynomial function of m.
Next we state, without proofs, some results involving central moments which are extensively used in statistics to study distributions. Let $z_1, z_2, ..., z_n$ be reals such that $\sum_{i=1}^{n} z_i = 0$, and $\sum z_i^2 = n$. Set $\propto_m = \frac{1}{n}\sum z_i^m$. Now we state the desired results for later use from Chakrabarti [2] and Lakshmarramurti [5].
Results: (a) $\propto_6 \geq \propto_4^2 + \propto_3^2$. Here $\propto_4$ and $\propto_3$ refer to Kurtosis and Skewness respectively which are used by statisticians to study "peakedness" and "symmetry" in empirical and theoretical distributions.
(b) $\propto_6 \leq (n^2 - 3n + 3) + \frac{2-n}{(n-1)^2}$
Remark: It is not difficult to see that (2.1)-(2.7) give us the moments of order $k (0 \leq k \leq 6)$ of the weights j (x_j representing their frequencies) of the columns of T in terms of m and the elements of the index set $\underline{\mu}'$. We use these together with results (a) and (b) given above to obtain some new results on the existence of B-arrays T.

Theorem 2.1. Consider a B-array $T(m, n, t = 6,\ s = 2,\ \underline{\mu}')$. Then the following holds:

$$\Big[N\sum j^2 x_j - (\sum j x_j)^2\Big]\Big[N^5\sum j^6 x_j - 6N^4\sum j^5 x_j \sum j x_j +$$
$$+ 15N^3\sum j^4 x_j (\sum j x_j)^2 - 20N^2\sum j^3 x_j (\sum j x_j)^3 + 15N\sum j^2 x_j (\sum j x_j)^4 -$$
$$- 5(\sum j x_j)^6\Big] \geq \Big[N^3\sum j^4 x_j - 4N^2\sum j^3 x_j \sum j x_j + 6N\sum j^2 x_j (\sum j x_j)^2 -$$
$$- 3(\sum j x_j)^4\Big]^2 + \Big[N\sum j^2 x_j - (\sum j x_j)^2\Big]\Big[N^2\sum j^3 x_j -$$
$$- 3N\sum j^2 x_j \sum j x_j + 2(\sum j x_j)^3\Big]^2 \qquad (2.8)$$

Proof: Here $\sum_{j=0}^{m} x_j = N$, x_j being the frequency of the vectors of weight $j (0 \leq j \leq m)$ in T. Let $\overline{j} = \frac{1}{N}\sum j x_j$, and $s^2 = \frac{1}{N}\sum (j - \overline{j})^2 x_j$. It is quite clear that the quantities $(\frac{j-\overline{j}}{s})$ are such that $\sum(\frac{j-\overline{j}}{s})x_j = 0$ and $\sum(\frac{j-\overline{j}}{s})^2 x_j = N$, and thus $(\frac{j-\overline{j}}{s})$ plays the role of z_j. Also we have $\propto_m = \frac{1}{N}\sum(\frac{j-\overline{j}}{s})^m x_j$. Using result (a), we get

$$\frac{1}{N}\sum(\tfrac{j-\overline{j}}{s})^6 x_j \geq \frac{1}{N^2}\Big[\sum(\tfrac{j-\overline{j}}{s})^4 x_j\Big]^2 + \frac{1}{N^2}\Big[\sum(\tfrac{j-\overline{j}}{s})^3 x_j\Big]^2$$

Simplifying, we obtain

$$N\sum(j-\overline{j})^2x_j\sum(j-\overline{j})^6x_j \geq N\left[\sum(j-\overline{j})^4x_j\right]^2 +$$
$$+\left[\sum(j-\overline{j})^2x_j\right]\left[\sum(j-\overline{j})^3x_j\right]^2$$

Expanding $(j-\overline{j})^k$ for $k = 2, 3, 4,$ and 6 and using $\frac{1}{N}\sum jx_j$ for $\overline{j}$, we obtain the desired result after some simplification.

Theorem 2.2. For the existence of a B-array $T(m,\ N,\ s=2,\ t=6,\ \underline{\mu}')$, we must have

$$N^5\sum j^6x_j - 6N^4\sum jx_j\sum j^5x_j + 15N^3(\sum jx_j)^2\sum j^4x_j -$$
$$-20N^2(\sum jx_j)^3\sum j^3x_j + 15N(\sum jx_j)^4\sum j^2x_j - 5(\sum jx_j)^6 \leq$$
$$\leq \left[N\sum j^2x_j - (\sum jx_j)^2\right]^3\left[N^2 - 3N + 3 + \tfrac{2-N}{(N-1)^2}\right] \qquad (2.9)$$

Proof: Using result (b), and substituting $\propto_6 = \frac{1}{N}\sum(\frac{j-\overline{j}}{s})^6x_j$ leads us to (2.9) after some simplification.

Remark: For a given $\underline{\mu}'$ and $m \geq 7$, it is easy to check conditions (2.8) and (2.9). If a condition is contradicted for $m = k+1$ (say), then clearly k is an upper bound on the number of constraints for such an array. Obviously these are necessary conditions for the existence of such arrays, and the array with parameters satisfying both the conditions may or may not exist. Next we include some illustrative examples obtained by using a computer program.

Example 2.1. Consider an array T with $\underline{\mu}' = (2,3,3,3,3,3,2)$. Clearly $N = 190$. This array is quite close to an orthogonal array of strength six with index set 3. Starting with the given $\underline{\mu}'$ and $m = 7$, we tested (2.8) and obtained a contradiction for $m = 10$. For this case, L.H.S. $= 1.059097 \times 10^{20}$ and the R.H.S. $= 2.094517 \times 10^{20}$. Thus the maximum number of constraints for this array is 9.

Example 2.2. Consider B-array $T(m \geq 7,\ \underline{\mu}' = (0,0,0,0,1,1,1)$. Here $N = 22$. We checked (2.9) for this array with $m = 7$, and obtained L.H.S. $= 2.4956 \times 10^8$ and R.H.S. $= 2.4950 \times 10^8$ which, obviously, is a contradiction. Hence the maximum number of constraints for this array is 6, and clearly this array exists.

References

[1] R.C. Bose and K.A. Bush, Orthogonal arrays of strength two and three, Ann. Math. Statist. 23 (1952), 508-524.

[2] M.C. Chakrabarti, A note on Skewness and Kurtosis, Bull. Calcutta Math. Soc. 38 (1946), 133-136.

[3] D.V. Chopra, Some Inequalities on balanced arrays arising from Kurtosis and Skewness, Congressus Numerantium 109(1995), 161-166.

[4] D.V. Chopra and R. Dio's, On the number of constraints for balnaced arrays of strength four, J. Comb., Inf. and Syst. Sci. 18 (1993), 203-208.

[5] M. Lakshmanamurti, On the upper bound of $\sum_{c=1}^{n} x_i^m$ subject to the conditions $\sum_{i=1}^{n} x_i = 0$ and $\sum_{i=1}^{n} x_i^2 = n$, Math. Student 18(1950), 111-116.

[6] J.Q. Longyear, Arrays of strength s on two symbols, J. Statist. Plann. Inf. 10(1984), 227-239.

[7] D.S. Mitrinović, Analytic Inequalities, Springer-Verlag, New York, 1970.

[8] J.A. Rafter and E. Seiden, Contributions to the theory and construction of balanced arrays, Ann. Statist. 2(1974), 1256-1273.

[9] C.R. Rao, Some combinatorial problems of arrays and applications to design of experiments, "A Survey of Combinatorical Theory," edited by J.N. Srivastava, et. al., North-Holland Publishing Co. (1973), 349-359.

[10] G.M. Saha, R. Mukerjee and S. Kageyama, Bounds on the number of constraints for balanced arrays of strength t, J. Statist. Plam. Inf. 18(1988), 255-265.

[11] E. Seiden and R. Zemach, On orthogonal arrays, Ann. Math. Statit. 37(1966), 1355-1370.

[12] W.D. Wallis, "Combinatorial Designs," Marcel-Dekker Inc., New York, 1988.

[13] J.E. Wilkins, A note on Skewness and Kurtosis, Ann. Math. Statist. 15(1994), 333-335.

[14] S. Yamamoto, M. Kuwada and F. Yuan, On the maximum number of constraints for s-symbol balanced arrays of strength t, Commun. Statist. Theory Meth. 14(1985), 2447-2456.