A Generalized Coloring of Graphs

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ABSTRACT. Let $\chi^*(G)$ denote the minimum number of colors required in a coloring c of the vertices of G where for adjacent vertices u, v we have $c(N_G[u]) \neq c(N_G[v])$ when $N_G[u] \neq N_G[v]$ and $c(u) \neq c(v)$ when $N_G[u] = N_G[v]$. We show that the problem of deciding whether $\chi^*(G) \leq n$, where $n \geq 3$, is NP-complete for arbitrary graphs. We find $\chi^*(G)$ for several classes of graphs including bipartite graphs, complete multipartite graphs, as well as, cycles and their complements. A sharp lower bound is given for $\chi^*(G)$ in terms of $\chi(G)$ and an upper bound is given for $\chi^*(G)$ in terms of $\Delta(G)$. For regular graphs with girth at least four we give substantially better upper bounds for $\chi^*(G)$ using random colorings of the vertices.

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1 Introduction

Addressing scheme used for electronic mail is hierarchical, and is represented by a tree. An individual, who belongs to more than one organization or group, may have more than one address called aliases. This can be modeled by an acyclic directed graph. On the other hand, two different individuals belonging to the same organization and with the same initial and last name may have the same address. In such a case, address resolution may be based on the set of aliases associated with each individual provided that these sets are distinct. In the context of generalized graph coloring, each vertex is to be distinguished from its neighbors based on the name or color assigned to it. Therefore, we require that if two adjacent vertices have the same closed neighborhood, then their colors be distinct; otherwise, it is enough to require that the set of colors assigned to the closed neighborhood of each vertex be distinct. Our discussion motivates the following definition.

For a graph G, a (not necessarily proper) coloring c of the vertices of G is a *good coloring* of G if and only if for all edges uv of G, $c(u) \neq c(v)$ when $N_G[u] = N_G[v]$ and $c(N_G[u]) \neq c(N_G[v])$ when $N_G[u] \neq N_G[v]$. A good coloring of G using k colors will be referred to as a *good k-coloring* of G. If we assign each vertex of G a different color we have a good coloring of G. Let $\chi^*(G)$ denote the minimum number of colors required in a good coloring of G. Hence, $\chi^*(G) \leq n$ when G has order n. It is readily seen that $\chi^*(G) \geq 3$ for any connected graph G with order at least 3.

A simple graph G has vertex set $V(G)$ and edge set $E(G)$. The order of G is $|V(G)|$ and the size of G is $|E(G)|$. For a vertex v in G, the open neighborhood $N_G(v)$ of v in G is the set of all vertices in G adjacent to v and the closed neighborhood $N_G[v] = N_G(v) \cup \{v\}$. The degree $d_G(v)$ of a vertex v in G is $|N_G(v)|$ and $\Delta(G)$ is the maximum degree of a vertex in G. For vertices u, v in G, the distance $d_G(u, v)$ between u, v in G is the length of a shortest (u, v)-path in G. We denote a cycle (path) of order n by $C_n(P_n)$ and the complement of a graph G by $\overline{G}$. The girth of G is the length of a shortest cycle in G. All other notation and terminology generally follows Bondy and Murty [1]. Unless otherwise noted, all logarithms are natural.

2 Complexity

We show that the problem of deciding whether $\chi^*(G) \leq n$, where $n \geq 3$, is NP-complete for arbitrary graphs.

Graph n-Colorability
INSTANCE: Graph $G = (V, E)$.
QUESTION: Is G n-colorable, i.e., does there exist a function c: $V \to \{1, \dots, n\}$ such that $c(u) \neq c(v)$ whenever $\{u, v\} \in E$?

It is known that graph n-colorability, for $n \geq 3$, is NP-Complete. (see [4].)

Graph Generalized n-Colorability
INSTANCE: Graph $G = (V, E)$.
QUESTION: Does G have a good n-coloring?

Theorem 1. *Graph generalized n-colorability for any $n \geq 3$ is NP Complete.*

Proof: Clearly, graph generalized n-colorability is in NP. Consider the reduction from the graph n-colorability problem. Let $G = (V, E)$ be a given graph. Construct graph $G' = (V', E')$ as follows. Replace each edge $uv \in E$ by $P_4 = u, u', v', v$ where u', v' are new vertices, and replace each vertex $u \in V$ by $C_4 = u, u_1, u_2, u_3, u$, where u_1, u_2, u_3 are new vertices.

Let c_1 be a proper n-coloring of G. We show that c_1 induces a good n-coloring c_2 of G'. For each $\{u, v\} \in E$, if $c_1(u) = i$ and $c_1(v) = j$ then $c_2(u) = i$, $c_2(v) = j$, and $c_2(u') = c_2(v') = k$ where k is any color other than i, j. For each $u \in V$ if $c_1(u) = i$ then $c_2(u) = i$, $c_2(u_1) = j$, $c_2(u_2) = i$, and $c_2(u_3) = k$ where j, k are distinct colors different than i. It is easily verified that c_2 is a good n-coloring of G'.

Conversely, let c_2 be a good n-coloring of G'. Consider edge $uv \in E$ and let $P_4 = u, u', v', v$ be the corresponding path in G'. If $c_2(u) = c_2(v)$ then $c_2(N_{G'}[u']) = c_2(N_{G'}[v'])$, hence, $c_2(u) \neq c_2(v)$. Therefore, by setting $c_1(u) = c_2(u)$ for each $u \in V$, we obtain a proper n-coloring of G. □

As our construction preserves planarity, graph generalized n-colorability, for any $n \geq 3$, is NP-complete for planar graphs.

3 Exact Values

In this section we find $\chi^*(G)$ for several classes of graphs.

Theorem 2. *For a connected bipartite graph G with order at least 3,*

$$\chi^*(G) = 3.$$

Proof: Let v be a vertex of G whose eccentricity is $r = \text{radius}(G) \geq 2$ ($r = 1$ being trivial). Let $N^k = N^k(v) = \{w \in V(G) : d_G(v, w) = k\}$, $O^k = O^k(v) = \{w \in N^k(v) : N_G(w) \subseteq N^{k-1}\}$ and $P^k = P^k(v) = N^k(v) - O^k(v)$ for $0 \leq k \leq r$. Each set N^k is independent since G is bipartite. Now color N^{4k} color 1; N^{4k+2} color 3; O^{4k+1} color 1; P^{4k+1} color 2; O^{4k+3} color 3, and P^{4k+3} color 2. It is easily seen that this is a good 3-coloring of G so that $\chi^*(G) = 3$. □

Theorem 3. *We have*

$$\chi^*(C_n) = \begin{cases} 4, & n = 5 \text{ or } 7 \\ 3, & n \text{ is odd and } n \geq 9. \end{cases}$$

Proof: It is easy to verify the result for $n = 5$ or 7. So we assume $n \geq 9$. It is sufficient to construct a good coloring c of C_n using three colors. Let $C_n = v_0, v_1, \ldots, v_{n-1}, v_0$. We define the good 3-coloring c of C_n as follows.

$$c(v_i) = \begin{cases} 1, & i \text{ is even} \\ 2, & i \equiv 1 \pmod 4 \\ 3, & i \equiv 3 \pmod 4 \end{cases}$$

with the exceptions that $c(v_2) = 3$, and $c(v_{n-3}) = 2$ when $n \equiv 1 (\mod 4)$; $c(v_0) = 2$, and $c(v_1) = c(v_2) = c(v_{n-3}) = 3$ when $n \equiv 3(\mod 4)$. □

Theorem 4. *If G is a complete t-partite graph with exactly m vertices of degree $|V(G)| - 1$, then*

$$\chi^*(G) = \begin{cases} 2t - 1, & m = 0 \\ 2t - m, & m > 0. \end{cases}$$

Proof: Let $P_1, P_2, \ldots, P_t$ be the t-partitions of $V(G)$ such that $|P_i| = 1$ if and only if $1 \leq i \leq m$. Let c be a good coloring of G using $\chi^*(G)$ colors. Define $C_i = c(P_i)$ for $1 \leq i \leq m$, and $C_i = c(P_i) - c(V(G) - P_i)$ for $m < i \leq t$. Then we can easily check the following three observations.

(i) $C_i \cap C_j = \emptyset$ whenever $i \neq j$.

(ii) If $m < i \neq j \leq t$, then either $|C_i| > 1$ or $|C_j| > 1$.

(iii) If $|C_i| \leq 1$ for some $i > m$, then $m = 0$.

We now consider two cases.

Case 1. $m = 0$.

In this case we may assume $|C_i| > 1$ for all $i > 1$ by (ii) and (iii). Note that $C_1 \cap C_j = \emptyset$ for all $j > 1$. Then we have $\chi^*(G) \geq |C_1| + \sum_{i=2}^{t} |C_i| \geq 1 + 2(t-1) = 2t - 1$.

Case 2. $m > 0$.

In this case we have $|C_i| > 1$ for all $i > m$. Then $\chi^*(G) \geq \sum_{i=1}^{t} |C_i| \geq m + 2(t-m) = 2t - m$.

This proves the lower bound.

Now let c be a good coloring of G such that (i) $C_i \cap C_j = \emptyset$ whenever $1 \le i \ne j \le t$; and (ii) $|C_i| = 2$ whenever $i > m$ with the exception that $|C_1| = 1$ if $m = 0$. Then clearly c is a good coloring of G. □

Lemma 5. *Let V be a finite set, and $S = \{B_1, B_2, \ldots, B_b\}$ be a collection of distinct subsets of V such that*

(i) $1 \le |B_i| \le 2$ *for each* i;

(ii) $1 \le r_v \le 2$ *for each* $v \in V$, *where* r_v *denotes the number of subsets in* S *containing* v.

Then $|V| \ge (2b + a_1)/3$, *where* a_1 *denotes the number of* v*'s so that* $r_v = 1$.

Proof: For $i = 1, 2$, let a_i denote the number of v's in V so that $r_v = i$ and b_i denote the number of subsets of size i in S. By counting the number of ordered pairs (v, B_i), where $v \in B_i \subseteq V$, in two ways we have

$$2a_2 + a_1 = 2b_2 + b_1. \tag{1}$$

Now let τ be the number of ordered pairs (v, B_i) such that $v \in B_i$, $|B_i| = 2$, and $r_v = 2$. Then we have $a_2 \le \tau \le 2b_2$ or $b_2 \ge \frac{1}{2}a_2$. This, together with (1), implies that $2a_2 + a_1 \ge b + a_2/2$, or

$$3|V| = 3(a_1 + a_2) \ge 2b + a_1.$$

Then $|V| \ge (2b + a_1)/3$. □

Theorem 6. *We have*

$$\chi^*(\overline{C}_n) = \begin{cases} \lceil (2n+2)/3 \rceil, & n \equiv 0, 5 \pmod 6, n \ne 12 \\ \lceil (2n-1)/3 \rceil, & n \not\equiv 0, 5 \pmod 6 \\ 8, & n = 12. \end{cases}$$

Proof: We first prove the lower bound. Assume $V(\overline{C}_n) = \{v_1, v_2, \ldots, v_n\}$ such that $v_i v_j \in E(\overline{C}_n)$ if and only if i and j are not consecutive, i.e., $j \not\equiv i + 1 \pmod n$. Let c be a good coloring of $\overline{C}_n$ with the color set $X = \{x_1, x_2, \ldots, x_t\}$, where $t = \chi^*(\overline{C}_n)$. Define $B_i = X - c(N_{\overline{C}_n}(v_i))$ for $1 \le i \le n$. Clearly $B_i = \emptyset$ for at most two i's, in which case they are consecutive. Hence, we assume $B_i \ne \emptyset$ if and only if $1 \le i \le b$, where $n-2 \le b \le n$, and let $S = \{B_1, B_2, \ldots, B_b\}$. It can be seen that all the sets in S are distinct. Now let r_x denote the number of B_i in S containing x for every $x \in X$. Thus $r_x \le 2$ for all $x \in X$. We then define $A_i = \{x_j : r_{x_j} = i\}$ and $a_i = |A_i|$ for $0 \le i \le 2$. Therefore, $\chi^*(\overline{C}_n) = t = a_0 + a_1 + a_2$.

Case 1. $a_0 > 0$.

Define $S_1 = \{B_1, B_3, \ldots, B_\alpha\}$, $S_2 = \{B_2, B_4, \ldots, B_\beta\}$, $X_1 = B_1 \cup B_3 \cup \cdots \cup B_\alpha$, $X_2 = B_2 \cup B_4 \cup \cdots \cup B_\beta$, where $\alpha = 2\lceil n/2 \rceil - 3$ and $\beta = 2\lfloor n/2 \rceil - 2$. Then we have $X_1 \cap X_2 = \emptyset$, and we can apply Lemma 5 to both (X_1, S_1) and (X_2, S_2), so that

$$|X_1| \geq \frac{2}{3}\left(\frac{\alpha+1}{2}\right) \text{ and } |X_2| \geq \frac{2}{3}\left(\frac{\beta}{2}\right).$$

Thus

$$\begin{aligned}\chi^*(\overline{C}_n) = t = |X_1| + |X_2| + a_0 &\geq \left\lceil \frac{\alpha+1}{3} \right\rceil + \left\lceil \frac{\beta}{3} \right\rceil + 1 \\ &= \left\lceil \frac{2}{3}(\lceil n/2 \rceil - 1) \right\rceil + \left\lceil \frac{2}{3}(\lfloor n/2 \rfloor - 1) \right\rceil + 1,\end{aligned}$$

which is at least as large as the lower bound.

Case 2. $a_0 = 0$.

In this case we can apply Lemma 5 to (X, S) so that

$$\chi^*(\overline{C}_n) = t \geq \frac{2b + a_1}{3} = \frac{2n + a_1 - 2(n-b)}{3}. \tag{2}$$

If $B_n = \emptyset$, then we have $c(v_{n-1}) \neq c(v_1)$, for otherwise $c(v_1)$ would be a color in A_0. Furthermore, we have $c(v_{n-1})$ and $c(v_1)$ are both contained in A_1. Similarly, if $B_n = B_{n-1} = \emptyset$, we can check that $c(v_{n-2})$, $c(v_{n-1})$, $c(v_n)$, and $c(v_1)$ are distinct and contained in A_1. Thus we have shown that $a_1 \geq 2(n-b)$. Therefore, by (2), we may assume

$$a_1 = 2(n-b) \leq 4. \tag{3}$$

Hence, $\chi^*(\overline{C}_n) \geq \frac{2n}{3}$ and we may assume $n \equiv 0 \pmod 6$ with $n \neq 6$.

Notice that each color in A_2 (A_1) is used exactly once (twice) in $V(\overline{C}_n)$. Then we have $2a_1 + a_n = n$, and (3) implies that

$$\chi^*(\overline{C}_n) = a_1 + a_2 = n - a_1 \geq n - 4,$$

which is at least as large as the lower bound unless $n \leq 11$.

This completes the proof of the lower bound.

We now prove the upper bound. The graph in Figure 1 covers the case $n = 12$, where the number next to each vertex represents the color of that vertex.

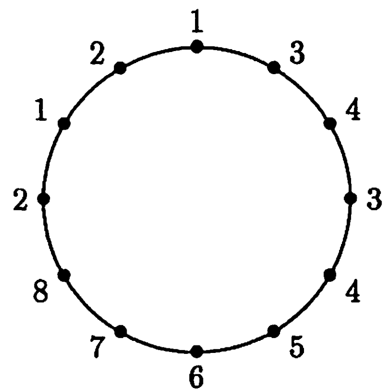

Figure 1

Write $n = 6m + r$, where $1 \le r \le 6$. Define $R = \emptyset$ if $r \le 2$; $R = \{x_i : 1 \le i \le r-2\}$ if $r \ge 3$; and $S = \{0\} \cup \{a_i, b_i, c_i, d_i : 0 \le i \le m-1\}$. We now define a good coloring c of G using colors in $S \cup R$ as follows.

(i) For $k = 6j + i$, where $0 \le j \le m-1$, and $1 \le i \le 6$,

$$c(v_k) = \begin{cases} 0, & 1 \le i \le 2 \\ a_j, & i = 3 \\ b_j, & i = 4 \\ c_j, & i = 5 \\ d_j, & i = 6. \end{cases}$$

(ii) $c(v_n) = 0$ if $r = 1$.

(iii) For $r \ge 2$, $k = 6m + i$, where $1 \le i \le r$,

$$c(v_k) = \begin{cases} x_i, & 1 \le i \le r-2 \\ 0, & i = r-1 \text{ or } r. \end{cases}$$

It is easily seen that c is a good coloring of $\overline{C}_n$. □

Definition: We say a vertex v *distinguishes* an edge $e = xy$ in G if and only if $(N_G[x] - N_G[y]) \cup (N_G[y] - N_G[x]) = \{v\}$.

Lemma 7. *For any graph G, there exists a vertex v in G that does not distinguish any edge of G.*

Proof: Suppose not. Let $V(G) = \{v_1, \ldots, v_n\}$ where v_i distinguishes the edge e_i for $1 \le i \le n$. Let H denote the subgraph of G induced by the

n edges $\{e_i\colon 1 \le i \le n\}$. Then H contains exactly n edges and at most n vertices. Then H contains a cycle $C = w_0, w_1, \ldots, w_{r-1}, w_0$. Without loss of generality, we may assume $e_i = w_{i-1}w_i$ for $1 \le i \le r$. Now for any $1 \le i \ne j \le r$, v_i is adjacent to exactly one of w_{i-1} and w_i, and v_i is adjacent to neither or both of w_{j-1} and w_j. Hence, if $v_iw_i \in E(G)$, then v_i would be adjacent to all the other vertices in C, which is impossible. □

Given any graph G, we use $s(G)$ to denote the maximum number of vertices in G all having the same closed neighborhood. Then we have $s(G) \le |V(G)|$, with equality if and only if G is a complete graph.

Theorem 8. *Let H be a graph of order n and K_n be vertex-disjoint from H, where $n \ge 2$. Define G to be the graph obtained from H and K_n by adjoining a 1-factor between H and K_n. Then we have $\chi^*(G) = n-1+s(H)$, unless each component of H is a P_3 or H contains an isolated vertex and $s(H) = 1$, in which case $\chi^*(G) = n+1$.*

Proof: We first partition $V(H)$ into m subsets $V_1, \ldots, V_m$ so that two vertices in H have the same closed neighborhood in H if and only if they are in the same subset. For any vertex x in H, we use $f(x)$ to denote the vertex in K_n adjacent to x. For $1 \le i \le m$, let $U_i = f(V_i)$ denote the set of vertices in K_n, adjacent to vertices in V_i.

We first prove the lower bound. Let c denote a good coloring of G using $\chi^*(G)$ colors. Then we have the following observations.

(i) No two vertices in H can receive the same color.

(ii) No two vertices in U_i can receive the same color for any fixed i, $1 \le i \le m$.

(iii) $|c(V(K_n)) \cap c(V(H))| \le 1$.

Therefore we have $\chi^*(G) \ge |c(V(H))| + |c(V_1)| - 1 = n + s(H) - 1$, where $|V_1| = \max\{|V_i|\colon 1 \le i \le m\} = s(H)$.

If each component of H is a P_3, then we can easily check that $\chi^*(G) \ge n+1$. If $s(H) = 1$ and H contains an isolated vertex u, then clearly $|c(V(K_n))| \ge 2$. Hence, $\chi^*(G) \ge n+2-1 = n+1$.

We now prove the upper bound. Let v be a vertex in H that does not distinguish any edge of H. Without loss of generality, we may assume u_1 is the vertex in $U_1 = f(V_1)$ adjacent to v. We also choose an arbitrary vertex $u_i \in U_i$ for $2 \le i \le m$. Let $S_1 = \{1, 2, \ldots, n\}$ and S_2 be a set of $s(H)-1$ colors other than those in S_1. We now define a good coloring of G as follows.

(i) Color vertices in H with those colors in S_1, so that $c(v) = 1$ and no two vertices in H receive the same color;

(ii) $c(u_i) = 1$ for $1 \le i \le m$;

(iii) $c(U_i - \{u_i\}) \subseteq S_2$ and no two vertices in $U_i - \{u_i\}$ receive the same color for $1 \le i \le m$;

with the exception that $c(u) = n+1$ for all $u \in V(K_n) - \{v_1\}$ if each component of H is a P_3 or H contains an isolated vertex and $s(H) = 1$. Then we can easily verify that c is a good coloring of G. □

4 Bounds

We first give a sharp lower bound for $\chi^*(G)$ in terms of $\chi(G)$. This is a special case of a more general result whose proof can be found in Zhang [5].

Theorem 9. *$1 + \lceil \log_2 \chi(G) \rceil \le \chi^*(G)$. Furthermore, for all $n \ge 1$ there exists a graph G such that $\chi(G) = n$ and $\chi^*(G) = 1 + \lceil \log_2 n \rceil$.*

For a positive integer k and a graph G, a coloring c of the vertices of G is a *good distance-k-coloring* of G if and only if $c(u) \ne c(v)$ for distinct vertices u, v with $d_G(u,v) \le k$. If we assign each vertex G a different color we have a good distance-k-coloring of G for all positive integers k. Let $\chi_k(G)$ denote the minimum number of colors required in a good distance-k-coloring of G. Hence, $\chi_k(G) \le n$ when G has order n. We require the following result.

Lemma 10. *For a graph G with $\Delta = \Delta(G) \ge 3$,*

$$\chi_k(G) \le \frac{\Delta(\Delta-1)^k - 2}{\Delta - 2}.$$

Proof: Let ℓ denote the integer $\frac{\Delta(\Delta-1)^k-2}{\Delta-2}$. We apply the Greedy Algorithm to any ordering $v_1, \dots, v_n$ of the vertices of G. First color v_1 with color 1. Assume we have a good distance-k-coloring of $v_1, \dots, v_j$ using the colors $1, \dots, \ell$. Now

$$\begin{aligned}\#\{v_i \colon d_G(v_i, v_{j+1}) \le k, 1 \le i \le j\} &\le \Delta + \Delta(\Delta-1) + \cdots + \Delta(\Delta-1)^{k-1} \\ &= \frac{\Delta(\Delta-1)^k - \Delta}{\Delta - 2} = \ell - 1.\end{aligned}$$

Hence, at most $\ell - 1$ colors are present among these vertices and we color v_{j+1} with the smallest color which has not been used. □

Remark: Our proof also shows that $\chi_k(G) \le 2k+1$ when $\Delta(G) = 2$.

Theorem 11. *For a graph G,*

$$\chi^*(G) \le \chi_3(G)$$

hence, for $\Delta = \Delta(G) \geq 3$,

$$\chi^*(G) \leq \frac{\Delta(\Delta-1)^3-2}{\Delta-2}.$$

Proof: Let c be a good distance-3-coloring of G using $\chi_3(G)$ colors. Hence, c is a proper coloring of G. For edge uv of G with $w \in N_G[u]-N_G[v]$, $c(w) \notin c(N_G[v])$ since $d_G(w,x) \leq 3$ for all $x \in N_G[u]$ and, hence, $c(N_G[u]) \neq c(N_G[v])$ ($w \in N_G[v]-N_G[u]$ is similar). Consequently, c is a good coloring of G. □

Remark: The bound given in Theorem 11 may be quite weak; we know of no graphs where equality is attained.

5 Bounds for Regular Graphs with Girth at Least Four

We us the following inequality,

$$1-x \leq e^{-x} \text{ for } x \in \mathbb{R}. \tag{4}$$

For integers k, n with $1 \leq k \leq n$, let $(n)_k = n(n-1)\ldots(n-k+1)$. Then (4) implies

$$(n)_k \leq n^k e^{-\binom{k}{2}/n} \leq e^{1/2} n^k e^{-k^2/2n}, n^k. \tag{5}$$

For nonnegative integers k, ℓ, n with $k \leq k+\ell \leq n-1$, we use

$$\binom{n-\ell}{k}\binom{n}{k}^{-1} \leq e^{-k\ell/n}. \tag{6}$$

We also use the following results for the Stirling number $S(n,k)$ of the second kind (see [2; p. 204–208]),

$$S(n,2) = 2^{n-1}-1 \text{ for } n \geq 1. \tag{7}$$

and

$$S(n,k) \leq \binom{n-1}{k-1} k^{n-k} \text{ for } 1 \leq k \leq n. \tag{8}$$

We make use of the Lovász Local Lemma (see [3]).

Lemma. (Erdös and Lovász) *Let* $A_1, A_2, \ldots, A_n$ *be events in an arbitrary probability space. Suppose that each event* A_i *is mutually independent of all, but at most* d, *of the other events* A_j *and that* $P(A_i) \leq p$ *for all* $1 \leq i \leq n$. *If*

$$ep(d+1) \leq 1$$

then

$$P(\bigwedge_{i=1}^{n} \overline{A}_i) > 0.$$

(As usual $\overline{A}$ denotes the complement of the event A.)

We are able to substantially improve the bound in Theorem 11 for regular graphs with girth at least four using random colorings of the vertices. Observe that if a graph G has girth$(G) \geq 4$ and $uv \in E(G)$, then $N_G[u] \neq N_G[v]$ when $d_G(u) + d_G(v) \geq 3$.

Theorem 12. *For an r-regular graph G of order n with $r \geq 61$ and girth $(G) \geq 4$,*

$$\chi^*(G) \leq \lceil er \rceil .$$

Proof: We initially assume only that $r \geq 3$. Independently, color the vertices of G randomly from $[k] = \{1, \ldots, k\}$ according to a uniform distribution. Hence, each coloring of G has probability k^{-n}. For $uv \in E(G)$, let A_{uv} denote the event "$c(N_G[u]) = c(N_G[v])$". Hence,

$$P(A_{uv}) = k^{-2r+1} + \binom{k}{2} S(r+1,2)2![2S(r-1,2)2!$$

$$+ 3S(r-1,1)1!]k^{-2r} + \sum_{j=3}^{r+1} \binom{k}{j} S(r+1,j)j![2S(r-1,j)j!$$

$$+ 3S(r-1,j-1)(j-1)! + S(r-1,j-2)(j-2)!]k^{-2r}.$$

Here $\binom{k}{j}$ is the number of ways to select j colors; $S(r+1,j)$ is the number of ways to partition $N_G[u]$ into j nonempty sets; $j!$ is the number of ways to assign the j colors to these sets; $2S(r-1,j)j!$ arises when the color(s) at u, v appear again in $N_G(v) - \{u\}$; $3S(r-1,j-1)(j-1)!$ arises when u, v have the same color which does not appear again in $N_G(v) - \{u\}$ or when u, v have different colors exactly one of which appears again in $N_G(v) - \{u\}$; and $S(r-1,j-2)(j-2)!$ arises when u, v have different colors neither of which appears in $N_G(v) - \{u\}$. (The cases $j = 1, 2$ are indicated separately.)

For $3 \leq j \leq r-1$, $0 \leq i \leq 2$, (8), as well as, analysis of the cases $i = 0, j = r, r+1$; $i = 1, j = r+1$ implies

$$S(r-1, j-i)(j-i)! \leq \binom{r-2}{j-i-1}(j-i)! j^{r+1-j}$$

while (5), as well as, analysis of the above cases implies

$$\binom{r-2}{j-i-1}(j-i)! \leq j r^{j-1}$$

so that

$$S(r-1,j-i)(j-i)! \leq r^{j-1}j^{r+2-j}.$$

Similarly, for $k \geq 1$, (5) and (8) imply

$$\binom{k}{j} S(r+1,j)j! \leq e^2(ek)^j r^{j-1} j^{r+2-2j} e^{-j^2/2r}.$$

Hence,

$$\begin{aligned} P_1 &:= \sum_{j=3}^{r+1} \binom{k}{j} S(r+1,j)j![2S(r-1,j)j! \\ &\quad + 3S(r-1,j-1)(j-1)! + S(r-1,j-2)(j-2)!]k^{-2r} \\ &\leq \frac{6e^2}{r^2} \sum_{j=3}^{r} \left(\frac{ekr^2}{j^3}\right)^j j^{2r+4} e^{-j^2/2r} + 8kr\left(\frac{r}{k}\right)^r. \end{aligned}$$

For $f(j) = \left(\frac{ekr^2}{j^3}\right)^j j^{2r+4} e^{-j^2/2r}$ we have
$f'(j) = f(j)\left[\log\left(\frac{kr^2}{e^2 j^3}\right) + \frac{2r+4}{j} - \frac{j}{r}\right] > f(j)\log\left(\frac{kr^2}{ej^3}\right) > 0$ on $[3,r]$ provided $k \geq er$. Then

$$P_1 \leq 6e^2r^3\left(\frac{e^{1/2}r}{k}\right)^r + 8kr\left(\frac{r}{k}\right)^r \leq 7e^2r^3\left(\frac{e^{1/2}r}{k}\right)^r \tag{9}$$

provided $er \leq k \leq e^{7/2}r^2/8$. Also (7) and (8) imply,

$$\binom{k}{2} S(r+1,2)2![2S(r-1,2)2! + 3S(r-1,1)1!]k^{-2r} \leq k^2\left(\frac{4}{k^2}\right)^r.$$

Hence,

$$P(A_{uv}) \leq k^{-2r+1} + k^2\left(\frac{4}{k^2}\right)^r + 7e^2r^3\left(\frac{e^{1/2}r}{k}\right)^r \leq 8e^2r^3\left(\frac{e^{1/2}r}{k}\right)^r$$

when $er \leq k \leq e^{7/2}r^2/8$. The event A_{uv} depends only on the events A_{wx} with $d_G(uv,wx) \leq 2$; hence, at most $2r^3 - 1$ such events A_{wx}. For $k = \lceil er \rceil$ with $r \geq 61$

$$16e^3r^6\left(\frac{e^{1/2}r}{k}\right)^r \leq 16r^6e^{-(r-6)/2} < 1$$

and the Lovász Local Lemma implies

$$P\left(\bigwedge_{uv\in E} \overline{A}_{uv}\right) > 0.$$

Consequently, there exists a good k-coloring of G and $\chi^*(G) \le \lceil er \rceil$. □

The proof of Theorem 12, using (9) with $er \le k \le e^{7/2}r^2/8$, immediately gives the following result.

Corollary 13. *For an r-regular graph G with $7 \le r \le 60$ and girth $(G) \ge 4$,*

$$\chi^*(G) \le \lceil c_r r \rceil$$

where

$$c_r = e^{1/2}(16e^3 r^6)^{1/r}.$$

Remark: Observe that c_r is a decreasing function of r on $[2, \infty)$ with $c_7 \doteq 19.937$, $c_{60} \doteq 2.7337 > e > 2.6496 \doteq c_{61}$.

We can extend Corollary 13 to $3 \le r \le 60$ by analyzing the proof of Theorem 12. However, for small values of r, the required probabilities can be found exactly rather than estimated. We give only the result for cubic graphs.

Theorem 14. *For a cubic graph G with girth$(G) \ge 4$,*

$$\chi^*(G) \le 15.$$

Proof: We use the probability space and notation of Theorem 12 and give only a brief outline of the proof. For $uv \in E(G)$, let the neighbors of u be u_1, u_2, v and the neighbors of v be u, v_1, v_2. Necessarily, u, v, u_1, u_2, v_1, v_2 are distinct. Let p denote the number of colors present among u, v, u_1, u_2, v_1, v_2 and $(c_1, \ldots, c_p)$ denote the event "among the colors of u, v, u_1, u_2, v_1, v_2, some color occurs c_1 times, another color occurs c_2 times, etc". Finally, observe that the event "the colors of u_1, u_2, v_1, v_2 occur exactly once among the colors of u, v, u_1, u_2, v_1, v_2" is a subevent of $\overline{A}_{uv}$ so that $p \ge 5$ is a subevent of $\overline{A}_{uv}$. By direct enumeration and independence,

$P(A_{uv}$ and $(6)) = \frac{k}{k^6}$, $P(A_{uv}$ and $(1,5)) = \frac{2k(k-1)}{k^6}$, $P(A_{uv}$ and $(2,4)) = \frac{13k(k-1)}{k^6}$, $P(A_{uv}$ and $(3,3)) = \frac{10k(k-1)}{k^6}$, $P(A_{uv}$ and $(1,1,4)) = \frac{k(k-1)(k-2)}{k^6}$, $P(A_{uv}$ and $(1,2,3)) = \frac{16k(k-1)(k-2)}{k^6}$, $P(A_{uv}$ and $(2,2,2)) = \frac{10k(k-1)(k-2)}{k^6}$, $P(A_{uv}$ and $(1,1,1,3)) = 0$, and $P(A_{uv}$ and $(1,1,2,2)) = \frac{2k(k-1)(k-2)(k-3)}{k^6}$.

Hence,

$$P(A_{uv}) = \frac{2k^3 + 15k^2 - 34k + 18}{k^5}.$$

Now A_{uv} depends on at most 28 other events and for $k \ge 15$,

$$29e\frac{2k^3 + 15k^2 - 34k + 18}{k^5} < 1$$

so that the Lovász Local Lemma implies

$$P(\bigwedge_{uv\in E} \overline{A}_{uv}) \geq 0.$$

Consequently, there exists a good 15-coloring of G and $\chi^*(G) \leq 15$. □

Suppose G has no isolated edges and girth$(G) \geq 4$. We say $uv \in E(G)$ is attached to $w \in V(G) - \{u, v\}$ provided uw or $vw \in E(G)$. Since girth$(G) \geq 4$, precisely one of uw or vw is in $E(G)$. A set $W \subseteq V(G)$ is an *attachment set* of G provided each edge in G is attached to some vertex in W. Clearly, $V(G)$ is an attachment set of G. Let $\tau(G)$ denote the smallest cardinality of an attachment set of G. Hence, $\tau(G) \leq n$ when G has order n. In fact, $\tau(G) \leq n-1$ according to Lemma 7. We require the following result.

Lemma 15. *If G has no isolated edges and girth$(G) \geq 4$, then*

$$\chi^*(G) \leq 1 + \tau(G).$$

Proof: Let W be an attachment set of G with $|W| = \tau(G) =: \tau$. Color each vertex of W with a distinct color from $\{1, \ldots, \tau\}$ and color each vertex of $\overline{W}$ with color $\tau + 1$. Clearly, this coloring is a good coloring of G. □

Our final result improves the bound in Theorem 12 for dense regular graphs with girth at least 4.

Theorem 16. *For an r-regular graph G of order $n \geq 48$ with $r \geq 3$ and girth$(G) \geq 4$,*

$$\chi^*(G) \leq 1 + \left\lceil \frac{n}{2r} \log 2er^3 \right\rceil.$$

Proof: Randomly choose $W \subseteq V(G)$ with $|W| = k$ according to a uniform distribution. Hence, $P(W) = \binom{n}{k}^{-1}$. For $uv \in E(G)$, let A_{uv} denote the event "uv is not attached to any vertex in W". Hence,

$$P(A_{uv}) = P(W \cap (N_G(u) \cup N_G(v)) = \emptyset) = \binom{n-2r}{k}\binom{n}{k}^{-1}.$$

For $n \geq 48$, $r \geq 3$ and $k = \lceil \frac{n}{2r} \log 2er^3 \rceil$, (6) implies

$$P(A_{uv}) \leq e^{-2rk/n} \leq (2er^3)^{-1}.$$

As in Theorem 12, A_{uv} depends on at most $2r^3 - 1$ other events A_{wx}, and the Lovász Local Lemma implies

$$P(\bigwedge_{uv\in E(G)} \overline{A}_{uv}) > 0.$$

Consequently, there exists an attachment set of G having cardinality k and our result now follows from Lemma 15. □

6 Open Problems

In this section we give three open problems.

(1) Find a good upper bound for $\chi^*(G)$ in terms of $\chi(G)$. As a consequence of Theorem 8, $\chi^*(2K_n + 1 -$ factor$) = 2\chi(G) - 1$, which is the largest such value we have been able to find.

(2) Find a good upper bound for $\chi^*(G)$ in terms of $\Delta(G)$. In Theorem 11, a cubic upper bound was given for $\chi^*(G)$ in terms of $\Delta(G)$. Perhaps, there is a linear upper bound for $\chi^*(G)$ in terms of $\Delta(G)$.

(3) Determine the quality of the upper bound for $\chi^*(G)$ given in Theorems 12 and 16.

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