# General Probabilistic Bounds for Dual Bin Packing Heuristics*

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ABSTRACT. Given $m$ unit-capacity bins and a collection $x(n)$ of $n$ pieces, each with a positive size at most one, the dual bin packing problem asks for packing a maximum number of pieces into the $m$ bins so that no bin capacity is exceeded. Motivated by the NP-hardness of the problem, Coffman et al. proposed a class of heuristics, the *prefix* algorithms, and analyzed its worst-case performance bound. Bruno and Downey gave a probabilistic bound for the FFI algorithm (which is a prefix algorithm proposed by Coffman et al.), under the assumption that piece sizes are drawn from the uniform distribution over [0,1]. In this article we generalize their result: Let $F$ be an *arbitrary* distribution over [0,1], and let $x(n)$ denote a random sample of a random variable $X$ distributed according to $F$. Then, for any $\epsilon > 0$, there are $\lambda > 0$ and $N > 0$, dependent only on $m$, $\epsilon$ and $F$, such that for all $n \geq N$,

$$Pr(\frac{OPT(x(n),m)}{PRE(x(n),m)} \leq 1+\epsilon) > 1 - Me^{-2\lambda n},$$

where $M$ is a universal constant. Another probabilistic bound is also given for $\frac{OPT(x(n),m)}{PRE(x(n),m)}$, under a mild assumption of $F$.

## 1 Introduction

Given $m$ unit-capacity bins and a collection $x(n) = (x_1, x_2, \ldots, x_n)$ of $n$ pieces, each with a positive size at most one, the dual bin packing problem asks for packing a maximum number of pieces into the $m$ bins so that no bin

*Research supported in part by the ONR Grant N00014-91-J-1383.

capacity is exceeded. This problem models a wide variety of applications in operations research and computer science. Example applications include the problem of storing a maximum number of variable length records on a disk with $m$ cylinders, or scheduling a maximum number of independent tasks on $m$ identical processors so as to meet a common deadline, or cutting as many wood pieces of specified length from $m$ equal-length stock pieces.

Like other bin packing problems, the dual bin packing problem is also NP-hard [7,9]; in fact, it is NP-hard for every fixed $m \geq 2$. Motivated by the computational complexity of the problem, Coffman et al. [4,9] considered a class of heuristics, the *prefix* algorithms, and showed that all prefix algorithms obey the worst-case performance bound of $2 - \frac{1}{m}$. A prefix algorithm is one that satisfies the following two properties: (1) no larger piece is packed without packing a smaller piece; (2) no unpacked piece will fit into any bin in the final packing.

To improve upon the worst-case performance bound of prefix algorithms, Coffman et al. [4,9] considered a special prefix algorithm, the *first-fit-increasing* (FFI) algorithm, which initially sorts the pieces in nondecreasing piece size, then packs successive pieces from the list into the lowest indexed bin into which it will fit, and finally terminates when it first fails to pack a piece. They showed that the FFI algorithm obeys a better worst-case performance bound of $\frac{4}{3}$.

To further improve the worst-case performance bound, Coffman and Leung [3,9] proposed the *iterated-first-fit-decreasing* (FFD*) algorithm which works as follows. Assuming that $x(n)$ has been sorted in nondecreasing piece size, the algorithm first scans $x(n)$ to find the maximum length prefix $x^{(1)}(n) = (x_1, x_2, \ldots, x_t) \subset x(n)$ such that $\sum_{i=1}^{t} x_i \leq m$. It then packs $x^{(1)}(n)$ into as many, say $m'$, bins as required, by scanning right to left and placing the next smaller piece into the lowest indexed bin into which it will fit. The algorithm terminates successfully if $m' \leq m$; otherwise, it constructs $x^{(2)}(n)$ by discarding the largest piece in $x^{(1)}(n)$ and then proceeds as above to pack $x^{(2)}(n)$. This process is repeated until for some $j$, $x^{(j)}(n)$ has been packed into $m' \leq m$ bins. They showed that the FFD* algorithm obeys an asymptotic worst-case performance bound of $\frac{7}{6}$. Furthermore, they showed that the FFD* algorithm never packs fewer pieces than the FFI algorithm, for all $m$, $n$ and $x(n)$.

We denote by $x_i$ both a piece and its size. Let $PRE(x(n), m)$, $FFI(x(n), m)$, $FFD*(x(n), m)$ and $OPT(x(n), m)$ denote the numbers of pieces packed into the $m$ bins by a prefix algorithm, the FFI algorithm, the FFD* algorithm and an optimization algorithm, respectively. The above results can be stated as follows:

$$\frac{OPT(x(n),m)}{PRE(x(n),m)} \leq 2 - \frac{1}{m}, \tag{1.1}$$

$$\frac{OPT(x(n),m)}{FFI(x(n),m)} \leq \frac{4}{3}, \tag{1.2}$$

$$OPT(x(n),m) \leq \frac{7}{6} FFD*(x(n),m) + 3, \tag{1.3}$$

and

$$FFD*(x(n),m) \geq FFI(x(n),m), \tag{1.4}$$

for all $m$, $n$ and $x(n)$.

The earliest result on probabilistic analysis of dual bin packing heuristic is due to Bruno and Downey [2]. Under the assumption that $x(n)$ is a random sample from a random variable distributed according to the uniform distribution over [0,1], they showed that for any $\varepsilon > 0$,

$$Pr(\frac{OPT(x(n),m)}{FFI(x(n),m)} < 1 + O(\frac{1}{\sqrt{n}})) \geq 1 - \varepsilon, \tag{1.5}$$

where the constant behind the $O$-notation depends only on $\varepsilon$ and $m$. We note that (1.5) also holds for *any* prefix algorithm.

Foster and Vohra [6] showed that for *any* distribution $F$ over [0,1],

$$\lim_{n\to\infty} \frac{OPT(x(n),m)}{PRE(x(n),m)} = 1,$$

almost surely. Probabilistic analysis of on-line dual bin packing was also considered. For a given $n$, choose a $c_n \in [0,1]$, and let $H(c_n)$ denote the number of pieces among $x(n) = (x_1, \ldots, x_n)$ that are packed into the $m$ bins by the following rule: If $x_i \leq c_n$, then pack it in a bin into which it will fit; otherwise, discard it. Assuming that $F$ is a continuous type of distribution over [0,1] with finite mean and zero in its compact support, they showed that

$$\lim_{n\to\infty} \frac{H(c_n)}{t} = 1$$

in probability, where $t$ is the largest integer such that the sum of the sizes of the first $t$ smallest pieces among $x(n)$ is no more than $m$.

In [10] Rhee and Talagrand studied the special case where $m$ is related to $n$, i.e., $m = \lfloor an \rfloor$ for some constant $0 < a < 1$. They showed that

$$b = \lim_{n\to\infty} \frac{E[OPT(x(n),m)]}{n}$$

exists, and $\frac{OPT(x(n),m)-bn}{\sqrt{n}}$ converges in distribution, whose limit distribution is the supremum of a certain Gaussian process canonically attached to $F$.

In this article we generalize the result of Bruno and Downey [2], by considering probabilistic bound for *any* prefix algorithm where piece sizes are drawn from *any* distribution $F$ over [0,1]. No further regularity assumption is made on $F$. Without loss of generality, we may assume that $m > 1$, since all prefix algorithms are optimal for just a single bin. Let $x(n)$ be a random sample of a random variable $X$ distributed according to $F$. We show that for any $\varepsilon > 0$, there are $\lambda > 0$ and $N > 0$, dependent only on $m$, $\varepsilon$ and $F$, such that for all $n \geq N$,

$$Pr(\frac{OPT(x(n),m)}{PRE(x(n),m)} \leq 1+\varepsilon) > 1 - Me^{-2\lambda n},$$

where $M$ is a universal constant, and $\lambda$ and $N$ can be determined for any given $\varepsilon > 0$ when $F$ is known. Another probabilistic bound for $\frac{OPT(x(n),m)}{PRE(x(n),m)}$ is given, under a mild assumption of $F$. Suppose there are constants $a \geq 1$, $b > 0$ and $C > 0$ such that $F(t) \geq Ct^a$ for all $0 \leq t \leq b$. Then, for any $\theta > 0$,

$$Pr(\frac{OPT(x(n),m)}{PRE(x(n),m)} \leq 1 + O(\frac{1}{n^{1/(2a)}})) > 1 - \theta,$$

where the constant behind the $O$-notation is dependent only on $\theta$, $m$, $b$ and $C$, and the constant can be calculated when $\theta$, $m$, $b$ and $C$ are known.

While the dual bin packing problem has the nice property that even very fast heuristics are asymptotically optimal on the average, regardless of the input distribution, this is not the case for other variants of the bin packing problem. Consider the NP-hard bin covering problem in which the objective is to pack $x(n)$ into a maximum number of bins such that each bin is filled to a level at least one [1]. For this problem the heuristic with the best known worst-case performance bound is the *iterated-lowest-fit-decreasing* (ILFD) algorithm due to Assmann et al. [1]. It was shown in [1] that

$$\lim_{N\to\infty} \sup(\max\{\frac{OPT(x(n))}{ILFD(x(n))} : n \geq 1 \text{ and } ILFD(x(n)) = N\}) = \frac{4}{3},$$

where $ILFD(x(n))$ and $OPT(x(n))$ denote the numbers of bins filled by the ILFD algorithm and an optimization algorithm, respectively.

Recently, Han et al. [8] showed that for any real number $\alpha \in [1, \frac{4}{3}]$, there is a distribution function $F_\alpha$ over [0,1] such that if $x(n)$ is a random sample of a random variable distributed according to $F_\alpha$, then

$$\lim_{n\to\infty} \frac{OPT(x(n))}{ILFD(x(n))} = \alpha,$$

almost surely. From this result we can conclude that the average-case performance of the ILFD algorithm depends strongly on the input distribution. Thus, in some sense, the bin covering problem is more difficult to approximate than the dual bin packing problem.

In the next section we will present a probabilistic bound for general distributions. Another probabilistic bound for a special class of distributions will be discussed in Section 3. Finally we draw some concluding remarks in the last section.

## 2 Probabilistic Bound for General Distributions

In this section we assume with probability one that the size of each piece is a positive number. That is, we assume that $F$ is an *arbitrary* distribution function over [0,1] with $F(0) = 0$. For a given $F$, define the function $F^{-1}$ from [0,1] to [0,1]:

$$F^{-1}(z) = \min\{t\colon F(t) \geq z \text{ and } 0 \leq t \leq 1\}. \tag{2.1}$$

Note that $F^{-1}$ is well defined because $F(t)$ is right continuous and nondecreasing. Also, $F^{-1}$ is nondecreasing since, for any $0 \leq z_1 < z_2 \leq 1$, $\{t\colon F(t) \geq z_1\} \supseteq \{t\colon F(t) \geq z-2\}$.

Let $U$ be a random variable uniformly distributed over [0,1], and let $Y = F^{-1}(U)$. If $X$ is a random variable distributed according to $F$, then $X$ and $Y$ have identical distributions; i.e., for any $0 \leq x \leq 1$,

$$Pr(Y \leq x) = Pr(X \leq x).$$

This is because $U$ is uniformly distributed and

$$\{u\colon 0 \leq F^{-1}(u) \leq x\} = \{u\colon 0 \leq u \leq F(x)\}.$$

Thus, $Pr(Y \leq x) = Pr(F^{-1}(U) \leq x) = F(x) = Pr(X \leq x)$. It follows that if $X_1, \ldots, X_n$ are $n$ independent and identically distributed random variables with distribution $F$, then for any $a_i < b_i$, $1 \leq i \leq n$,

$$Pr(\cap_{i=1}^{n}\{a_i < X_i \leq b_i\}) = Pr(\cap_{i=1}^{n}\{a_i < F^{-1}(U_i) \leq b_i\}), \tag{2.2}$$

where $U_1, \ldots, U_n$ are independent and identically distributed according to the uniform distribution over [0,1].

We need a well-known result in Kolmogorov-Smirnov statistics. Let $u_1, \ldots, u_n$ be a random sample from a random variable $U$ which is uniformly distributed over [0,1], and let $u_{(1)}, \ldots, u_{(n)}$ be its order statistics. Then there is a universal constant $M$ such that for all $n \geq 1$ and $s > 0$,

$$Pr(\cap_{i=1}^{n}\{u_{(i)} - \frac{i}{n} \leq \frac{s}{\sqrt{n}}\}) > 1 - Me^{-2s^2}. \tag{2.3}$$

Let $\beta$ be a real number in (0,1]. From (2.3), it is easy to see that for all $s > 0$ and $n \geq 1$,

$$Pr(\max\{i: 1 \leq i \leq n \text{ and } 0 < u_{(i)} \leq \beta\} \geq \beta n - s\sqrt{n} - 1) > 1 - Me^{-2s^2}. \tag{2.4}$$

We also need a result in combinatorial analysis of the dual bin packing problem, which was proved in [2]. Suppose we pack $x(n)$ into a *single* bin with capacity $m$ by the FFI algorithm. Let $S(x(n), m)$ denote the number of pieces packed. Then, for *any* prefix algorithm, we have

$$OPT(x(n), m) \leq S(x(n), m) \leq PRE(x(n), m) + m - 1. \tag{2.5}$$

The idea of proving our probabilistic bound is as follows. From (2.2) we can regard every random sample $x_1, \ldots, x_n$ as a random sample $y_1 = F^{-1}(u_1), \ldots, y_n = F^{-1}(u_n)$, where $u_1, \ldots, u_n$ is a random sample from the uniform distribution over [0,1]. Since $F(0) = 0$, there are three cases to consider: (1) $F$ is continuous at zero; (2) there is a positive number $r_0$ such that $F(t) = 0$ for all $t < r_0$ and $F$ jumps at $r_0$; (3) $F$ is continuous at the point $r_0$. In the first case, it is clear that when $n$ gets large, there would be many small $u_i$'s, and hence many small $y_i$'s. Thus a prefix algorithm would pack many pieces into the $m$ bins, and hence by (2.5), $\frac{OPT(x(n),m)}{PRE(x(n),m)}$ would be very close to 1. In the second case, there would be many pieces with size $r_0$ (which is the smallest size) when $n$ gets large. Thus a prefix algorithm and an optimization algorithm would pack exactly $\left\lfloor \frac{1}{r_0} \right\rfloor$ pieces into each bin, and hence $OPT(x(n), m) = PRE(x(n), m)$. In the last case, there would be many pieces with size $r_0$ or slightly larger, as $n$ gets large. In this case, a prefix algorithm and an optimization algorithm would also pack the same number of pieces into the $m$ bins.

**Theorem 1.** *Let $F$ be an* arbitrary *distribution over [0,1], and let $x(n)$ denote a random sample of a random variable $X$ distributed according to $F$. Then, for any $\varepsilon > 0$, there are $\lambda > 0$ and $N > 0$, dependent only on $m$, $\varepsilon$ and $F$, such that for all $n \geq N$,*

$$Pr(\frac{OPT(x(n), m)}{PRE(x(n), m)} \leq 1 + \varepsilon) > 1 - Me^{-2\lambda n}, \tag{2.6}$$

*where $M$ is a universal constant.*

**Proof:** Let $x(n) = (x_1, \ldots, x_n)$ be a random sample of a random variable distributed according to $F$. As discussed above, $x(n)$ can be regarded as a random sample $y(n) = (F^{-1}(u_1), \ldots, F^{-1}(u_n))$, where $u_1, \ldots, u_n$ is a random sample from the uniform distribution over (0,1]. Define $r_0 = \sup\{t: F(t) = 0\}$. Clearly, $0 \leq r_0 \leq 1$, since $F(t)$ is a distribution over [0,1]. We consider the following three cases.

*Case I*: $r_0 = 0$.

From (2.5), we have

$$\frac{OPT(x(n),m)}{PRE(x(n),m)} \leq 1 + \frac{m-1}{PRE(x(n),m)} \leq 1 + \frac{m-1}{S(x(n),m)-m+1}. \quad (2.7)$$

Thus, if we could prove that for any $\varepsilon > 0$, there is an $N > 0$, dependent only on $m$, $\varepsilon$ and $F$, such that for all $n \geq N$,

$$Pr(S(x(n),m) \geq (m-1)(\frac{1+\varepsilon}{\varepsilon})) > 1 - Me^{-2\lambda n}, \quad (2.8)$$

the theorem would follow immediately from (2.7) and (2.8). Observe that for any fixed $m$, $S(x(n),m)$ is computable, and hence, as a function of $x(n)$, it is measurable. By (2.2), (2.8) holds if and only if

$$Pr(S(y(n),m) \geq (m-1)(\frac{1+\varepsilon}{\varepsilon})) > 1 - Me^{-2\lambda n} \quad (2.9)$$

holds.

We now proceed to prove (2.9). In this case we have $F(\frac{m\varepsilon}{(m-1)(1+\varepsilon)}) > 0$. Thus there are positive numbers $\lambda$ and $N$, dependent only on $m$, $\varepsilon$ and $F$, such that by letting $s = \sqrt{\lambda n}$ in (2.4),

$$F(\frac{m\varepsilon}{(m-1)(1+\varepsilon)})n - \sqrt{\lambda n} - 1 > (m-1)(\frac{1+\varepsilon}{\varepsilon}), \quad (2.10)$$

for all $n \geq N$. From (2.4) and (2.10), we have

$$Pr(\max\{i\colon 1 \leq i \leq n \text{ and } 0 < u_{(i)} \leq F(\frac{m\varepsilon}{(m-1)(1+\varepsilon)})\}$$
$$> (m-1)(\frac{1+\varepsilon}{\varepsilon})) > 1 - Me^{-2\lambda n},$$

for all $n \geq N$. By the definition and monotonicity of $F^{-1}$, we have for all $n \geq N$,

$$Pr(|\{i\colon 1 \leq i \leq n \text{ and } 0 < F^{-1}(u_i) \leq \frac{m\varepsilon}{(m-1)(1+\varepsilon)}\}|$$
$$> (m-1)(\frac{1+\varepsilon}{\varepsilon})) > 1 - Me^{-2\lambda n},$$

which means that the event in which at least $\lceil (m-1)(\frac{1+\varepsilon}{\varepsilon}) \rceil$ $y_i$'s have sizes at most $\frac{m\varepsilon}{(m-1)(1+\varepsilon)}$ occurs with probability greater than $1 - Me^{-2\lambda n}$. Clearly, in this event, $S(y(n),m) \geq (m-1)(\frac{1+\varepsilon}{\varepsilon})$ and hence (2.9) holds.
*Case II*: $r_0 > 0$ and $F(r_0) > 0$.

Let $\lambda$ and $N$ be positive numbers such that for all $n \geq N$,

$$F(r_0)n - \sqrt{\lambda}n - 1 > \frac{m}{r_0}. \tag{2.11}$$

For each $n \geq N$, we have, by (2.11) and letting $s = \sqrt{\lambda n}$ in (2.4),

$$Pr(\max\{i: 1 \leq i \leq n \text{ and } 0 < u_{(i)} \leq F(r_0)\} > \frac{m}{r_0}) > 1 - Me^{-2\lambda n},$$

and by the definition and monotonicity of $F^{-1}$,

$$Pr(|\{i: 1 \leq i \leq n \text{ and } 0 < F^{-1}(u_i) \leq r_0\}| > \frac{m}{r_0}) > 1 - Me^{-2\lambda n}. \tag{2.12}$$

But in this case, for all $n \geq 1$, we have

$$Pr(\{(F^{-1}(u_1), \ldots, F^{-1}(u_n)) : \text{ there is an } i, 1 \leq i \leq n, \text{ such that } F^{-1}(u_i) < r_0\}) = 0.$$

Thus, from (2.12), we have for each $n \geq N$,

$$Pr(|\{i: 1 \leq i \leq n \text{ and } F^{-1}(u_i) = r_0\}| > \frac{m}{r_0}) > 1 - Me^{-2\lambda n},$$

which means that the event in which at least $\lceil \frac{m}{r_0} \rceil$ $y_i$'s have sizes exactly $r_0$ occurs with probability greater than $1 - Me^{-2\lambda n}$. Since $r_0$ is the smallest size, a prefix algorithm and an optimization algorithm would both pack exactly $\lfloor \frac{1}{r_0} \rfloor$ pieces per bin, and hence $OPT(y(n), m) = PRE(y(n), m)$. Observe that for any fixed $m$, $\frac{OPT(x(n),m)}{PRE(x(n),m)}$ is computable, and hence, as a function of $x(n)$, it is measurable. Thus, by (2.2), we have for all $n \geq N$,

$$Pr(\frac{OPT(x(n), m)}{PRE(x(n), m)} = 1) = Pr(\frac{OPT(y(n), m)}{PRE(y(n), m)} = 1) > 1 - Me^{-2\lambda n}.$$

*Case III*: $r_0 > 0$ and $F(r_0) = 0$.

In this case we have $r_0 < 1$; otherwise, $F(1) = 0$, contradicting the assumption that $F$ is a distribution over [0,1]. Moreover, we may assume that $r_0 < \frac{1}{2}$; otherwise, $Pr(\{x(n) = (x_1, \ldots, x_n): x_i > \frac{1}{2} \text{ for all } i = 1, \ldots, n\}) = 1$. Consequently, a prefix algorithm and an optimization algorithm would both pack exactly one piece per bin, and hence the theorem would hold vacuously.

Define

$$\delta = \begin{cases} \frac{r_0}{1-r_0} & \text{if } \frac{1}{r_0} \text{ is an integer} \\ \lfloor \frac{1}{r_0} \rfloor^{-1} & \text{if } \frac{1}{r_0} \text{ is not an integer} \end{cases}$$

Note that $r_0 < \delta < 1$ when $0 < r_0 < \frac{1}{2}$. Let $\lambda$ and $N$ be positive numbers such that for all $n \geq N$,

$$F(\delta)n - \sqrt{\lambda}n - 1 > \frac{m}{r_0}. \tag{2.13}$$

By (2.13) and letting $s = \sqrt{\lambda n}$ in (2.4), we have for all $n \geq N$,

$$Pr(\max\{i\colon 1 \leq i \leq n \text{ and } 0 < u_{(i)} \leq F(\delta)\} > \frac{m}{r_0}) > 1 - Me^{-2\lambda n}, \tag{2.14}$$

and by the definition and monotonicity of $F^{-1}$,

$$Pr(\max\{i\colon 1 \leq i \leq n \text{ and } 0 < F^{-1}(u_{(i)}) \leq \delta\} > \frac{m}{r_0}) > 1 - Me^{-2\lambda n}. \tag{2.15}$$

Since $F(r_0) = 0$, we have

$$Pr(\{(F^{-1}(u_1), \ldots, F^{-1}(u_n))\colon \text{ there is an } i, 1 \leq i \leq n, \text{ such that } F^{-1}(u_i) \leq r_0\}) = 0$$

Together with (2.15), we have for all $n \geq N$,

$$Pr(\max\{i\colon 1 \leq i \leq n \text{ and } r_0 < F^{-1}(u_{(i)}) \leq \delta\} > \frac{m}{r_0}) > 1 - Me^{-2\lambda n}, \tag{2.16}$$

which means that the event in which at least $\left\lceil \frac{m}{r_0} \right\rceil$ $y_i$'s have sizes in the range $(r_0, \delta]$ occurs with probability greater than $1 - Me^{-2\lambda n}$.

Now consider the packings of $y(n)$ in the above event by a prefix algorithm and an optimization algorithm. Without loss of generality, we may assume that $y(n) = (y_{(1)}, \ldots, y_{(n)})$, where $y_{(1)}, \ldots, y_{(n)}$ is the order statistics of $y_1, \ldots, y_n$. Since the sum of the first $\left\lceil \frac{m}{r_0} \right\rceil$ elements of $y(n)$ is greater than $m$, only the first $\left\lceil \frac{m}{r_0} \right\rceil$ elements would be considered by both algorithms. Since the sizes of these pieces are in the range $(r_0, \delta]$, both algorithms would pack exactly $\left\lfloor \frac{1}{r_0} \right\rfloor$ pieces into each bin if $\frac{1}{r_0}$ is not an integer, and $(\frac{1}{r_0} - 1)$ pieces if $\frac{1}{r_0}$ is an integer.

With the above argument and (2.16), we have for all $n \geq N$,

$$Pr(\frac{OPT(y(n), m)}{PRE(y(n), m)} = 1) > 1 - Me^{-2\lambda n}. \tag{2.17}$$

Observe that for any fixed $m$, $\frac{OPT(x(n),m)}{PRE(x(n),m)}$ is computable, and hence, as a function of $x(n)$, it is measurable. By (2.2) and (2.17), we have for all

$n \geq N$,

$$Pr(\frac{OPT(x(n),m)}{PRE(x(n),m)} = 1) = Pr(\frac{OPT(y(n),m)}{PRE(y(n),m)} = 1) > 1 - Me^{-2\lambda n}.$$

□

## 3 Probabilistic Bound for Special Distributions

In this section we will consider any distribution $F$ satisfying the following two properties:

**(A1)** $F(t)$ is continuous at $t = 0$.

**(A2)** $F(t)$ has a lower bound $Ct^a$ in the neighborhood $[0, b]$ of 0, where $a \geq 1$, $b > 0$ and $C > 0$.

We will give a probabilistic bound for this kind of distribution.

**Theorem 2.** *Let $F$ be a distribution over $[0,1]$ satisfying (A1) and (A2), and let $x(n)$ be a random sample from $F$. Then, for any $\theta > 0$ and any $m > 1$,*

$$Pr(\frac{OPT(x(n),m)}{PRE(x(n),m)} \leq 1 + O(\frac{1}{n^{1/(2a)}})) > 1 - \theta, \tag{3.1}$$

*where the constant behind the O-notation is dependent only on $\theta$, $m$, $b$ and $C$.*

**Proof:** Recall the inequality (2.4). For any given $\theta > 0$, let $s(\theta) > 0$ be such that $Me^{-2s^2(\theta)} \leq \theta$. Fix $s(\theta)$ and consider the inequality.

$$Cn\gamma^a - s(\theta)\sqrt{n} - 1 > \frac{m}{\gamma}, \tag{3.2}$$

where $\gamma = \frac{D}{n^k}$ for some undetermined constants $D > 0$ and $k > 0$. Let $\Delta = Cn\gamma^{a+1} - s(\theta)\gamma\sqrt{n} - \gamma$. By (A2), we have $a \geq 1$. Clearly, for any fixed $m > 1$, when $k \leq \frac{1}{2a}$, there are constants $D > 0$ and $N > 0$ such that $\Delta > m$ for all $n > N$, and when $k > \frac{1}{2a}$, $\Delta \to \infty$ as $n \to \infty$, for any constant $D > 0$. Thus, for any fixed $m > 1$, the largest value of $k$ for (3.2) to hold (as $n \to \infty$) is $\frac{1}{2a}$. Consequently, we will choose $k$ to be $\frac{1}{2a}$. Now, using $s(\theta)$, $m$ and $C$, we choose carefully the constant $D$ so that there is an $N_0 > 0$ such that (3.2) holds for all $n > N_0$. Let $\varepsilon = \frac{(m-1)\gamma}{m-(m-1)\gamma}$; i.e., $\gamma = \frac{m\varepsilon}{(m-1)(1+\varepsilon)}$. Since $\frac{\varepsilon}{\gamma} \to \frac{m-1}{m}$ as $\gamma \to 0$, we have

$$\varepsilon = O(\frac{1}{n^{1/(2a)}}), \tag{3.3}$$

where the constant behind the $O$-notation is dependent only on $\gamma$ and $m$, and hence is dependent only on $\theta$, $C$, and $m$. From (3.3) it is easy to see that there is an $N > N_0$ such that $0 < \frac{m\varepsilon}{(m-1)(1+\varepsilon)} \leq b$ for all $n \geq N$. Since $F(t) \geq Ct^a$ for all $0 \leq t \leq b$, we have

$$F(\frac{m\varepsilon}{(m-1)(1+\varepsilon)}) \geq C(\frac{m\varepsilon}{(m-1)(1+\varepsilon)})^a$$

for all $n \geq N$. By (3.2), we have for all $n > N$,

$$F(\frac{m\varepsilon}{(m-1)(1+\varepsilon)})n - s(\theta)\sqrt{n} - 1 > (m-1)\frac{(1+\varepsilon)}{\varepsilon}. \tag{3.4}$$

Starting from (3.4) and using a similar argument as in the proof of Case I in Theorem 1, we have for all $n > N$,

$$Pr(\frac{OPT(x(n),m)}{PRE(x(n),m)} \leq 1+\varepsilon) > 1-\theta.$$

By (3.3) and by choosing an appropriate constant depending on $N$, we finally obtain

$$Pr(\frac{OPT(x(n),m)}{PRE(x(n),m)} \leq 1 + O(\frac{1}{n^{1/(2a)}})) > 1-\theta,$$

where the constant behind the $O$-notation is dependent only on $\theta$, $m$, $b$ and $C$. □

## 4 Conclusions

In this article we have given a probabilistic bound for an *arbitrary* distribution and a probabilistic bound for a special class of distributions. Following the proof of Theorem 1, we see that the constants $\lambda$ and $N$ mentioned in Theorem 1 can be calculated for any $\varepsilon$ when the distribution $F$ is known. Similarly, the constants behind the $O$-notation in (3.1) and (3.5) can be calculated for any $\theta$ and $m$, when $a$, $b$ and $C$ are known, or when $F$ is known. The universal constant $M$ in Theorem 1 is from the Kolmogorov-Smirnov statistics, which is close to one; $M$ was simply taken as 1 in [2]. Thus the probabilistic bounds presented in this article can be used to obtain a more detailed picture about the average-case behaviors of $\frac{OPT(x(n),m)}{PRE(x(n),m)}$.

All of our results are applicable to the FFI and FFD* algorithms as well. Note that our results can be useful *even* if the distribution is not known; in this case we simply obtain a sample distribution from random samples of the instances.

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