Graphs of order at most 10 which are critical with respect to the total chromatic number

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Abstract

A catalogue is presented which contains the graphs having order at most 10 which are critical with respect to the total chromatic number. A number of structural properties which cause these graphs to be critical are discussed, and a number of infinite classes of critical graphs are identified.

A total colouring of a graph G is a function assigning colours to the vertices and edges of G in such a way that no two adjacent or incident elements are assigned the same colour. The total chromatic number, $\chi''(G)$, is the minimum number of colours which need to be assigned to obtain a total colouring of the graph G.

A longstanding conjecture, made independently by Behzad [3] and Vizing [17], claims that

$$\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$$

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where $\Delta(G)$ is the maximum degree of G. The lower bound is sharp, the upper bound remains to be proved. A graph G is said to be Type 1 if $\chi''(G) = \Delta(G)+1$ and is said to be Type 2 if $\chi''(G) \geq \Delta(G)+2$.

We define a graph G to be critical with respect to the total chromatic number if G is connected and $\chi''(G-e) < \chi''(G)$ for every edge e in G. In Section 1 of this paper we identify all small order critical graphs, the catalogue of graphs is presented as a table of diagrams. In Section 2 we study structural properties of these graphs in order to identify features which cause a graph to be Type 2.

1 A Catalogue of Critical Graphs

In this section we present a catalogue of the critical graphs having order at most ten. Associated with each graph is a triple (n, Δ, m), where n is the order, Δ is the maximum degree and m is the size of the graph. The graphs are listed in increasing order of n, then Δ, then m. The catalogue is complete up to and including the graphs of order eight, and is believed to be complete up to and including the graphs of order ten.

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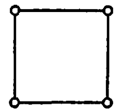

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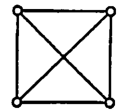

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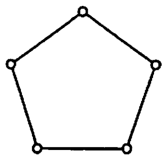

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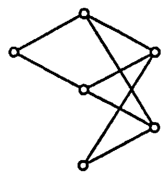

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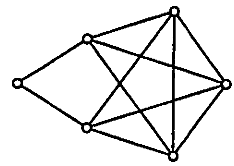

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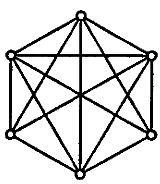

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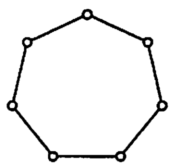

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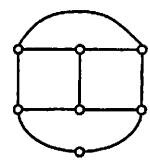

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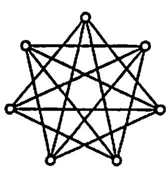

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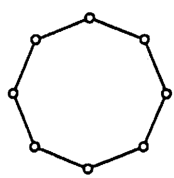

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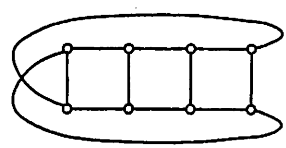

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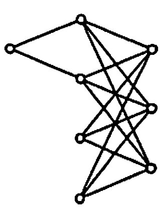

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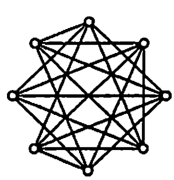

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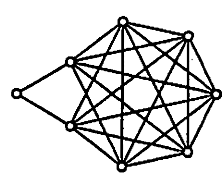

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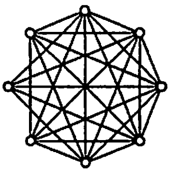
Catalogue of Critical Graphs (Nos. 1-15)



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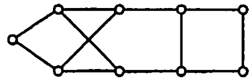

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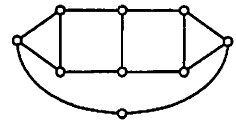

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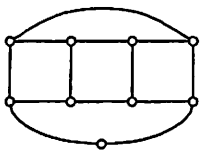

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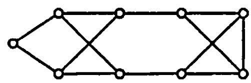

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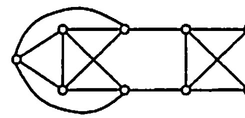

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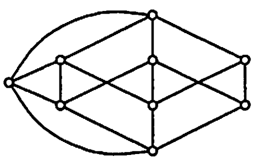

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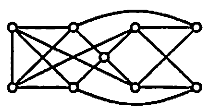

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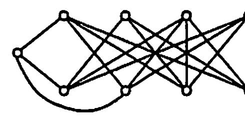

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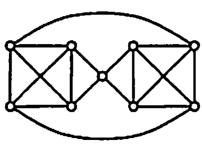

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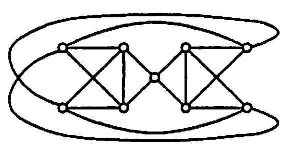

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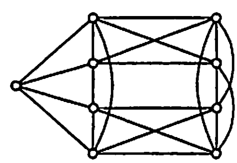

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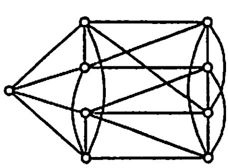

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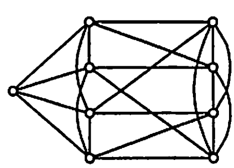

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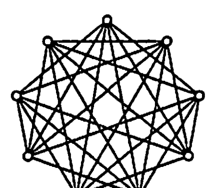

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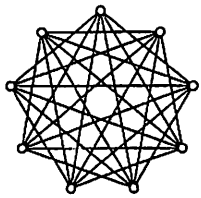
Catalogue of Critical Graphs (Nos.16-30)



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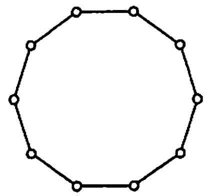

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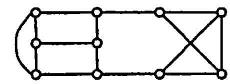

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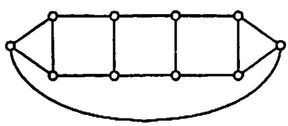

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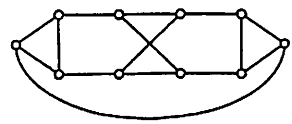

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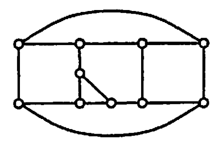

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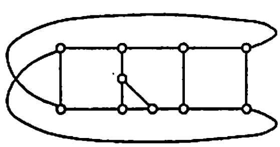

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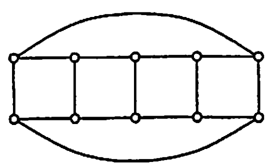

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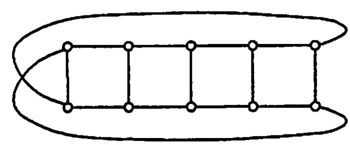

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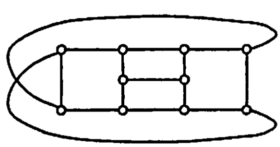

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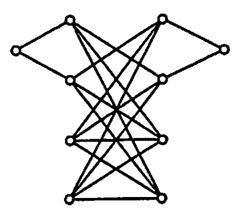

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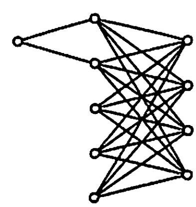

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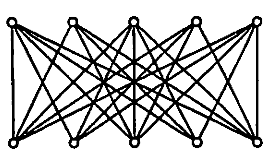

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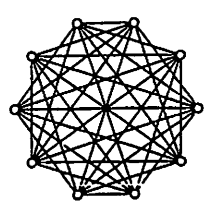

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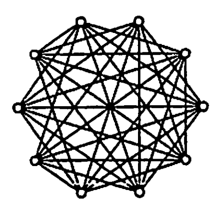

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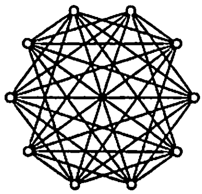
Catalogue of Critical Graphs (Nos.31-45)



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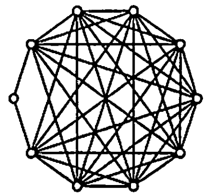

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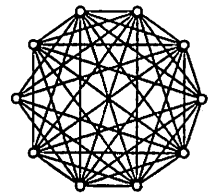

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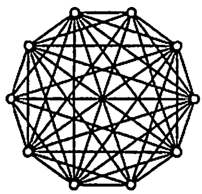

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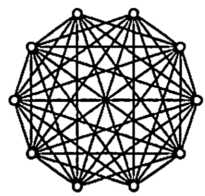

No.50 (10,9,43)

Catalogue of Critical Graphs (Nos.46-50)

2 Classification of the Critical Graphs

In this section, we try to give short (structural) reasons why the critical graphs listed in the catalogue are Type 2. Certain graphs will fall into more than one of the categories (for example graph 2 is both a cycle and a Chen and Fu graph). We have, however, simply listed such graphs in the first category which applies to them.

2.1 Nonconformability

A graph G is conformable if it has a vertex colouring

$$\varphi : V(G) \longrightarrow \{1, 2, \ldots, \Delta(G) + 1\}$$

for which at most $\mathrm{def}(G) = \sum_{v \in V(G)}(\Delta(G) - d_G(v))$ colour classes have parity different from that of $|V(G)|$. It is easily seen (for example [7]) that a being conformable is a necessary condition for a graph to be Type 1, or equivalently if a graph is non-conformable, then it is Type 2. The following graphs in the catalogue are non-conformable 1, 3, 7, 10, 14, 16, 27, 28, 29, 30, 31, 44, 45, 46, 48, 49 and 50.

2.2 Cycles

It is well known (see for example [4]) that a cycle which has an order which is not a multiple of 3 is critical. The following graphs fall in this class: 2, 4, 8, 11 and 32.

2.3 Graphs having maximum degree three

These graphs are presented, and their structure discussed, in an earlier paper by Hamilton and Hilton [10]. They will not be discussed in detail in this section. The following graphs are in this class: 9,12,17,18, 19, 20 and 33 to 40.

2.4 Chen and Fu graphs

Chen and Fu [9] showed that any graph obtained from an odd order complete graph by subdividing one edge is critical even though the graph (and all subgraphs with the same maximum degree) is conformable. This class contains graphs numbers 6, 15 and 47.

2.5 Non-biconformability

In [12] Hilton showed that if J is a subgraph of $K_{n,n}$ with $e = |E(J)|$ and j independent edges in J, then $G = K_{n,n} \backslash E(J)$ has total chromatic number $n+2$ if and only if $e + j \leq n - 1$. This explains graphs numbers 5, 13, 42 and 43. Later in this section we will give a necessary condition for a bipartite graph to be Type 1, namely biconformability. The graphs 5, 13, 42 and 43, and also graph number 41, are not biconformable.

2.6 'Same and different' graphs

A number of graphs are Type 2 because they have the following structure: The graph G possesses a two edge-cutset $\{e_1, e_2\}$ and two (connected) subgraphs, G_1 and G_2 say, such that $E(G) = E(G_1) \cup E(G_2)$, $E(G_1) \cap E(G_2) = \{e_1, e_2\}$ and $V(G_1) \cap V(G_2)$ comprises the endvertices of e_1 and e_2. Furthermore, any $\Delta + 1$-total colouring of G_1 assigns the same colour to both e_1 and e_2, while any $\Delta + 1$-total colouring of G_2 assigns different colours to e_1 and e_2. Since a total colouring of G induces a total colouring of G_1 and a total colouring of G_2, it follows that G must be Type 2. Graph number 21 has this structure. A number of graphs having maximum degree three also fall into this class.

2.7 Further critical graphs

The graphs numbered 22, 23, 24, 25 and 26 are not included in any of the above categories. These graphs seem to fit in two general groups: Graphs 23, 25 and 26 are constructed by taking the union of two complete even order graphs with an additional isolated vertex, then matching some corresponding vertices in the complete graphs. Graphs 22 and 24 are complete equibipartite graphs from which a matching has been removed and to which one further vertex has been added; these graphs appear to be similar to graph number 41.

The following lemma explains why graph number 25 is Type 2 (graphs numbers 23 and 26 appear to be close variants of graph number 25, so we can expect to find explanations why these graphs are Type 2 based on similar ideas). The lemma actually gives an infinite family of critical graphs of which graph number 25 is the smallest (order) member.

Lemma 1 *Let G be a graph having order $2n+1$, where $n \geq 4$ and n is even, formed from two vertex disjoint K_n's by adding a matching of size $n-2$ joining $n-2$ vertices from each K_n, and by joining the remaining two vertices in each K_n to a new vertex. Then G is critical.*

Proof Suppose that G is Type 1 and consider a $\Delta(G)+1$ total colouring of G. Since $\Delta(G) = n$, at most $n+1$ colours can be used in the colouring. Each of the $n+1$ colours must be present at each of the vertices of degree n. Suppose, without loss of generality, that the vertices of one of the subgraphs isomorphic to K_n, subgraph H_1 say, are assigned colours $c_1, c_2, \ldots, c_n$. By counting, it follows that in H_1 there are $\frac{n}{2}$ edges coloured c_{n+1}, and $\frac{n}{2}-1$ edges coloured with each of $c_1, \ldots, c_n$. Consequently each of the n edges of G, which are incident with exactly one of the vertices in H_1, must be assigned a distinct colour from $c_1, c_2, \ldots, c_n$. Repeating this argument with the second complete subgraph, H_2 say, we deduce that each of the n edges of G, which are incident with exactly one of the vertices in H_2, must be assigned a distinct colour from $c_1, c_2, \ldots, c_n, c_{n+1}$. Since $n-2$ of the colours from $c_1, c_2, \ldots, c_n, c_{n+1}$ are assigned to edges of the matching joining H_1 and H_2, it follows that the remaining four edges (each of which is incident with the vertex not in $V(H_1) \cup V(H_2)$) must be assigned one of three remaining colours. Thus G cannot be Type 1.

To show that G is critical, we must show that the removal of any edge gives a Type 1 graph. In this section we assume for convenience that $n \geq 8$; the cases $n = 4$ and $n = 6$ are easily checked. Notice that there are five types of non-isomorphic edges in $E(G)$.

Let H_1 be one of the subgraphs of G isomorphic to K_n and let H_1 have vertices $x_1, x_2, \ldots, x_n$. Let H_2 be the other subgraph of G isomorphic to K_n and let H_2 have vertices $y_1, y_2, \ldots, y_n$. Let z be the vertex of degree four. Let $x_1y_1, x_2y_2, \ldots, x_{n-2}y_{n-2}$ be the edges joining vertices in H_1 to vertices in H_2 and let $x_{n-1}z, x_nz, y_{n-1}z, y_nz$ be the edges joining vertices in H_1 or H_2 to z.

Consider $G_1 = G - y_{n-1}z$. Colour the vertices of H_1 with colours $c_1, c_2, \ldots, c_n$ and the vertices of H_2 with $c_1, c_2, \ldots, c_{n-1}$ and c_{n+1}. It is possible (provided the vertex-colourings are chosen appropriately) to colour the edges of H_1 and H_2 with $c_1, c_2, \ldots, c_{n+1}$ in such a way that colour c_i is missing at x_i for $i = 1, 2, \ldots, n$, colour c_j is missing at y_j for $j = 1, 2, \ldots, n-1$, and colour c_{n+1} is missing at y_n. Colour $x_i y_i$ with c_i for $i = 1, 2, \ldots, n-2$, $x_{n-1}z$ with c_{n-1}, $x_n z$ with c_n and $y_n z$ with c_{n+1}. Finally colouring the vertex z with a colour c_i not used on an adjacent vertex or incident edge gives a total colouring of G_1 using $\Delta + 1 = n + 1$ colours.

The other cases are all similar to this. If $G_2 = G - x_{n-2}y_{n-2}$, colour H_1 as above, and modify the colouring of H_2 so that all the missing colours are as before, except that colours c_{n-1}, c_{n-2} and c_{n+1} are missing from y_{n-2}, y_{n-1} and y_n, respectively. Then a total colouring of G_2 is obtained in the obvious way.

If $G_3 = G - y_{n-1}y_n$, colour H_1 as before, and colour all of H_2 with the same colours missing at the vertices as in G_1, but with $y_{n-1}y_n$ coloured c_n, and y_{n-1} coloured c_1. Then remove the edge $y_{n-1}y_n$, recolour y_{n-1} with c_n, and extend the colourings of H_1 and $H_2 \backslash y_{n-1}y_n$ to all of G_3 in the obvious way, with zy_{n-1} and zy_n coloured c_1 and c_{n+1} respectively.

If $G_4 = G - y_{n-2}y_{n-1}$, colour H_1 as before, and colour all of H_2 with the same colours missing at the vertices as in G_1, but with $y_{n-2}y_{n-1}$ coloured c_n, and y_{n-1} coloured c_1. Then remove the edge $y_{n-2}y_{n-1}$, recolour y_{n-1} with c_n, and extend the colourings of H_1 and $H_2 \backslash y_{n-2}y_{n-1}$ to all of G_4 in the obvious way, with zy_{n-1} and zy_n coloured c_1 and c_{n+1} respectively.

Finally, if $G_5 = G - y_{n-3}y_{n-2}$, colour all of H_2 with the same colours missing at the vertices as in G_1, but with $y_{n-3}y_{n-2}$, $y_{n-1}y_{n-2}$, $y_1 y_{n-1}$ and $y_1 y_{n-3}$ coloured c_n, c_1, c_n and c_{n-1}, respectively. Since the existence of such a colouring is a little more complicated than for the earlier cases, we provide a detailed argument: Note that an edge colouring of K_{n+2} using $n+1$ colours is equivalent to a total colouring of K_{n+1} using the same $n+1$ colours, and that a total colouring of K_{n+1} using $n+1$ colours induces a total colouring of K_n using the same $n+1$ colours. Let H_2^{**} be a complete graph on K_{n+2} vertices constructed from H_2 by adding two new vertices y^* and y^{**}. A result of Andersen and Hilton [2] (Corollary 4.3.9) states that any edge colouring of $K_{\frac{n+2}{2}}$ with $n+1$ colours can be extended to an edge-colouring of K_{n+2} with the same set of colours. Thus it is possible to find an edge colouring of H_2^{**} with the edges $y_{n-3}y_{n-2}$, $y_{n-1}y_{n-2}$, y_1y_{n-1}, y_1y_{n-3}, y^*y_1, y^*y_{n-3}, y^*y_{n-2}, and y^*y_{n-1} coloured c_n, c_1, c_n, c_{n-1}, c_1, c_{n-3}, c_{n-2}, and c_{n-1}, respectively. Deleting y^* and y^{**} and assigning the colour of the the edge $y^{**}y_i$ to the vertex y_i for $1 \leq i \leq n$, we obtain a total colouring of H_2 with edges $y_{n-3}y_{n-2}$, $y_{n-1}y_{n-2}$, y_1y_{n-1} and y_1y_{n-3} coloured c_n, c_1, c_n and c_{n-1}, respectively, and with the colours c_1, c_{n-3}, c_{n-2}, and c_{n-1} missing at the vertices y_1, y_{n-3}, y_{n-2}, and y_{n-1}, respectively. By permuting the remaining colours (if necessary) we can obtain a total colouring of H_2 with c_i missing at y_i for $1 \leq i \leq n-1$ and c_{n+1} missing at y_n. This is the desired colouring of H_2. Next choose a colouring of H_1 with the same colours missing at the vertices as in G_1 and with the colour of x_i distinct from that of y_i. Finally remove the edge $y_{n-3}y_{n-2}$, recolour y_1y_{n-1} with c_{n-1}, $y_{n-2}y_{n-1}$ with c_n and y_1y_{n-3} with c_n. Then extend the colourings of H_1 and $H_2\backslash y_{n-3}y_{n-2}$ to all of G_5 in an obvious way, with zy_{n-1} and zy_n coloured c_1 and c_{n+1}, respectively.

Since the removal of any other edge leaves a graph isomorphic to one of the five graphs considered above, it follows that G is critical. □

In the remainder of this section we present a lemma which explains why graph number 41 is not biconformable, and also why it is critical. Once again an infinite family of critical graphs is identified having graph number 41 as the smallest (order) member. Since the graphs described in the lemma are bipartite, it is worth mentioning that there is a vertex-colouring condition for bipartite graphs which plays a role similar to that of conformability for general graphs. If G is a bipartite graph, we call G equibipartite if it has bipartition (A, B) of the vertex-set such that $|A| = |B|$, and each edge joins a vertex of A to a vertex of B.

Given an equibipartite graph G and given a vertex-colouring which assigns the colours $c_1, c_2, \ldots, c_{\Delta(G)+1}$, let A_i be the set of vertices of A coloured c_i and let B_i be the set of vertices of B coloured c_i. Let $a_i = |A_i|$ and $b_i = |B_i|$. If W is a subset of $V(G)$, let $V_{<\Delta}(W)$ denote the set of vertices in W which have degree less than Δ in the graph G. Call G *biconformable* if G has a vertex-colouring such that for $1 \le i \le \Delta(G)+1$,

$$|V_{<\Delta}(A \backslash A_i)| \ge b_i - a_i,$$
$$|V_{<\Delta}(B \backslash B_i)| \ge a_i - b_i,$$

and

$$\mathrm{def}(G) \ge \sum_{i=1}^{\Delta(G)+1} |a_i - b_i|.$$

If G is not equibipartite, we say that G is not biconformable. Note that this definition of biconformability differs from that proposed by Chetwynd and Hilton in [7] by the inclusion of the inequalities for $|V_{<\Delta}(A \backslash A_i)|$ and $|V_{<\Delta}(B \backslash B_i)|$. A modification to the original definition of biconformability was suggested by J. Wojciechowski. A further modification was then suggested by L. Andersen to give the current definition. Further modifications may be necessary, if biconformability is to play the actual role for bipartite graphs that conformability plays for general graphs. The following lemma shows that like conformability for general graphs, biconformability is a necessary condition for a bipartite graph to be class 1:

Lemma 2 *Let G be an equibipartite graph which is not biconformable. Then G is Type 2*

Proof Suppose that G is Type 1. Consider a total colouring of G with colours $c_1, c_2, \ldots, c_{\Delta(G)+1}$. Recall that if v is a vertex of maximum degree, then each colour c_i is used to colour v or an edge incident with v. Suppose that $a_i \ge b_i$ for some i. Then $a_i - b_i$ vertices of B must have colour c_i missing at them. These vertices are in $B \backslash B_i$, and have degree less than Δ. Consequently, $|V_{<\Delta}(B \backslash B_i)| \ge a_i - b_i$. Similarly $|V_{<\Delta}(A \backslash A_i)| \ge b_i - a_i$ for each i. Furthermore, for each i, $|a_i - b_i|$ vertices have colour c_i missing at them, so $\mathrm{def}(G) \ge \sum_{i=1}^{\Delta(G)+1} |a_i - b_i|$. Thus G is biconformable and the lemma follows. □

Lemma 3 *Let G be a graph of order $2n+2$, where $n \ge 4$, which is obtained by removing $n-2$ independent edges $x_1y_1, x_2y_2, \ldots, x_{n-2}y_{n-2}$ from a $K_{n,n}$, and joining a new vertex y^* to $x_1, x_2, \ldots, x_{n-2}$ and a second new vertex x^* to $y_1, y_2, \ldots, y_{n-2}$. Then G is not biconformable and is critical.*

Proof Let the independent vertex-sets of the bipartite graph G be $A = \{x^*, x_1, x_2, \ldots, x_n\}$ and $B = \{y^*, y_1, y_2, \ldots, y_n\}$.

We show first that G is not biconformable. Suppose G has a vertex-colouring with colours $\{1, 2, \ldots, \Delta + 1\}$. Let A_i, B_i denote the set of vertices of A, B, respectively, coloured i. Suppose the vertex-colouring satisfies the two conditions

$$|V_{<\Delta}(A \backslash A_i)| \geq b_i - a_i \qquad (\forall i)$$

and

$$|V_{<\Delta}(B \backslash B_i)| \geq a_i - b_i \qquad (\forall i).$$

Let c be an arbitrary colour. If c is the colour of either x_{n-1} or x_n, then the only vertex in B which can be coloured c is y^*. If y^* is not coloured c, then $b_c = 0$ so

$$|V_{<\Delta}(B \backslash B_c)| = 1 \geq a_c - b_c = a_c - 0,$$

so $a_c = 1$, and so no other vertex in A is coloured c. If y^* is coloured c, then $b_c = 1$ and so

$$|V_{<\Delta}(B \backslash B_c)| = 0 \geq a_c - b_c = a_c - 1;$$

therefore $a_c = 1$, so no other vertex in A is coloured c. Thus if x_{n-1} or x_n is coloured c then the only other vertex which can be coloured c is y^*. Similarly, if y_{n-1} or y_n is coloured c, the only other vertex which can be coloured c is x^*.

If x_j is coloured c for some $j \in \{1, 2, \ldots, n-2\}$, the only vertex of B which can be coloured c is y_j. If y_j is not coloured c, then $b_c = 0$, so

$$|V_{<\Delta}(B \backslash B_c)| = 1 \geq a_c - b_c = a_c - 0,$$

so $a_c = 1$, and so no other vertex in A is coloured c. If y_j is coloured c, then again no other vertex in A is coloured c. Thus if x_j is coloured c for some $j \in \{1, 2, \ldots, n-2\}$, then only one other vertex can be coloured c, namely y_j.

Thus $n-2$ colours are needed for the vertices $x_1, x_2, \ldots, x_{n-2}$ and $y_1, y_2, \ldots, y_{n-2}$, and a further four colours are needed to colour x_{n-1}, x_n, y_{n-1} and y_n. Thus at least $n+2$ colours are needed altogether, and so G is not biconformable. Since G is not biconformable, it follows that G is Type 2.

Next we show that G is critical. We give a formal proof which is valid only for $n \geq 10$. The remaining cases were checked on a computer, but we do not give the details here. To show that G is critical we have to show that $G \backslash e$ can be totally coloured with $n+1$ colours for each edge e of G. There are exactly four non-isomorphic graphs which can be obtained from G by removing a single edge; they can be obtained by removing the edges $x_{n-2}y^*$, x_1y_2, $x_{n-2}y_{n-1}$ and $x_{n-1}y_{n-1}$ respectively.

Let G^* be the complete bipartite graph $K_{n+1,n+1}$ obtained from G be adding the edges x^*y_{n-1},x^*y_n, y^*x_{n-1}, y^*x_n, x^*y^* and x_iy_j for $1 \leq i \leq n-2$. In each case we shall give G^* an appropriate edge-colouring with colours $c_1, c_2, \ldots, c_{n+1}$, and then derive from it a total colouring of $G \backslash e$.

First consider the graph $G_1 = G - x_{n-2}y^*$. In G^* colour the edges x^*y^*, x^*y_{n-1}, x^*y_n, y^*x_{n-2}, y^*x_{n-1}, y^*x_n, $x_{n-2}y_{n-2}$, $x_{n-2}y_{n-1}$ with colours c_{n-2}, c_1, c_{n+1}, c_1, c_n, c_{n-1}, c_{n-2}, c_{n+1}, respectively. Now extend this partial edge-colouring of G^* to an edge-colouring of all of G^* with colours $c_1, \ldots, c_{n+1}$ in such a way that the edge x_iy_i, for $1 \leq i \leq n-3$, receives colour c_i. Bearing in mind the correspondence between an edge-coloured $K_{n+1,n+1}$ and a latin square of side $n+1$ (where the two maximal independent vertex-sets in the $K_{n+1,n+1}$ correspond to the row and column labels of the latin square, and the coloured edges correspond to the symbols in the latin square), it follows from [1] that such an extension of the partial edge-colouring exists for all $n \geq 10$. We can now obtain the required total colouring of G_1 as follows. We remove the edges of G^* which are not in G_1. All the edges of G_1 are coloured. We recolour the edge $x_{n-2}y_{n-1}$ with colour c_1, and we colour the vertices $x^*, y^*, x_{n-2}, x_{n-1}, x_n, y_{n-2}, y_{n-1}, y_n$ with colours c_{n+1}, c_{n-2}, c_{n-2}, c_n, c_{n-1}, c_{n-2}, c_{n+1}, c_{n+1} respectively, and colour x_i and y_i with colour c_i for $1 \leq i \leq n-3$. It is easy to see that this yields a total colouring of G_1.

Next consider the graph $G_2 = G - x_1y_2$. In G^* colour the edges x_1y_1, x_2y_2, x_1y_2, x_1y_{n-1}, x^*y_{n-1}, x^*y_n, y^*x_{n-1}, y^*x_n with colours c_1, c_2, c_3, c_{n+1}, c_3, c_{n+1}, c_{n-1}, c_n, respectively and, for $3 \leq i \leq n-2$, colour the edge x_iy_i with colour c_i. Then extend this partial edge-colouring of G^* to an edge-colouring of all of G^* with colours $c_1, \ldots, c_{n+1}$. For $n \geq 10$ this can be done by [1]. Now we obtain the required total colouring of G_2 as follows. We remove the edges of G^* which are not in G_2. We recolour the edge x_1y_{n-1} with colour c_3. Then we colour $x^*, y^*, x_1, x_2, x_{n-1}, x_n, y_1, y_2, y_{n-1}, y_n$ with colours c_{n+1}, c_n, c_1, c_2, c_{n-1}, c_n, c_1, c_2, c_{n+1}, c_{n+1} respectively, and colour x_i and y_i with c_i for $3 \leq i \leq n-2$. It is easy to check that this yields a total colouring of G_2.

Next consider the graph $G_3 = G - x_{n-2}y_{n-1}$. In G^* colour the edges x^*y_n, y^*x_{n-1}, y^*x_n, $x_{n-2}y_{n-2}$, $x_{n-2}y_{n-1}$ with colours c_{n+1}, c_{n-1}, c_n, c_{n-2}, c_{n+1} respectively, and for $1 \leq i \leq n-3$, colour the edge x_iy_i with c_i. Then extend this partial edge-colouring of G^* to an edge-colouring of all of G^*. This can be done for all $n \geq 6$ by [1]. Now obtain a total colouring of G_3 as follows. Remove the edges of G^* which are not in G_3. Then colour vertices $x^*, y^*, x_{n-1}, x_n, y_{n-1}, y_n$ with colours c_{n+1}, c_n, c_{n-1}, c_n, c_{n+1}, c_{n+1} respectively, and colour x_i and y_i with c_i for $1 \leq i \leq n-2$. This yields a total colouring of G_3.

Finally consider the graph $G_4 = G - x_{n-1}y_{n-1}$. In G^* colour the edges x^*y_n, y^*x_n, $x_{n-1}y_{n-1}$ with colours c_{n+1}, c_n, c_{n-1} respectively, and, for $1 \leq i \leq n-2$, colour the edge x_iy_i with c_i. Then extend this partial edge-colouring to an edge-colouring of all of G. This can be done for $n \geq 6$ by [1]. Now obtain the required total colouring of G_4 as follows. Remove the edges of G^* which are not in G_4. Then colour the vertices $x^*, y^*, x_{n-1}, x_n, y_{n-1}, y_n$ with c_{n+1}, c_n, c_{n-1}, c_n, c_{n-1}, c_{n+1} respectively, and, for $1 \leq i \leq n-2$, colour x_i and y_j with colour c_i. This yields a total colouring of G_4.

Since all subgraphs of G which can be obtained by removing one edge are isomorphic to one of G_1, G_2, G_3 or G_4, it follows that G is critical. □

There seems to be considerable similarity between graph 41 and graphs 22 and 24, even though the latter graphs are not bipartite. It seems probable that results like Lemma 3 could be proved for general classes of graphs which include graphs 22 or 24.

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