Matchings in the leave of equitable partial Steiner triple systems

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ABSTRACT. In this note, necessary and sufficient conditions are given for the existence of an equitable partial Steiner triple system (S, T) on n symbols with exactly t triples, such that the leave of (S, T) contains a 1-factor if n is even and a near 1-factor if n is odd.

1 Introduction

A partial Steiner triple system of order n ($STS(n)$) is an ordered pair (S, T) where T is a set of edge-disjoint copies of K_3, or *triples*, that together form a subgraph $G(S)$ of K_n with vertex set S. The *leave* of (S, T) is the complement of $G(S)$ in K_n. For each $s \in S$ let $r(s)$ be the number of *triples* in T containing s. A partial $STS(n)$ (S, T) is said to be *equitable* if $|r(s_1) - r(s_2)| \leq 1$ for all s_1, $s_2 \in S$. A *maximum* partial $STS(n)$ is a $STS(n)$ (S, T) in which T is as large as possible among all partial $STS(n)$s. If (S, T) is a maximum partial $STS(n)$ then let $\mu(n) = |T|$. A *near 1-factor* is a graph on n vertices consisting of $(n-1)/2$ independent edges (so n is necessarily odd).

Schönheim has shown [6] that:

$$\mu(n) = \begin{cases} \lfloor\lfloor(n-1)/2\rfloor n/3\rfloor - 1 & \text{if } n \equiv 5 \pmod 6\text{, and} \\ \lfloor\lfloor(n-1)/2\rfloor n/3\rfloor & \text{otherwise.} \end{cases}$$

Furthermore, the leave of any maximum $STS(n)$:

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(a) has no edges if $n \equiv 1$ or 3 (mod 6),

(b) is a 1-factor if $n \equiv 0$ or 2 (mod 6),

(c) has its edges forming one 4-cycle if $n \equiv 5$ (mod 6), and

(d) is the spanning subgraph $F \cup K_{1,3}$ if $n \equiv 4$ (mod 6),

where F consists of $(n-4)/2$ independent edges.

It has been shown by Andersen, Hilton and Mendelsohn [1] that for all $t(n) \leq \mu(n)$ there exists an equitable partial $STS(n)$ (S,T) containing exactly $t(n)$ triples (see [5] for a generalization to triple systems of higher index). Here we present a variation of their result where the leave of (S,T) is required to contain a 1-factor or a near 1-factor. This result is very useful in embedding partial totally symmetric quasigroups [4]. While Theorem 2.2 can be proved directly by making use of several different constructions, here it is proved using an extension of the proof used by Andersen, Hilton and Mendelsohn [1].

It is worth noting some related results. Necessary and sufficient conditions have been found [3] for the existence of partial triple systems (S,T) of order n and index λ with $t(n)$ triples whose leave has a 1-factorization (so $G(S)$ must be regular). Also, Colbourn and Rosa [2] have characterized the graphs in which all vertices have degree 0 and 2 that are the leave of a partial STS.

2 The Main Result

We begin with a result that can be applied with any matching (not necessarily a maximum matching) in the leave.

Theorem 2.1 *If there exists a partial $STS(n)$ with $t(n)$ triples that contains a matching M in its leave, then there exists an equitable partial $STS(n)$ with $t(n)$ triples whose leave contains M.*

Proof: Suppose that $(S = \{1,2,\ldots,n\}, T')$ is a partial $STS(n)$ that contains $t(n)$ triples, and whose leave contains a matching M. If (S,T') is equitable then we are finished. Otherwise let $r'(i)$ be the number of triples in T' that contain symbol i, and assume that $r'(1) \leq r'(2) \leq \cdots \leq r'(n)$, and that $r'(n) - r'(1) \geq 2$. If vertex 1 is incident with an edge in M then let the edge be $\{1,\ell\}$. Form a simple graph H on the vertices $2,\ldots,n-1$ by joining i to j if and only if either $\{1,i,j\} \in T'$ or $\{i,j,n\} \in T'$, and color $\{i,j\}$ with 1 or n respectively. Clearly this is a proper 2-edge-coloring of H in which for each $x \in \{1,n\}$ $r'(x)$ edges are colored x if there is no triple in T' that contains both 1 and n, and $r'(x)-1$ edges are colored x otherwise. Since $r'(n) - r'(1) \geq 2$, at least 2 components of H consist of a path in which the first and last edges are colored n. Clearly at least one of these paths, say $P = (s_1, s_2, \ldots, s_{2k})$ does not begin or end with the vertex ℓ. So the edges $\{1, s_1\}$ and $\{1, s_{2k}\}$ occur in no triple in T', nor in M. Therefore if we define $T = T' \cup \{\{1, s_{2i-1}, s_{2i}\} \mid 1 \leq i \leq k\} \cup \{\{s_{2i}, s_{2i+1}, n\} \mid 1 \leq i \leq k-1\} \setminus (\{\{s_{2i-1}, s_{2i}, n\} \mid 1 \leq i \leq k\} \cup \{\{1, s_{2i}, s_{2i+1}\} \mid 1 \leq i \leq k-1\})$, then (S, T) is a partial $STS(n)$ that contains $t(n)$ triples, whose leave contains M, and in which $r(1) = r'(1) + 1$, $r(n) = r'(n) - 1$, and $r(i) = r'(i)$ for $2 \leq i \leq n-1$. Repetition of this process produces the required equitable partial $STS(n)$. □

Now we can apply Theorem 2.1 to the most interesting case where the matching in the leave is a (near) 1-factor.

Theorem 2.2 *There exists an equitable partial $STS(n)$ (S,T) with $t(n)$ triples such that its leave contains a 1-factor if n is even and a near 1-factor if n is odd if and only if $t(n) \leq T(n)$, where*

$$T(n) = \begin{cases} \mu(n) & = n(n-2)/6 & \text{if } n \equiv 0 \pmod 6, \\ \mu(n) - (n-1)/3 & = (n-1)(n-2)/6 & \text{if } n \equiv 1 \text{ mod } 6), \\ \mu(n) & = n(n-2)/6 & \text{if } n \equiv 2 \pmod 6, \\ \mu(n) - n/3 & = n(n-3)/6 & \text{if } n \equiv 3 \pmod 6, \\ \mu(n) - 1 & = (n+2)(n-4)/6 & \text{if } n \equiv 4 \pmod 6, \text{ and} \\ \mu(n) - (n-5)/3 & = (n-1)(n-2)/6 & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

Proof: The necessity when n is even is obvious once one notes that when $n \equiv 4 \pmod 6$ the leave of any maximum partial $STS(n)$ is $F \cup K_{1,3}$ (see (d) above) which does not contain a 1-factor, so in this case $T(n) < \mu(n)$. The necessity when n is odd follows directly from the leave of (S,T) having all vertices of even degree, so since the leave contains a near 1-factor, at least $n-1$ vertices in the leave have degree at least two, so the leave has at least $n-1$ edges.

To prove the sufficiency, we first show the result is true if we set $t(n) = T(n)$. Secondly, if $t(n) < T(n)$ then by starting with a partial $STS(n)$ with $T(n)$ triples whose leave contains a (near) 1-factor, clearly we can throw away triples to form a partial $STS(n)$ (S,T') with $t(n)$ triples whose leave contains a (near) 1-factor. The result then follows from Theorem 2.1.

To obtain the result when $t(n) = T(n)$ proceed as follows. If $n \equiv 0$ or $2 \pmod 6$ then define (S,T) to be a maximum partial $STS(n)$, since its leave is a 1-factor (see (b) above). If $n \equiv 4 \pmod 6$ then let (S,T') be a maximum partial $STS(n)$, in which the $K_{1,3}$ (see (d) above) in the leave consists of the edges $\{1,2\}$, $\{1,3\}$ and $\{1,4\}$ and let $\{3,4,x\} \in T'$; then $(S, T = T' \setminus \{\{3,4,x\}\})$ has $T(n)$ triples, and its leave contains the 1-factor $F \cup \{\{1,2\},\{3,4\}\}$. If n is odd then let (S',T') be a maximum partial $STS(n+2)$. Let s_1, $s_2 \in S$, where if $n+2 \equiv 5 \pmod 6$ then $\{s_1, s_2\}$

is an edge in the leave (the edges in the leave form a cycle of length 4 if $n+2 \equiv 5 \pmod 6$). Let $t(s_i)$ be set of the triples in T' containing s_i. Then $(S = S' \backslash \{s_1, s_2\}, T = T' \backslash (t(s_1) \cup t(s_2))$ is a partial triple system with $|T| = T(n)$ and with leave containing the near 1-factor consisting of the edges in $\{\{x,y\} \mid \{s_1,x,y\} \in T'\}$ (this could also be obtained from results in [2], for example). In any case, the leave contains a near 1-factor as required. □

References

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