Existence of Frame SOLS of Type $3^n u^1$

Xu Yunqing
Department of Mathematics
Xinyang Teachers College
Xinyang, 464000, Henan, P.R. China

Lu Qinglin
Department of Mathematics
Xuzhou Teachers College
Xuzhou, 221009, Jiangsu, P.R. China

ABSTRACT. An SOLS (self orthogonal latin square) of order n with n_i missing sub-SOLS (holes) of order h_i ($1 \le i \le k$), which are disjoint and spanning (i.e. $\sum_{i=1}^{k} n_i h_i = n$), is called a frame SOLS and denoted by FSOLS($h_1^{n_1} h_2^{n_2} \dots h_k^{n_k}$). In this article, it is shown that an FSOLS($3^n u^1$) exists if and only if $n \ge 4$ and $n \ge 1 + \frac{2u}{3}$, with seventeen possible exceptions $(n,u) = (5,1)$ and $(n,u) = (n, \lfloor \frac{3(n-1)}{2} \rfloor)$ for $n \in \{6, 10, 14, 18, 22, 30, 34, 38, 42, 46, 54, 58, 62, 66, 70, 94\}$.

1 Introduction

A *self-orthogonal* latin square of order v, or SOLS(v), is a latin square of order v which is orthogonal to its transpose. It is well known [4] that an SOLS(v) exists for all values of v, $v \ne 2$, 3 or 6.

Let S be a set and $H = \{S_1, S_2, \dots S_n\}$ be a set of subsets of S. A *holey latin square* having *hole set* H is a $|S| \times |S|$ array, L, indexed by S, which satisfies the following properties:

1. every cell of L is either empty or contains a symbol of S,
2. every symbol of S occurs at most once in any row or column of L,
3. the subarrays $S_i \times S_i$ are empty for $1 \le i \le n$ (these subarrays are referred to as *holes*),

JCMCC 24 (1997), pp. 129-146

4. symbol $s \in S$ occurs in row or column t if and only if $(s,t) \in (S \times S) \setminus \cup_{i=1}^{n}(S_i \times S_i)$.

The *order* of L is $|S|$. Two holey latin squares on symbol set S and hole set H, say L_1 and L_2, are said to be orthogonal if their superposition yields every ordered pair in $(S \times S) \setminus \cup_{i=1}^{n}(S_i \times S_i)$. We shall use the notation IMOLS$(s; s_1, s_2, \dots, s_n)$ to denote a pair of orthogonal holey latin squares on symbol set S and hole set $H = \{S_1, S_2, \dots, S_n\}$, where $s = |S|$ and $s_i = |S_i|$ for $1 \le i \le n$. If $H = \emptyset$, we obtain an MOLS(s). If $H = \{S_1\}$, we simply write IMOLS(s, s_1) for the orthogonal pair of holey latin squares.

If L_1 and L_2 form an IMOLS$(s; s_1, s_2, \dots, s_n)$ such that L_2 is the transpose of L_1, then we call L_1 a *holey* SOLS, denoted by ISOLS$(s; s_1, s_2, \dots, s_n)$. If $H = \emptyset$, then a holey SOLS is an ISOLS(s, s_1).

If $H = \{S_1, S_2, \dots, S_n\}$ is a partition of S, then a holey SOLS is called a *frame* SOLS. The *type* of the frame SOLS is defined to be the multiset $\{|S_i|: 1 \le i \le n\}$. We shall use an "exponential" notation to describe types: so type $h_1^{n_1} h_2^{n_2} \dots h_k^{n_k}$ denote n_i occurrences of h_i, $1 \le i \le k$, in the multiset. We briefly denote a frame SOLS of type $h_1^{n_1} h_2^{n_2} \dots h_k^{n_k}$ FSOLS$(h_1^{n_1} h_2^{n_2} \dots h_k^{n_k})$.

We observe that the existence of an SOLS(n) is equivalent to the existence of an FSOLS(1^n), and the existence of an ISOLS(n, h) is equivalent to the existence of an FSOLS$(1^{n-h} h^1)$.

FSOLS has been very useful in recursive constructions of various combinatorial designs, such as 2 perfect m-cycle systems [16], intersection of transversal designs [7], and skew Room frames [6]. For some results on IMOLS, we refer to [1,2,8,15,17,18,21,26]. The following are known results concerning FSOLS(h^n) and FSOLS$(h^n u^1)$.

Theorem 1.1. [22] *If there exists an FSOLS$(h^n u^1)$, then $n \ge 1 + \frac{2u}{h}$.*

Theorem 1.2.

(1) [4] *There exists an FSOLS(1^n) if and only if $n \ge 4$, $n \ne 6$.*

(2) [20,22] *For $h \ge 2$, there exists an FSOLS(h^n) if and only if $n \ge 4$.*

(3) [27] *There exists an FSOLS$(1^{v-u} u^1)$ if $v \ge 3u+1$ and $(v,u) \ne (6,1)$ or $(3u+2, u)$ where $u \in \{2, 4, 6, 8, 10, 14, 16, 18, 20, 22, 26, 28, 32, 34, 46\}$.*

(4) [24] *There exists an FSOLS$(2^n u^1)$ if and only if $n \ge 4$ and $n \ge 1 + u$.*

The following is another necessary condition for the existence of FSOLS $(h^n u^1)$.

Theorem 1.3. *For $h \ne u$, if there exists an FSOLS$(h^n u^1)$ then $n \ge 4$.*

Proof: The proof of $n > 2$ is trivial, so we need only to prove that $n \ne 3$.

Assume that there exists an FSOLS(h^3u^1) and denoted by L, with row and column indices and entries taken from the set $X \cup Y \cup Z \cup T$, where $X = \{x_1, x_2, \ldots, x_h\}$, $Y = \{y_1, y_2, \ldots, y_h\}$, $Z = \{z_1, z_2, \ldots, z_h\}$, and $T = \{t_1, t_2, \ldots, t_u\}$. The entries in the first row (indexed by x_1) of L with column indices y_i $(1 \le i \le h)$ must be from $Z \cup T$. Suppose they are $t_1, t_2, \ldots, t_k, z_1, z_2, \ldots, z_{h-k}$. Suppose the entries of the first row with the column indices z_i $(1 \le i \le h)$ are $t_{k+1}, \ldots, t_u, y_1, y_2, \ldots, y_{h-(u-k)}$. We observe the self-orthogonality of L, suppose the entries in the first column with row indices y_i $(1 \le i \le h)$ are $z'_1, z'_2, \ldots, z'_k, t'_1, t'_2, \ldots, t'_{h-k}$ and the entries in the first column with row indices z_i $(1 \le i \le h)$ are $y'_1, y'_2, \ldots, y'_{u-k}, t'_{h-k+1}, \ldots, t'_{h-k+(h-u+k)}$, where $\{z'_1, \ldots, z'_k\} \subset Z$, $\{y'_1, \ldots, y'_{h-k}\} \subset Y$ and $\{t'_1, t'_2, \ldots, t'_{h-k+(h-u+k)}\} = T$. Then it must be $h - k + (h - u + k) = u$. This means that $h = u$. This is a contradiction and the proof is complete. □

In this article, we shall show that for an FSOLS($3^n u^1$) exists if and only if $n \ge 4$ and $n \ge 1 + \frac{2u}{3}$, with seventeen possible exceptions that $(n, u) = (5, 1)$ and $(n, u) = (n, \left\lfloor \frac{3(n-1)}{2} \right\rfloor)$ where $n \in \{6, 10, 14, 18, 22, 30, 34, 38, 42, 46, 54, 58, 62, 66, 70, 94\}$.

2 Constructions and transversals

Construction 2.1. (Filling in holes) [24] *Suppose there exists an FSOLS of type $\{s_i : 1 \le i \le n\}$ and for $1 \le i \le n$, $s_i = \sum_{j=1}^{t_i} s_{ij}$.*

(1) *If there exists an FSOLS of type $\{s_{nj} : 1 \le j \le t_n\}$, then there exists an FSOLS of type $\{s_i : 1 \le i \le n-1\} \cup \{s_{nj} : 1 \le j \le t_n\}$.*

(2) *Let $a \ge 0$ be an integer. For $1 \le i \le n-1$, if there exist FSOLS of type $\{a\} \cup \{s_{ij} : 1 \le j \le t_i\}$, then there is an FSOLS of type $\{a + S_n\} \cup (\cup_{i=1}^{n-1} \{S_{ij} : 1 \le j \le t_i\})$.*

Construction 2.2. (Weighting) *Suppose $(X, \mathbf{G}, \mathbf{A})$ is a GDD and let w be a map: $X \to \mathbf{Z}^+ \cup \{0\}$. Suppose there exist FSOLS of type $\{w(x) : x \in A\}$ for every $A \in \mathbf{A}$. Then there exists an FSOLS of type $\{\sum_{x \in G} : G \in \mathbf{G}\}$.*

The following recursive construction is referred to as the Inflation Construction. It essentially "blows up" every occupied cell of an FSOLS into a latin square such that if one cell is filled with a certain latin square, then its symmetric cell is filled with the transpose of an orthogonal mate of the latin square. We mention the work [5,19] which can be thought of as sources of the Inflation Construction.

Construction 2.3. (Inflation Construction) *Suppose there exists an FSOLS $(h_1^{n_1} h_2^{n_2} \ldots h_k^{n_k})$ and an MOLS(h), then there exists an FSOLS($(hh_1)^{n_1}$*

$(hh_2)^{n_2} \dots (hh_k)^{n_k})$. *In particular, the existence of FSOLS*(1^n) *and MOLS*(h) *implies the existence of an FSOLS*(h^n).

In order to get an FSOLS$(m^n u^1)$, we generalize the Inflation Construction. The following recursive constructions rely on information regarding the location of (holey) transversal in certain latin squares. Suppose L is a holey latin square on symbol set S with hole S_1. A ***holey transversal*** with hole S_1 is a set T of $|S| - |S_1|$ (occupied) cells in L such that every symbol of $S \setminus S_1$ is contained in exactly one cell of T and the $|S| - |S_1|$ cells in T intersect each row and each column indexed by $S \setminus S_1$ in exactly one cell. $|S|$ is called the *size* of the hole of the holey transversal. A hole transversal T is *symmetric* if $(i, j) \in T$ implies $(j, i) \in T$. Two holey transversals T_1 and T_2 with the same hole are called a *symmetric pair of holey transversals* if $(i, j) \in T_1$ if and only if $(j, i) \in T_2$. If $S_1 = \emptyset$, then we call the holey transversal a (*complete*) transversal. m holey transversals are said to be *disjoint* if they have no cell in common.

The following construction is a generalization of [27, Construction 3.3].

Construction 2.4. *Suppose there is an FSOLS*(t^n) *which has* $p + 2q$ *disjoint transversals,* p *of them being symmetric and the rest being* q *symmetric pairs. For* $1 \le i \le p$ *and* $1 \le j \le q$*, let* $v_i \ge 0$ *and* $w_j \ge 0$ *be integers. Let* h *be a positive integer, where* $h \ne 2$ *or 6 if* $p + 2q < t(n-1)$. *Suppose there exist IMOLS*$(h + v_i, v_i)$ *for* $1 \le i \le p$ *and IMOLS*$(h + w_j, w_j)$ *for* $1 \le j \le q$. *Then there exists an FSOLS*$((ht)^n (v + 2w)^1)$, *where* $v = \sum v_i$ *and* $w = \sum w_j$.

Construction 2.5. *Suppose there is an FSOLS*(t^n) *which has* $p + 2q$ *disjoint transversals,* p *of them being symmetric and the rest being* q *symmetric pairs. For* $1 \le i \le p$ *and* $1 \le j \le q$*, let* $v_i \ge 0$ *and* $w_j \ge 0$ *be integers. Let* s *and* h *be positive integers, where* $sh \ne 2$ *or 6 if* $p + 2q < t(n-1)$. *Suppose there exist FMOLS*$(s^h v_i^1)$ *for* $1 \le i \le p$, *FSOLS*$(s^h w_j^1)$ *for* $1 \le j \le q$ *and FSOLS*$(s^{tn} k^1)$. *Then there exists an FSOLS*$((tsh)^n u^1)$, *where* $u = k + \sum v_i + 2 \sum w_j$.

Proof: Applying Inflation Construction, take FSOLS(t^n) as initial square, filling every occupied cell into the given FSOLS$(s^h w_j^1)$ we obtain an FSOLS $((tsh)^n (v + 2w)^1)$ with h sub-SOLS of order nts missing, where $v = \sum v_i$, $w = \sum w_j$. Filling the holes of order nts with the given FSOLS$(s^{tn} k^1)$ we obtain an FSOLS$((tsh)^n u^1)$ as desired. □

The following is a modification of Construction 2.4, in which holey transversals are used.

Construction 2.6. [27, Construction 3.4] *Suppose there is an FSOLS*$(t^g h^1)$, *where* H *is the size* h *hole, having* $k+2p$ *disjoint holey transversals with hole* H *such that* k *of them are symmetric and the rest form* p *symmetric pairs. For* $1 \le i \le k$ *and* $1 \le j \le p$*, let* v_i *and* w_j *be non-negative integers. Let* m *be a positive integer,* $m \ne 2$ *or 6, and suppose there exist IMOLS*$(m+v_i, v_i)$

for $1 \le i \le k$ *and IMOLS*$(m+w_j, w_j)$ *for* $1 \le j \le p$. *Then there is an FSOLS*$((mt)^g(mh+u+2w)^1)$, *where* $v = \sum v_i$ *and* $w = \sum w_j$.

To apply these recursive constructions we need some "ingredients" provided in the following theorems.

Theorem 2.7. [3] *There exists an MOLS*(v) *for any positive integer* v, $v \neq 2, 6$.

Theorem 2.8. [13] *There exist IMOLS*(v,n) *for all values of* v *and* n *satisfying* $v \ge 3n$ *except that IMOLS*$(6,1)$ *does not exist.*

Theorem 2.9. [27] *There exist IMOLS*(v,n) *if* $v \ge 3n+1$ *and* $(v,n) \neq (6,1)$, $(v,n) \neq (3n+2, n)$, *where* $n \in \{2,4,6,8,10,14,16,18,20,22,26,28,32,34,46\}$.

Theorem 2.10. [11] *If* $n \ge 5$ *is an odd prime power, then there exists an FSOLS*(1^n) *with* $n-1$ *disjoint transversals occurring as* $\frac{n-1}{2}$ *pairs of symmetric transversals.*

Theorem 2.11. [9] *For all even* n, $n \notin \{2,6,10,14,46,54,58,62,66,70\}$, *there exists an FSOLS*(1^n) *with* $n-1$ *disjoint symmetric transversals.*

A *transversal design* TD(k,n) is a GDD with kn points, k groups of size n, and n^2 blocks of size k. It is well known that a TD(k,n) is equivalent to $k-2$ MOLS (mutually orthogonal latin squares) of order n and that for any prime power p, there exist $p-1$ MOLS of order p. So we have the following lemma.

Lemma 2.12. *For any prime power* p, *there exists a TD*(k,p), *where* $3 \le k \le p+1$.

Theorem 2.13. [11, Theorem 2.5] *If* n *is a prime power,* $n \ge 7$, *then there exist FSOLS*(1^n) *with a pair of symmetric holey transversals with a hole of size one.*

Theorem 2.14. *Suppose* $n \ge 5$ *is an odd prime, then there exists an FSOLS*(1^n) *with one symmetric holey transversal and* $\frac{n-5}{2}$ *symmetric pairs of holey transversals with a hole of size one, all these holey transversals are disjoint.*

Proof: From [12, Lemma 1.4] we know that there exist $n-1$ MOLS(n) and occurring as $\frac{n-3}{2}$ pairs of squares, each pair being mutually transposes, plus a symmetric square and a square with constant on the main diagonal. Then the first pair gives an ISOLS$(n,1)$ and each of the remaining determines a symmetric pair of holey transversals in the ISOLS$(n,1)$. Each of the extra squares determines a holey symmetric transversal, and the holey symmetric transversal determined by the last square is on the main diagonal. This means that there exists an FSOLS(1^n) as has been desired. □

Lemma 2.15. *Suppose there is a TD$(s+2,t)$, where $s \geq 4$. If there exist FSOLS of type $h^s k_i^1$, $h^{s+1} k_i^1$ $(1 \leq i \leq t)$, $h^m r^1$ and $h^t r^1$, then there exist FSOLS$(h^{st+m} u^1)$ for $u = r + \sum_{i=1}^{t} k_i$.*

Proof: Applying Construction 2.2, in the given TD$(s+2,t)$, give weight h to each point of the first s groups. For the $(s+1)$th group, give weight h to m point and weight zero to the rest points. For the last group, give weight k_i to the ith point $(1 \leq i \leq t)$. Using the input FSOLSs, we obtain an FSOLS of type $(ht)^s (hm)^1 (\sum k_i)^1$. Filling the holes of size ht and hm with FSOLS$(h^t r^1)$ and FSOLS$(h^m r^1)$ we obtain the desired FSOLS. □

All the construction given above are recursive constructions. In the following we give a direct construction. It is a modification of the starter-adder type constructions. The idea has been described by several authors including Horton [14], Hedayat and Seiden [10], Zhu [25], Heinrich and Zhu [12] and Xu and Zhu [24].

Construction 2.16. *Let* e$=(\emptyset, a_{01}, a_{02}, \ldots, a_{0(n-1)}, \emptyset, a_{0(n+1)}, \ldots, a_{0(2n-1)}, \ldots, \emptyset, a_{0(hn-n+1)}, \ldots, a_{0(hn_1)})$ *be a vector of length hn with entries in $(\mathbf{Z}_{hn} \setminus \{0, n, \ldots, (h-1)n\}) \cup X$, where $X = \{x_1, x_2, \ldots, x_u\}$, "$\emptyset$" means that the cell it occupies is empty. Let* f $= (a_{0x_1}, a_{0x_2}, \ldots, a_{0x_u})$ *and* g $= (a_{x_1 0}, a_{x_2 0}, \ldots, a_{x_u 0})$ *be vectors of length u with entries in $\mathbf{Z}_{hn} \setminus \{0, n, \ldots, (h-1)n\}$. These vectors are use to construct an array $A = (a_{ij})$ of order $hn+u$ with n empty subarrays of order h and one empty subarray of order u having row and column indices and entries in $\mathbf{Z}_{hn} \cup X$. The array is constructed as follows, where all the elements including indices are calculated modulo hn and x_i acts as "infinite" elements.*

(1) If $a_{ij} = \emptyset$, $0 \leq i, j \leq hn-1$, then $a_{(i+1)(j+1)} = \emptyset$.

(2) If $a_{ij} \in \mathbf{Z}_{hn}$, $0 \leq i, j \leq hn-1$, then $a_{(i+1)(j+1)} = a_{ij} + 1$.

(3) If $a_{ij} \in X$, $0 \leq i \leq hn-1$, then $a_{(i+1)(j+1)} = a_{i,j}$

(4) If $0 \leq i \leq hn-1$, and $j \in X$, then $a_{(i+1)j} = a_{ij} + 1$.

(5) If $0 \leq j \leq hn-1$, and $i \in X$, then $a_{i(j+1)} = a_{ij} + 1$.

Conditions can be described for the vectors e, f and g so that the array as constructed is an FSOLS$(h^n u^1)$. However, we shall simply give the vectors and the reader can check for himself that they do yield the desired FSOLS$(h^n u^1)$.

Example 2.17. [23] Let e $= (\emptyset, 2, x, 1, \emptyset, 7, 3, 6)$, f $= (5)$, g $= (3)$, these vectors generate an FSOLS$(2^4 1^1)$ shown in Figure 1, where $X = \{x\}$.

From Figure 1 we can easily see that there are two pairs of symmetric holey transversals with a hole of size one in the FSOLS$(2^4 1^1)$. It is clear that there are there are $\frac{h(n-1)}{2} - b$ pairs of symmetric holey transversals with a hole of size b in an FSOLS($h^n b^1$) generated by Construction 2.16 if $h(n-1)$ is even; if $h(n-1)$ is odd, there will be $\frac{h(n-2)}{2} - b$ pairs of symmetric holey transversals and one holey symmetric transversal with a hole of size b in an FSOLS($h^n b^1$) generated by Construction 2.16. We list in Table 2.2 some FSOLS generated by Construction 2.16, we only give the type and the vectors e, f and g. So from [24, Lemma 3.17], [23], [12] and Table 2.2 we have the following lemma which is very useful in constructing FSOLS($3^n u^1$).

	2	x	1		7	3	6	5
7		3	x	2		0	4	6
5	0		4	x	3		1	7
2	6	1		5	x	4		0
	3	7	2		6	x	5	1
6		4	0	3		7	x	2
x	7		5	1	4		0	3
1	x	0		6	2	5		4
3	4	5	6	7	0	1	2	

Figure 1. an FSOLS($2^4 1^1$)

Lemma 2.18. *There exist FSOLS($h^n b^1$) with $h(n-1) - 2b$ disjoint holey transversals with a hole of size b occurring as $\frac{h(n-1)}{2} - b$ of symmetric pairs when $h(n-1)$ is even and with $h(n-2) - 2b + 1$ disjoint holey transversals with a hole of size b occurring as a symmetric one and $\frac{h(n-2)}{2} - b$ of symmetric pairs when $h(n-1)$ is odd for the parameters shown in Table 2.1.*

h	n	b
2	6	0,1,3,4
2	7	1
2	11	1
1	10	1,3
1	13	3
1	14	2,5
1	15	0,2,6
1	17	1,3
1	19	1,3
1	46	17,21
1	58	23,27
1	62	26,29

Table 2.1. FSOLS($h^n b^1$) have holey transversals

2^6: $\mathbf{e} = (\emptyset, 10, 3, 1, 9, 7, \emptyset, 11, 4, 8, 5, 2), \mathbf{f} = \mathbf{g} = \emptyset.$

$1^{17}1^1$: $\mathbf{e} = (\emptyset, 2, 11, 6, 14, 13, 1, 9, 4, 7, 15, 10, 16, 3, 8, 12, x), \mathbf{f} = (5), \mathbf{g} = (6).$

1^{15}: $\mathbf{e} = (\emptyset, 14, 13, 11, 10, 8, 7, 2, 12, 6, 9, 5, 4, 3, 1), \mathbf{f} = \mathbf{g} = \emptyset.$

$1^{19}1^1$: $\mathbf{e} = (\emptyset, x, 18, 17, 16, 15, 14, 6, 9, 13, 12, 7, 4, 11, 8, 3, 2, 1, 5), \mathbf{f} = (10),$
$\mathbf{g} = (9).$

$1^{46}17^1$: $\mathbf{e} = (\emptyset, 13, 9, 43, 42, 19, 27, 23, 37, x_1, 7, x_2, 34, 32x_3, 30, 12, x_4, 28, x_5,$
$x_6, x_7, 25, 14, x_8, 24, 44, 40, x_9, 17, x_{10}, x_{11}, 22, x_{12}, x_{13}, 36, x_{14}, 39,$
$x_{15}, x_{16}, x_{17}, 6, 4, 3, 2, 26),$
$\mathbf{f} = (31, 11, 38, 35, 29, 21, 10, 16, 8, 1, 33, 41, 15, 45, 5, 18, 20),$
$\mathbf{g} = (30, 9, 35, 31, 23, 28, 20, 5, 41, 32, 17, 24, 33, 26, 25, 39, 44).$

$1^{46}21^1$: $\mathbf{e} = (\emptyset, 13, 7, x_1, x_2, 19, 27, 23, 37, x_3, 7, x_4, 34, 32, x_5, 30, 12, x_6, 28, x_7$
$x_8, x_9, 25, 14, x_{10}, 24, 44, 40, x_{11}, 17, x_{12}, x_{13}, 22, x_{14}, x_{15}, 36, x_{16}, 39,$
$x_{17}, x_{18}, x_{19}, x_{20}, 4, 3, x_{21}, 26),$
$\mathbf{f} = (31, 11, 38, 35, 43, 29, 21, 42, 2, 10, 16, 6, 8, 1, 33, 41, 15, 45, 5, 18, 20),$
$\mathbf{g} = (30, 9, 35, 31, 38, 23, 28, 4, 11, 20, 5, 40, 41, 32, 17, 24, 33, 26, 25, 39, 44).$

$1^{58}23^1$: $\mathbf{e} = (\emptyset, 57, 56, 55, 54, 53, x_1, 51, x_2, x_3, 48, 47, 45, 34, 43, 38, 42, 41, 35, x_4,$
$x_5, 37, x_6, 33, x_7, 32, 31, 49, x_8, 13, 7, x_9, x_{10}, x_{11}, 21, x_{12}, 50, x_{13}, 14, 20,$
$x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, 40, 19, x_{23}, 9, 6, 4, 3, 2, 1),$
$\mathbf{f} = (27, 10, 16, 52, 18, 5, 28, 26, 44, 46, 24, 23, 11, 22, 39, 25, 8, 36, 29, 30,$
$15, 12, 17),$
$\mathbf{g} = (28, 12, 20, 57, 25, 13, 18, 37, 31, 32, 9, 40, 51, 3, 19, 46, 30, 1, 53, 55,$
$41, 43, 47).$

$1^{58}17^1$: $\mathbf{e} = (\emptyset, 57, 56, 55, x_1, x_2, x_3, 51, x_4, x_5, 48, 47, 45, 34, 43, 38, 42, 41, 35, x_6,$
$x_7, 37, x_8, 33, x_9, 32, 31, 49, x_{10}, 13, 7, x_{11}, x_{12}, x_{13}, 21, x_{14}, 50, x_{15}, 14, 20,$
$x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{24}, 40, 19, x_{25}, 9, 1, 4, x_{26}, x_{27}, 10),$
$\mathbf{f} = (27, 2, 53, 16, 52, 54, 18, 5, 3, 28, 26, 44, 46, 24, 6, 23, 11, 22, 39, 25, 8,$
$36, 29, 30, 15, 12, 17),$
$\mathbf{g} = (28, 4, 50, 20, 57, 2, 25, 13, 12, 18, 37, 31, 32, 9, 48, 40, 51, 3, 19, 46, 30,$
$1, 53, 55, 41, 43, 47).$

$1^{62}26^1$: $\mathbf{e} = (\emptyset, 61, 60, 59, 58, 57, x_1, 55, 54, 53, 52, 51, 50, 49, 48, x_2, 46, 45, 44, 43,$
$42, 41, 40, 33, 38, 37, 36, 35, 33, x_3, 29, 34, x_4, 4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11},$
$x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, 16, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, 47,$
$x_{26}, 5, 3, 2, 1),$
$\mathbf{f} = (6, 9, 11, 8, 10, 13, 15, 12, 7, 18, 14, 21, 22, 17, 24, 30, 20, 27, 32, 23, 26,$
$19, 28, 31, 56, 25),$
$\mathbf{g} = (7, 11, 15, 13, 17, 21, 25, 23, 19, 32, 29, 37, 39, 35, 43, 50, 41, 49, 55, 47,$
$51, 45, 1, 59, 27, 57).$

$1^{62}29^1$: $\mathbf{e} = (\emptyset, 61, 60, 59, 58, 57, x_1, 55, 54, 53, 52, 51, 50, 49, 48, 22, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, x_3, x_4, 29, 34, x_5, 2, 9, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, 10, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}, x_{24}, x_{25}, x_{26}, 47, x_{26}, x_{27}, x_{28}, x_{29}, 8)$,

$\mathbf{f} = (3, 7, 18, 11, 1, 5, 6, 15, 13, 12, 21, 4, 33, 14, 16, 22, 17, 24, 30, 20, 27, 32, 23, 26, 19, 28, 31, 56, 25)$,

$\mathbf{g} = (2, 5, 21, 15, 6, 11, 13, 7, 4, 23, 33, 17, 19, 29, 32, 39, 35, 43, 50, 41, 49, 55, 47, 51, 45, 1, 59, 27, 57)$.

Table 2.2. Some FSOLSs constructed by Construction 2.16

3 Existence of FSOLS($3^n u^1$) for n is even

Let $E = \{2, 6, 10, 14, 46, 54, 58, 62, 66, 70\}$, $F = \{6, 10, 14, 18, 22, 30, 34, 38, 42, 46, 54, 58, 66, 70, 94\}$ and $G = E \cup F = \{2, 6, 10, 14, 18, 22, 30, 34, 38, 42, 46, 54, 58, 62, 66, 70, 94\}$.

Lemma 3.1. *If n is even and $n \notin G$, then there exist FSOLS($3^n u^1$) for $0 \le u \le \frac{3(n-1)}{2}$.*

Proof: From Theorem 2.11 we know that there is an FSOLS(1^n) with $n-1$ disjoint symmetric transversals. Applying Construction 2.4 with $t = 1$, $p = n-1$, $q = 0$, $h = 3$, $v_i = 0$, or 1, we obtain an FSOLS($3^n u^1$), where $0 \le u \le n-1$.

Applying Construction 2.5 with $t = 1$, $p = n-1$, $q = 0$, $s = 1$, $h = 3$, $v_i = 1$, $0 \le k \le \frac{n}{2} - 1$, the input designs are from Theorem 1.2, we obtain an FSOLS($3^n u^1$) for $n - 1 \le u \le \frac{3(n-1)}{2}$. □

Lemma 3.2. *There exist FSOLS($3^6 u^1$) for $1 \le u \le 6$.*

Proof: Applying Theorem 2.14 and Construction 2.6 with $n = 7$, $g = 6$, $t = h = 1$, $k = p = 1$, $m = 3$ and $0 \le v_i, w_j \le 1$, we obtain an FSOLS($3^6 u^1$) for $3 \le u \le 6$.

FSOLS($3^6 2^1$) can be constructed by vectors $\mathbf{e} = (\emptyset, 17, 15, 14, 13, 9, \emptyset, 10, x_1, 5, 11, x_2, \emptyset, 2, 4, 7, 3, 1)$, $\mathbf{f} = (6, 8)$, $\mathbf{g} = (17, 15)$, and FSOLS($3^6 1^1$) by $\mathbf{e} = (\emptyset, x_1, 17, 16, 8, 13, \emptyset, 14, 10, 7, 3, 2, \emptyset, 5, 15, 11, 1, 4)$, $\mathbf{f} = (9)$, $\mathbf{g} = (17)$. □

Lemma 3.3. *There exist FSOLS($3^{10} u^1$) for $1 \le u \le 12$.*

Proof: From Lemma 2.18 we know that there exists an FSOLS($1^{10} b^1$) with $10 - 2b - 1$ disjoint holey transversals with a hole of size b and occurring as a symmetric one and $4 - b$ symmetric pairs for $b = 1$ or 3. Applying Construction 2.6 with $t = 1$, $g = 10$, $h = 1$, $k = 1$, $p = 3$, $m = 3$, $0 \le v_i, w_j \le 1$ we obtain an FSOLS($3^{10} u^1$) for $3 \le u \le 10$; with

$t=1$, $g=10$, $h=3$, $k=1$, $p=1$, $m=3$, $0 \le v_i, w_j \le 1$ we obtain an FSOLS($3^{10}u^1$) for $9 \le u \le 12$. FSOLS($3^{10}1^1$) can be constructed by vectors $\mathbf{e} = (\emptyset, 29, 28, 27, 26, 24, 23, 19, x_1, 22, \emptyset, 18, 17, 8, 13, 16, 25, 14, 21, 5, \emptyset, 12, 15, 7, 9, 6, 4, 3, 2, 1)$, $\mathbf{f} = (11)$, $\mathbf{g} = (18)$, and FSOLS($3^{10}2^1$) by $\mathbf{e} = (\emptyset, 29, 28, 27, 26, 24, 23, 25, 17, 22, \emptyset, 18, 11, x_1, x_2, 16, 21, 12, 15, 5, \emptyset, 14, 13, 7, 9, 6, 4, 3, 2, 1)$, $\mathbf{f} = (8, 19)$, $\mathbf{g} = (3, 12)$. □

Lemma 3.4. *There exist FSOLS($3^{14}u^1$) for $1 \le u \le 18$.*

Proof: From Lemma 2.18 we know that there exist FSOLS($1^{14}b^1$) with $14 - 2b - 1$ disjoint holey transversals with a hole of size b and occurring as a symmetric one and $6 - b$ symmetric pairs for $b = 2$ and 5. Applying Construction 2.6, the proof is the same as that of Lemma 3.3, we obtain an FSOLS($3^{14}u^1$) for $6 \le u \le 18$.

FSOLS($3^{14}u^1$) for $u = 1, 2, 4, 5$ can be constructed by Construction 2.16:

$3^{14}1^1$: $\mathbf{e} = (\emptyset, 41, 40, 39, 38, 37, 36, 34, 33, 35, 29, 27, 23, 31, \emptyset, 18, x, 30, 17, 26, 25, 22, 15, 32, 21, 20, 13, 16, \emptyset, 7, 10, 12, 11, 24, 9, 8, 6, 5, 4, 3, 2)$, $\mathbf{f} = (19), \mathbf{g} = (24)$.

$3^{14}2^1$: $\mathbf{e} = (\emptyset, 41, 40, 39, 38, 37, 36, 34, 33, 35, 31, 29, 32, \emptyset, 26, 25, 30, 23, 12, 21, 18, x_1, x_2, 27, 20, 7, 16, \emptyset, 11, 17, 22, 10, 13, 9, 8, 6, 5, 4, 3, 2, 1)$, $\mathbf{f} = (15, 24), \mathbf{g} = (16, 41)$.

$3^{14}4^1$: $\mathbf{e} = (0, 41, 40, 39, 38, 37, 36, 34, 33, 35, 31, 30, 25, 29, 0, 22, 27, 26, 7, 18, 23, 16, x_1, x_2, x_3, x_4, 19, 24, \emptyset, 11, 17, 12, 10, 13, 9, 8, 6, 5, 4, 3, 2, 1)$, $\mathbf{f} = (32, 20, 15, 21), \mathbf{g} = (33, 18, 5, 1)$.

$3^{14}5^1$: $\mathbf{e} = (\emptyset, 41, 40, 39, 38, 37, 36, 34, 33, 35, 31, 30, 25, 29, \emptyset, 24, 23, 16, 21, 18, 27, 22, x_1, x_2, x_3, x_4, x_5, 26, \emptyset, 11, 17, 12, 10, 13, 9, 8, 6, 5, 4, 3, 2, 1)$, $\mathbf{f} = (32, 20, 19, 7, 15), \mathbf{g} = (31, 18, 11, 39, 35)$.

FSOLS($3^{14}3^1$) can be get from Theorem 1.2. □

Lemma 3.5. *There exist FSOLS($3^n u^1$) for $n \in \{18, 22, 30, 34, 38, 42, 94\}$ and $0 \le u \le \frac{3(n-1)}{2}$.*

Proof: From Theorem 2.11 we know that there is an FSOLS(1^n) with $n-1$ disjoint symmetric transversals. Applying Construction 2.4 with $t=1$, $p=n-1$, $q=0$, $h=3$, $v_i = 0$ or 1, we obtain an FSOLS($3^n u^1$), where $0 \le u \le n-1$.

Applying Construction 2.5 with $t=1$, $p=n-1$, $q=0$, $s=1$, $h=3$, $v_i = 1$, $0 \le k \le \frac{n}{2} - 2$, the input designs are from Theorem 1.2(3), we obtain an FSOLS($3^n u^1$) for $n-1 \le u \le \frac{3(n-1)}{2}$. □

Lemma 3.6. *There exist FSOLS($3^{46}u^1$) for $2 \le u \le 66$.*

Proof: In a TD(7,9), give weight three to each point of the first five groups. In the sixth group, give one point weight three and other points weight zero. In the last group, give the ith point weight w_i $(0 \le w_i \le 6, w_i \ne 1)$ $(1 \le i \le 9)$. Applying Construction 2.2 we obtain an FSOLS$(27^5 3^1 u^1)$ for $2 \le u \le 54$. Filling the holes of size 27 with FSOLS(3^9) we obtain an FSOLS$(3^{46} u^1)$ for $2 \le u \le 54$.

From Lemma 2.18 we know that there exists an FSOLS$(1^{46} b^1)$ with $45 - 2b$ disjoint holey transversals with a hole of size b and occurring as a symmetric one and $22 - b$ symmetric pairs for $b = 17$ and 21. Applying Construction 2.6 with $t = 1$, $g = 46$, $h = 17$, $k = 1$, $p = 22 - 17$, $m = 3$, $0 \le v_i, w_j \le 1$ we obtain FSOLS$(3^{46} u^1)$ for $54 \le u \le 65$; with $t = 1$, $g = 46$, $h = 21$, $k = 1$, $p = 1$, $m = 3$, $v_1 = w_1 = 1$ we obtain an FSOLS$(3^{46} 66^1)$. □

Lemma 3.7. *There exists an FSOLS*$(3^{54} u^1)$ *for* $2 \le u \le 78$.

Proof: From Theorem 2.10 we know that there is an FSOLS(1^9) with eight disjoint transversals occurring as four symmetric pairs. Applying Construction 2.4 with $t = 1$, $n = 9$, $p = 0$, $q = 4$, $h = 18$, $2 \le w_j \le 9$, the input designs, IMOLS$(h + w_j, w_j)$, are from Theorem 2.8, we obtain an FSOLS$(18^9 v^1)$ for $0 \le v \le 72$, where v is even. Filling the holes of size 18 with FSOLS$(3^6 k^1)$ $(2 \le k \le 6)$ we obtain an FSOLS$(3^{54} u^1)$ for $2 \le u \le 78$. □

Lemma 3.8. *There exists an FSOLS*$(3^{58} u^1)$ *for* $2 \le u \le 84$.

Proof: In a TD(9,11), give weight three to each point of the first five groups. Give weight three to one point and weight zero to other points in the sixth, seventh and eighth group. In the last group, give the ith point weight w_i $(0 \le w_i \le 6, w_i \ne 1)$ $(1 \le i \le 11)$. Applying Construction 2.2 we obtain an FSOLS$(33^5 3^1 u^1)$ for $2 \le u \le 66$. Filling the holes of size 33 with FSOLS$(3^1 1)$ we obtain an FSOLS$(3^{58} u^1)$ for $2 \le u \le 66$.

In a TD(7,11), give weight three to each point of the first five groups. In the sixth group, give three points weight three and other points weight zero. In the last group, give the ith point weight w_i $(0 \le w_i \le 6, w_i \ne 1)$ $(1 \le i \le 11)$, then we obtain an FSOLS$(33^5 9^1 v^1)$ for $2 \le v \le 66$. Filling the holes of size 33 with FSOLS(3^{12}) and the hole of size 9 with FSOLS(3^4) we obtain an FSOLS$(3^{58} u^1)$ for $5 \le u \le 69$.

From Lemma 2.18 we know that there exists an FSOLS$(1^{58} b^1)$ with $57 - 2b$ disjoint holey transversals with a hole of size b and occurring as a symmetric one and $28 - b$ symmetric pairs for $b = 23$ and 27. Applying Construction 2.6 with $t = 1$, $g = 58$, $h = b$, $k = 1$, $p = 28 - b$, $m = 3$, $0 \le v_i, w_j \le 1$. With $b = 23$ we obtain FSOLS$(3^{58} u^1)$ for $69 \le u \le 80$; with $b = 27$ we obtain an FSOLS$(3^{58} u^1)$ for $81 \le u \le 84$. □

Lemma 3.9. *There exist FSOLS*$(3^{62} u^1)$ *for* $2 \le u \le 90$.

Proof: In a TD(9,8), give weight three to each point of the first seven groups. In the eighth group, give weight three to six points and weight zero to other points. In the last group, give the ith point weight w_i $(0 \le w_i \le 9, w_i \ne 1)$ $(1 \le i \le 8)$. Applying Construction 2.2 we obtain an FSOLS($24^7 18^1 v^1$) for $2 \le u \le 72$. Filling the holes of size 24 with FSOLS($3^8 k^1$) and the holes of size 18 with FSOLS($3^6 k^1$) $(0 \le k \le 6, k \ne 1)$ we obtain an FSOLS($3^{62} u^1$) for $2 \le u \le 78$.

From Lemma 2.18 we know that there exists an FSOLS($1^{62} b^1$) with $61 - 2b$ disjoint holey transversals with a hole of size b and occurring as a symmetric one and $30 - b$ symmetric pairs for $b = 26$ and 29. Applying Construction 2.6 with $t = 1$, $g = 62$, $h = b$, $k = 1$, $p = 30 - b$, $m = 3$, $0 \le v_i, w_j \le 1$. With $b = 26$ we obtain FSOLS($3^{62} u^1$) for $78 \le u \le 87$; with $b = 29$ we obtain an FSOLS($3^{62} u^1$) for $87 \le u \le 90$. □

Lemma 3.10. *There exist an FSOLS($3^{66} u^1$) for $2 \le u \le 96$.*

Proof: From Theorem 2.10 we know that there exists an FSOLS(1^{11}) with ten disjoint transversals occurring as a five symmetric pairs. Applying Construction 2.4 with $t = 1$, $n = 11$, $p = 0$, $q = 5$, $h = 18$, $0 \le w_j \le 9$. We obtain an FSOLS($18^{11} v^1$) for $0 \le u \le 90$, where v is even. Filling the holes of size 18 with FSOLS($3^6 k^1$) $(2 \le k \le 6)$ we obtain an FSOLS($3^{66} u^1$) for $2 \le u \le 96$. □

Lemma 3.11. *There exists an FSOLS($3^{70} u^1$) for $2 \le u \le 102$.*

Proof: Using Theorem 2.10 and Construction 2.4 with $t = 1$; $n = 7$, $p = 0$, $q = 3$, $h = 30$, $0 \le w_j \le 15$. we obtain an FSOLS($30^7 v^1$) for $0 \le v \le 90$, where v is even. Filling the holes of size 30 with FSOLS($3^{10} k^1$) $(2 \le k \le 12)$ we obtain an FSOLS($3^{70} u^1$) for $2 \le u \le 102$. □

From Lemmas 3.1–3.11 we get the following theorem.

Theorem 3.12. *There exists an FSOLS($3^n u^1$) for even n and $2 \le u \le \frac{3(n-1)}{2}$, with possible exceptions that $u = \left\lfloor \frac{3(n-1)}{2} \right\rfloor$ for $n \in \{6, 10, 14, 18, 22, 30, 34, 38, 42, 46, 54, 58, 62, 66, 70, 94\}$.*

4 Existence of FSOLS($3^n u^1$) for n is odd

Lemma 4.1. *If n is an odd prime power and $n \ge 7$, then there exist FSOLS($3^n u^1$) for $n - 1 \le u \le \frac{3(n-1)}{2}$.*

Proof: Applying Construction 2.5 with $t = 1$, $p = 0$, $q = \frac{n-1}{2}$, $s = 1$, $h = 3$, $w_j = 1$, $0 \le k \le \frac{n-1}{2}$, the input designs are from Theorems 1.2 and 2.8, we obtain an FSOLS($3^n u^1$) for $n - 1 \le u \le \frac{3(n-1)}{2}$. □

Lemma 4.2. *There exist FSOLS($3^5 u^1$) for $2 \le u \le 6$.*

Proof: Applying Theorem 2.10 and Construction 2.4 with $t = 1$, $n = 5$, $p = 0$, $q = 2$, $h = 3$, $0 \le w_j \le 1$, we obtain FSOLS(3^5u^1) for $u = 2$ and 4.

Applying Construction 2.5 with $t = 1$, $n = 5$, $p = 0$, $q = 2$, $s = 1$, $h = 3$, $w_j = 1$, and $k = 2$, the input designs are from Theorems 1.2 and 2.8, we obtain an FSOLS(3^56^1).

FSOLS(3^55^1) can be constructed by Construction 2.16 with vectors $\mathbf{e} = (\emptyset, 13, x_1, 7, x_2, \emptyset, x_3, x_4, 9, 11, \emptyset, 3, x_5, 4, 8)$, $\mathbf{f} = (12, 1, 14, 2, 6)$, and $\mathbf{g} = (13, 3, 11, 8, 14)$.

FSOLS(3^6) is from Theorem 1.2(2). □

Lemma 4.3. *There exist FSOLS(3^nu^1) for $n = 7, 9, 11, 13$ and $2 \le u \le n-1$.*

Proof: Applying Theorem 2.10 and Construction 2.4 with $t = 1$, $p = 0$, $q = \frac{n-1}{2}$, $h = 3$, $0 \le w_j \le 1$, we obtain FSOLS(3^nu^1) for $n = 7, 9, 11, 13$ and $2 \le u \le n-1$, where u is even.

Applying Theorem 2.13 and Construction 2.6 with $n = 8$, $t = h = 1$, $g = 7$, $k = 0$, $p = 1$, $m = 3$, $w_j = 1$, we obtain an FSOLS(3^75^1).

FSOLS(3^nu^1) for $n = 9, 11, 13$ and $u = 5, 7$ can be constructed by Construction 2.16 with vectors e, f and g as follows:

3^95^1: $\mathbf{e} = (\emptyset, 26, 25, 24, 23, 22, 21, 20, x_1, \emptyset, 13, x_2, x_3, x_4, 19, 4, 17, x_5, \emptyset, 16, 15, 8, 7, 6, 5, 2, 1)$,
$\mathbf{f} = (10, 3, 11, 12, 14), \mathbf{g} = (11, 7, 6, 20, 26)$.

3^97^1: $\mathbf{e} = (\emptyset, 26, 25, 24, 23, 22, 20, 12, 21, \emptyset, x_1, 17, x_2, x_3, 11, 10, x_4, 16, \emptyset, x_5, x_6, x_7, 7, 6, 5, 2, 1)$,
$\mathbf{f} = (4, 14, 3, 15, 19, 13, 8), \mathbf{g} = (3, 16, 7, 20, 11, 1, 15)$.

$3^{11}5^1$: $\mathbf{e} = (\emptyset, 21, 31, 30, 29, 28, 27, 26, 15, 24, 7, \emptyset, x_1, 16, x_2, x_3, x_4, 12, 23, x_5, 19, \emptyset, 14, 17, 10, 9, 8, 5, 4, 3, 2, 1)$,
$\mathbf{f} = (18, 13, 6, 25, 20), \mathbf{g} = (17, 9, 1, 32, 12)$.

$3^{11}7^1$: $\mathbf{e} = (\emptyset, 21, 31, 30, 29, 28, 27, 26, x_1, x_2, \emptyset, 24, x_3, 13, 19, x_4, x_5, 32, x_6, x_7, \emptyset, 18, 6, 10, 9, 8, 5, 4, 3, 2, 1)$,
$\mathbf{f} = (25, 7, 20, 12, 23, 17, 16), \mathbf{g} = (26, 5, 24, 17, 30, 9, 32)$.

$3^{13}5^1$: $\mathbf{e} = (\emptyset, 38, 37, 36, 35, 19, 33, 32, 31, 10, 29, 28, 15, \emptyset, 34, 12, x_1, 24, x_2, x_3, 25, 11, x_4, 27, x_5, 16, \emptyset, 22, 7, 18, 23, 30, 9, 6, 5, 17, 21, 20, 1)$,
$\mathbf{f} = (8, 14, 2, 3, 4), \mathbf{g} = (6, 9, 8, 11, 15)$.

$3^{13}7^1$: $\mathbf{e} = (\emptyset, 38, 37, 36, 35, 34, 33, 32, 31, 30, 29, 28, 23, \emptyset, x_1, 22, x_2, 14, 11, x_3, 21, x_4, x_5, 27, x_6, 24, \emptyset, x_7, 19, 18, 25, 12, 9, 6, 5, 4, 3, 16, 1)$,
$\mathbf{f} = (17, 8, 2, 7, 15, 10, 20), \mathbf{g} = (22, 14, 9, 15, 5, 24, 3)$.

FSOLS(3^n9^1) for $n = 11$ and 13 can be generated by Construction 2.3 with FSOLS(1^n3^1) and MOLS(3).

From Lemma 2.18 we know that there exists an FSOLS($1^{13}3^1$) with six disjoint holey transversals with a hole of size three and occurring as three symmetric pairs. Applying Construction 2.6 with $t = 1$, $g = 13$, $h = 3$, $k = 0$, $p = 3$, $m = 3$, $0 \le w_j \le 1$, we obtain an FSOLS($3^{13}11^1$). FSOLS(3^n3^1) is from Theorem 1.2(2). □

Lemma 4.4. *There exist FSOLS*($3^{15}u^1$) *for* $2 \le u \le 21$.

Proof: Applying Construction 2.5 with FSOLS(1^5) and $h = 9$, $p = 0$, $q = 2$, $s = 1$, $0 \le w_j \le 4$, $k = 0$ or 2, we obtain an FSOLS(9^5v^1), where v is even and $0 \le v \le 18$.

Filling the holes of size 9 with FSOLS(3^4) we obtain an FSOLS($3^{15}u^1$), where u is odd and $3 \le u \le 21$.

From Lemma 2.18 we know that there exist FSOLS($1^{15}v^1$) with $7 - v$ pairs of symmetric holey transversals with hole of size v for $v = 0, 2$, or 6, Applying Construction 2.6 with $t = 1$, $g = 15$, $h = v$, $k = 0$, $p = 7 - v$, $m = 3$ and $0 \le w_j \le 1$, we obtain an FSOLS($3^{15}u^1$), where u is even and $2 \le u \le 20$. □

Lemma 4.5. *For* $n = 17$ *and* 19, *there exist FSOLS*(3^nu^1) *for* $2 \le u \le n - 1$.

Proof: Applying Theorem 2.10 and Construction 2.4 with $t = 1$, $p = 0$, $q = \frac{n-1}{2}$, $h = 3$, $0 \le w_j \le 1$, we obtain FSOLS(3^nu^1) for $n = 17, 19$, and $0 \le u \le n - 1$, where u is even.

From Lemma 2.18 we know that there is an FSOLS(1^{18}) with $\frac{17-1}{2} - 1 = 7$ pairs of symmetric holey transversals with hole of size one. Applying Construction 2.6 with $t = 1$, $g = 17$, $h = 1$, $k = 0$, $p = 7$, $m = 3$ and $0 \le w_j \le 1$, we obtain an FSOLS($3^{17}u^1$) for $3 \le u \le 17$, where u is odd.

From Lemma 2.18, there is an FSOLS($1^{19}1^1$) with $\frac{19-1}{2} - 1 = 8$ pairs of symmetric holey transversals with hole of size one and an FSOLS($1^{19}3^1$) with $\frac{19-1}{2} - 3 = 6$ pairs of symmetric holey transversals with hole of size 3. Applying Construction 2.6 as above we obtain an FSOLS($3^{19}u^1$) for $3 \le u \le 21$, where u is odd. □

Lemma 4.6. *If* n *is odd and* $21 \le n \le 47$, *then there exist FSOLS*(3^nu^1) *for* $2 \le u \le n - 2$.

Proof: Applying Lemma 2.15 with $s = 4$, $t = 5$, $h = 3$, $k_i \in \{0, 2, 3, 4\}$ and $m = 1$, $r = 0$ we obtain FSOLS($3^{21}u^1$) for $2 \le u \le 20$; with $m = 3$ or 5 and $r = 3$ we obtain FSOLS($3^{23}u^1$) and FSOLS($3^{25}u^1$) for $3 \le u \le 23$, $u \ne 4$.

Applying Construction 2.4 with $n = 27$, $t = 1$, $p = 0$, $q = 13$, $h = 3$, $0 \le w_j \le 1$ we obtain an FSOLS($3^{27}u^1$) for $0 \le u \le 26$, where u is even;

with $n = 9$, $t = 1$, $p = 0$, $q = 4$, $h = 9$, $0 \le w_j \le 3$ we obtain an FSOLS$(9^9 v^1)$ for $0 \le v \le 24$, where u is even. Filling the holes of size 9 with FSOLS(3^4) we obtain an FSOLS$(3^{27} u^1)$ for $3 \le u \le 27$, where u is odd.

Applying Lemma 2.15 with $s = 4$, $t = 7$, $h = 3$, $k_i \in \{0, 2, 3, 4\}$ and $m = 1$, $r = 0$ we obtain FSOLS$(3^{29} u^1)$ for $2 \le u \le 28$; with $3 \le m \le 5$, $r = 3$ we obtain FSOLS$(3^n u^1)$ for $31 \le n \le 33$ and $3 \le u \le 31$, where $u \ne 4$.

Applying Lemma 2.15 with $s = 4$, $t = 8$, $h = 3$, $k_i \in \{0, 2, 3, 4\}$, $3 \le m \le 5$ and $r = 3$ we obtain FSOLS$(3^n u^1)$ for $35 \le n \le 37$ and $3 \le u \le 35$, where $u \ne 4$.

Applying Lemma 2.15 with $s = 5$, $t = 7$, or 8, $h = 3$, $k_i \in \{0, 2, 3, 4, 5, 6\}$, $3 \le m \le 7$ and $r = 3$ we obtain FSOLS$(3^n u^1)$ for $38 \le n \le 42$, $3 \le u \le 42$, $u \ne 4$ and for $43 \le n \le 47$, $3 \le u \le 51$, $u \ne 4$.

Filling the holes of size 14 of an FSOLS$(3^m 14^1)$ $(21 \le m \le 43)$ with an FSOLS$(3^4 2^1)$ we obtain an FSOLS$(3^n 2^1)$ for $25 \le n \le 47$. Filling the holes of size 16 of an FSOLS$(3^m 16^1)$ $(21 \le m \le 43)$ with an FSOLS$(3^4 4^1)$ we obtain an FSOLS$(3^n 4^1)$ for $25 \le n \le 47$.

Applying Construction 2.4 with $n = 23$, $t = 1$, $p = 0$, $q = 11$, $h = 3$, $0 \le w_j \le 1$ we obtain an FSOLS$(3^{23} u^1)$ for $u = 2$ or 4. □

Lemma 4.7. *Then there exist FSOLS*$(3^n u^1)$ *for* $n \ge 49$ *and* $2 \le u \le n-2$.

Proof: Write $n = 4t + k$, where $4 \le k \le 7$. Since $n \ge 49$, we have $t \ge 11$ and $N(t) \ge 4$ or $N(t-1) \ge 4$, where $N(t)$ is the maximum number of mutually orthogonal latin squares of order t.

If $N(t) \ge 4$, applying Lemma 2.15 with $s = 4$, $h = 3$, $m = k$, $r = 0$, $0 \le k_i \le 5$, $k_i \ne 1$ we obtain an FSOLS$(3^n u^1)$ for $2 \le u \le n - 1$.

If $N(t) < 4$, then it must be $N(t-1) \ge 4$. Applying Lemma 2.15 with a TD$(6, t-1)$, $h = 3$, $m = k + 4$, $r = 0$, $0 \le k_i \le 5$, $k_i \ne 1$ $(1 \le i \le t-1)$ we obtain an FSOLS$(3^n u^1)$ for $2 \le u \le n - 1$. □

Combining Lemmas 4.1–4.3 and 4.5–4.7 we have the following lemma.

Lemma 4.8. *If n is an odd prime power exceeding 3, then there exist FSOLS*$(3^n u^1)$ *for* $2 \le u \le \frac{3(n-1)}{2}$.

Theorem 4.9. *If n is odd and $n \ge 5$, then there exist FSOLS*$(3^n u^1)$ *for* $2 \le u \le \frac{3(n-1)}{2}$.

Proof: The Theorem is true for $n = 15$ or n is an odd prime power by Lemmas 4.2, 4.4, and 4.8.

If $n = 3p$, where n is an odd prime power and $n \ge 7$. Applying Construction 2.5 with $s = t = 1$, $h = 9$ we obtain an FSOLS$(9^p v^1)$ for $0 \le v \le \frac{9(p-1)}{2}$. Filling the holes of size 9 with an FSOLS(3^4) we obtain an

FSOLS($3^n u^1$) for $3 \le u \le \frac{3(3p-1)}{2}$. FSOLS($3^n 2^1$) is from Lemmas 4.6 and 4.7.

If $n = pq$, where $p = p_1^a > 5$, $q = p_2^b > 3$ and p_1, p_2 are different prime numbers. Applying Theorem 2.10 and Construction 2.5 with an FSOLS(1^p), $s = 1$, $h = 3q$, $0 \le w_j \le \frac{3q-1}{2}$, $0 \le k \le \frac{p-1}{2}$ we obtain an FSOLS($(3q)^p v^1$) for $0 \le v \le \frac{3q(p-1)}{2}$. Filling the holes of size $3q$ with FSOLS($3^q k^1$), $2 \le h \le \frac{3(q-1)}{2}$ we obtain an FSOLS($3^{pq} u^1$) for $2 \le u \le \frac{3(pq-1)}{2}$.

If $n = 3pq$, p and q are as above, applying Theorem 2.10 and Construction 2.5 with an FSOLS(1^p), $h = 9p$ we obtain an FSOLS($(9q)^p v^1$) for $0 \le v \le \frac{9q(p-1)}{2}$. Filling the holes of size $9q$ with FSOLS($3^{3q} k^1$), $2 \le k \le \frac{3(3q-1)}{2}$ we obtain an FSOLS($3^n u^1$) for $2 \le u \le \frac{3(n-1)}{2}$.

Applying the Induction Principle we can complete the proof. □

5 Conclusion of the existence of FSOLS($3^n u^1$)

Lemma 5.1. *There exist FSOLS($3^n u^1$) for all $n \ge 4$, $n \ne 5$.*

Proof: FSOLS of type $3^4 1^1$, $3^6 1^1$, $3^8 1^1$, $3^{10} 1^1$, $3^{12} 1^1$, are from Lemmas 3.1–3.3. FSOLS of type $3^7 1^1$, $3^9 1^1$, $3^{11} 1^1$, $3^{13} 1^1$, can be generated by Construction 2.16:

$$3^7 1^1: \mathrm{e} = (\emptyset, x, 1, 4, 6, 15, 11, \emptyset, 20, 12, 19, 17, 9, 3, \emptyset, 10, 8, 13, 5, 2, 18),$$
$$\mathrm{f} = (16), \mathrm{g} = (15).$$
$$3^9 1^1: \mathrm{e} = (\emptyset, x, 1, 4, 6, 15, 26, 21, 11, \emptyset, 23, 8, 19, 25, 22, 20, 12, 5, \emptyset, 14, 10, 13, 16, 3, 2, 24)$$
$$\mathrm{f} = (17), \mathrm{g} = (16).$$
$$3^{11} 1^1: \mathrm{e} = (\emptyset, x, 32, 31, 30, 29, 27, 26, 25, 10, 20, \emptyset, 28, 21, 8, 17, 12, 23, 16, 18, 24, 13, 0, 5, 9, 15, 7, 6, 4, 3, 2, 1, 19),$$
$$\mathrm{f} = (14), \mathrm{g} = (13).$$
$$3^{13} 1^1: \mathrm{e} = (\emptyset, x, 38, 37, 36, 35, 34, 82, 31, 33, 28, 25, 29, \emptyset, 20, 9, 24, 16, 14, 21, 8, 30, 23, 27, 22, 15, \emptyset, 19, 11, 10, 12, 7, 4, 6, 5, 3, 2, 1, 18),$$
$$\mathrm{f} = (17), \mathrm{g} = (16).$$

From Theorem 3.12 and Theorem 4.9 we know that for ever integer $m \ge 10$, there exists an FSOLS($3^m 13^1$). Filling the holes of size 13 with an FSOLS($3^4 1^1$) we obtain an FSOLS($3^{m+4} 1^1$). □

We are now in a position to give the main result of this article.

Theorem 5.1. *An FSOLS($3^n u^1$) exists if and only if $n \ge 4$ and $n \ge$*

$1+\frac{2u}{3}$, with seventeen possible exceptions that $(n,u)=(5,1)$ and $(n,u)=(n,\left\lfloor\frac{3(n-1)}{2}\right\rfloor)$ for $n\in\{6,10,14,18,22,30,34,38,42,46,54,58,62,66,70,94\}$.

Proof: The necessity comes from Theorems 1.1 and 1.3. The sufficiency comes from Theorems 3.12, 4.9 and Lemma 5.1. □

Acknowledgement. The authors are thankful to Professor L. Zhu for introducing them to this work.

References

[1] F.E. Bennet, L. Wu and L. Zhu, Conjugate orthogonal Latin squares with equal sized holes, *Ann. Discrete Math.* **34** (1987), 65–79.

[2] F.E. Bennet, L. Wu and L. Zhu, On the existence of SOLS with equal sized holes, *Ars Combinatoria* **26B** (1988), 5–36.

[3] R.C. Bose, S.S. Shrikhande and E.T. Parker, Further results on the construction of mutually orthogonal latin squares and falsity of Euler's conjecture, *Canad. J. Math.* **12** (1990), 189–203.

[4] R K. Brayton, D. Coppersmith and A.J. Hoffmam, Self-orthogonal latin squares, *Teorie Combinatorie, Proc. Rome Conf.*, 1976, 509–517.

[5] A.E. Brouwer and G.H.J. van Rees, More mutually orthogonal latin squares, *Discrete Math.* **39** (1982), 283–281.

[6] Chen Kejun and Zhu Lie, On the existence of skew Room frames of typy t^u, *Ars Combinatoria*, to appear.

[7] C.J. Colbourn and G.F. Royle, Supports of $(v,4,2)$ designs, *Le. Matematiche* **45** (1990), 39–60.

[8] J.H. Dinitz and D.R. Stinson, MOLS with holes, *Discrete Math.* **44** (1983), 145–154.

[9] B. Du, A few more resolvable spouse-avoiding mixed-double round robin tournaments, *Ars Combinatoria*, **36** (1993), 309–314.

[10] A. Hedayat and E. Seiden, On the theory and application of sum composition of latin squares, *Pacific J. Math.* **54** (1974), 85–113.

[11] K. Heinrich, L. Wu and L. Zhu, Incomplete self-orthogonal latin squares ISOL$(6m+6,2m)$ exist for all m, *Discrete Math.* **87** (1991), 281–290.

[12] K. Heinrich and L. Zhu, Incomplete self-orthogonal latin squares, *J. Australia Math. Soc. Ser. A* **42** (1987), 365–384.

[13] K. Heinrich and L. Zhu, Existence of orthogonal latin squares with aligned subsquares, *Discrete Math.* **59** (1986), 69–87.

[14] J.D. Horton, Sub-latin squares and incomplete orthogonal arrays, *J. Combin. Theory (A)* **16** (1974), 23–33.

[15] E.R. Lamken, The existence of three partitioned incomplete latin squares of type t^u, *Discrete Math.* **89** (1991), 231–251.

[16] C.C. Lindner and C.A. Rodger, "Decomposition into cycles II: Cycle systems", *Contemporary Design Theory*, Wiley, New York, 1992, pp. 325–369.

[17] C.C. Lindner, R.C. Mullin and D.R. Stinson, On the Spectrum of resolvable orthogonal arrays invariant under the Klein group K_4, *Aequations Math.* **26** (1983), 176–183.

[18] R.C. Mullin and D.R. Stinson, Holey SOLSSOMs, *Utilitas Math.* **25** (1984), 159–169.

[19] D.R. Stinson A general construction for group divisible designs, *Discrete Math.* **33** (1981), 89–94.

[20] D.R. Stinson and L. Zhu, On the existence of MOLS with equal-sized holes, *Aequations Math.* **33** (1987), 96–105.

[21] D.R. Stinson and L. Zhu, On the existence of MOLS with equal-sized holes, *Australasian J. Combinatorics* **4** (1991), 33–47.

[22] D.R. Stinson and L. Zhu, On the existence of certain SOLS with holes, *JCMCC* **15** (1994), 33–45.

[23] Y. Xu, G. Ge and L. Zhu, Existence SOLS with holes of type $2^n u^1$, *J. Statist. Plann. Inference*, to appear.

[24] Y. Xu and L. Zhu, Existence of frame SOLS of type $2^n u^1$, *J. Combin. Designs*, **3** (1995), 115–133.

[25] Zhu Lie, A short disproof of Euler's conjecture concerning orthogonal latin squares (with editorial comment by A.D. Keedwell), *Ars Combinatoria* **14** (1982), 47–55.

[26] L. Zhu, Existence of holey SOLSSOMs of type 2^n, *Congresses Numerantium* **45** (1984), 295–304.

[27] L. Zhu, Existence of self-orthogonal latin squares ISOLS$(6m+2, 2m)$, *Ars Combinatoria* **39** (1995), 65–74.