# Triangle-Free and Triangle-Saturated Graphs

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ABSTRACT. A graph $G$ is *triangle-saturated* if every possible edge insertion creates at least one new triangle. Furthermore, if no proper spanning subgraph has this property, then $G$ is *minimally triangle-saturated*. (Minimally triangle-saturated graphs of order $n$ are the diameter 2 critical graphs when $n \geq 3$.) The maximally triangle-free graphs of order $n$ are a proper subset of the minimally triangle-saturated graphs of order $n$ when $n \geq 6$. All triangle-saturated graphs are easily derivable from the minimally triangle-saturated graphs which are *primitive*, that is, have no duplicate vertices. We determine the 23 minimally triangle-saturated graphs of orders $n \leq 7$ and identify the 6 primitive graphs among them.

## 1 Introduction

A *triangle* in a graph is a complete subgraph of order 3. In this paper we study the structure of two classes of graphs defined by restrictions on the presence of triangles. All graphs under study are finite, simple graphs.

For any graph $G$ of order $n$, let $G\uparrow$ denote the set of all graphs of order $n$ which contain a subgraph isomorphic to $G$, and let $G\downarrow$ denote the set of all graphs of order $n$ which are subgraphs of $G$. More generally, if $\mathcal{G}$ is any set of graphs of order $n$, we define $\mathcal{G}\uparrow$ to be the union of all the sets $G\uparrow$, and $\mathcal{G}\downarrow$ to be the union of all the sets $G\downarrow$, each taken over every $G \in \mathcal{G}$.

Let $G$ be any graph containing $t \geq 0$ triangles. Then $G$ is *triangle-free* if $t = 0$, and is *maximally triangle-free* if it is the only triangle-free graph in

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$G\uparrow$. Also $G$ is *triangle-saturated* (or *saturated,* where no confusion results) if it is the only graph in $G\uparrow$ containing $t$ triangles. If $G$ is not a complete graph, it follows that it is triangle-saturated precisely if addition of any edge to $G$ creates at least one new triangle. Thus any triangle-saturated graph which is not complete has diameter 2, and conversely. We call $G$ *minimally triangle-saturated* if it is the only triangle-saturated graph in $G\downarrow$. Let $\mathcal{F}_n$ and $\mathcal{S}_n$ respectively denote the set of all maximally triangle-free graphs of order $n$, and the set of all minimally triangle-saturated graphs of order $n$ (that is, the set of all diameter 2 critical graphs when $n \geq 3$). Note that if $G$ is triangle-free, so is every graph in $G\downarrow$; therefore $\mathcal{F}_n\downarrow$ is the set of all triangle-free graphs of order $n$. Again, if $G$ is triangle-saturated, so is every graph in $G\uparrow$; therefore $\mathcal{S}_n\uparrow$ is the set of all triangle-saturated graphs of order $n$.

**Remark 1.** *Note that every graph which is maximally triangle-free is minimally triangle-saturated, that is,* $\mathcal{F}_n \subseteq \mathcal{S}_n$ *for every* $n \geq 1$.

Bollobás [1] discusses many results from extremal graph theory which are related to the study of $\mathcal{F}_n$ and $\mathcal{S}_n$. Note that his terms 3-*saturated* and *strongly* 3-*saturated* correspond in our terminology to *maximally triangle-free* and *triangle-saturated* respectively. Plesnik [10] constructs families of graphs which are triangle-saturated, and refers to other relevant papers.

## 2 Triangle-free and triangle-saturated graphs of small order

By systematic examination of all graphs of order $n \leq 7$, we determined all triangle-free and triangle-saturated graphs in this range. Our search was considerably simplified by use of the excellent diagrams published by Peter Steinbach [11]. Since $\mathcal{F}_n\downarrow$ is the set of all triangle-free graphs of order $n$, the triangle-free graphs are adequately specified by listing the $f_n := |\mathcal{F}_n|$ members of the maximal set $\mathcal{F}_n$. Likewise, $\mathcal{S}_n\uparrow$ is the set of all triangle-saturated graphs of order $n$, so the triangle-saturated graphs are adequately specified by the $s_n := |\mathcal{S}_n|$ members of the dominated set $\mathcal{S}_n$. The cardinalities of these sets for $n \leq 7$ are given in Table 1. In Figure 1 we show the 10 non-bipartite members of $\mathcal{F}_n$ and $\mathcal{S}_n$ for $n \leq 7$.

## 3 Minimally triangle-saturated graphs need not be triangle-free

From Table 1 we note the important fact that there are minimally triangle-saturated graphs which are not triangle-free. The smallest of these is $B_1$. Let $B_r$ be as illustrated in Figure 2. Then we have

**Remark 2.** *For every order* $n \geq 6$, *there is at least one minimally triangle-saturated graph which is not triangle-free, so* $\mathcal{S}_n \setminus \mathcal{F}_n$ *is nonempty if* $n \geq 6$.

| $n$ | $f_n$ | $s_n$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 1 | 1 |
| 3 | 1 | 1 |
| 4 | 2 | 2 |
| 5 | 3 | 3 |
| 6 | 4 | 5 |
| 7 | 6 | 10 |

**Table 1.**
The numbers of maximally triangle-free graphs ($f_n$) and minimally triangle-saturated graphs ($s_n$) of order $n \leq 7$.

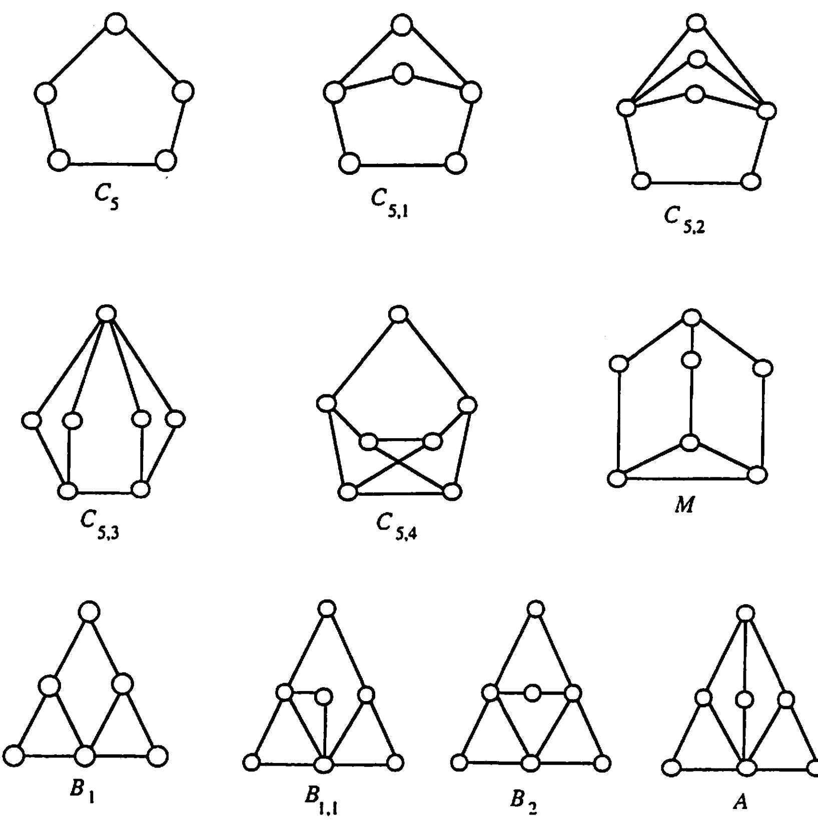


**Figure 1.**
Non-bipartite members of $\mathcal{F}_n$ and $\mathcal{S}_n$ for $n \leq 7$.

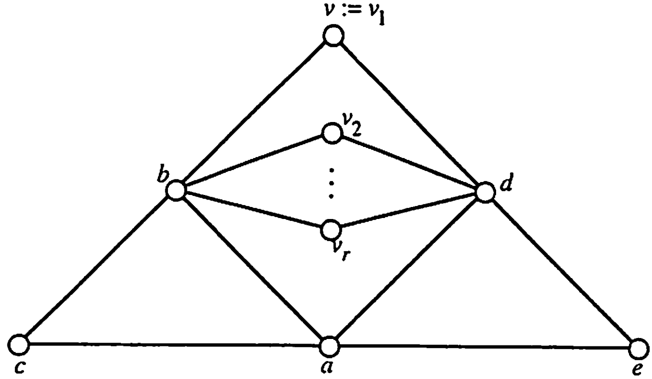


**Figure 2.**
The minimally triangle-saturated graph $B_r$.

What local structure characterises a minimally triangle-saturated graph? Some notation and terminology will be helpful. The *neighbourhood* of any vertex $v$ in a graph $G$ is the set $N(v)$ of all vertices adjacent to $v$ in $G$, and the *non-neighbourhood* of $v$ is the set $\tilde{N}(v)$ of all vertices distinct from $v$ which are not adjacent to $v$ in $G$. A vertex in $\tilde{N}(v)$ is a *non-neighbour* of $v$. If explicit reference to $G$ is appropriate, these sets will be written $N_G(v)$ and $\tilde{N}_G(v)$. A vertex or edge in $G$ is *triangular* if it belongs to a triangle. If $ab$ is a triangular edge in $G$, its endpoint $a$ is *selective for ab* if $a$ has a non-neighbour $x$ which is adjacent to $b$ but not to any other neighbour of $a$, that is, there is at least one vertex $x \in \tilde{N}_G(a)$ such that $N(a) \cap N(x) = \{b\}$. A triangular edge $ab$ is *selective* if at least one of $a$ and $b$ is selective for $ab$. For example, the edge $bc$ in the graph $B_1$ is selective.

**Theorem 1.** *A triangle-saturated graph is minimally saturated if and only if all its triangular edges are selective.*

**Proof:** Let $G$ be any triangle-saturated graph. Suppose $G$ is minimally saturated and $ab$ is any one of its triangular edges. Then there are nonadjacent vertices $x, y$ in $F := G \setminus ab$ such that insertion of the edge $xy$ in $F$ does not create a triangle. Since $ab$ is triangular in $G$, its insertion in $F$ would create a triangle, so $xy \neq ab$ and $x, y$ are nonadjacent in $G$. But $G$ is saturated, so insertion of $xy$ in $G$ would create a triangle. Such a triangle must contain $ab$, since insertion of $xy$ and deletion of $ab$ are commutative operations and their cumulative result does not produce a triangle. Hence with appropriate labelling we may take $y = a$. It follows that $x$ is adjacent to $b$ in $G$, so insertion of $xy$ in $G$ forms the triangle $abxa$. On the other hand $x$ is not adjacent to any other neighbour of $a$ in $G$, for if it were, such adjacencies would also be present in $F$ and insertion of $xy$ in $F$ would create a triangle. Thus $x \in \tilde{N}_G(a)$ and $N_G(a) \cap NG(x) = \{b\}$, so $a$ is selective for $ab$ in $G$. It follows that every triangular edge in $G$ is selective.

Conversely, suppose all triangular edges of $G$ are selective. Let $ab$ be any edge of $G$ and let $F := G \setminus ab$. If $ab$ is not triangular, its insertion in $F$ retrieves $G$ without creating a triangle. On the other hand, if $ab$ is triangular, we may suppose without loss of generality that $a$ is selective for $ab$, so there is a vertex $x \in \tilde{N}_G(a)$ such that $N_G(a) \cap N_G(x) = \{b\}$. Thus $x$ is nonadjacent to $a$ in $G$, and $b$ is the unique common neighbour of $a$ and $x$ in $G$, so insertion of $ax$ in $G$ creates precisely one triangle, namely $abxa$. Therefore insertion of $ax$ in $F$ does not create a triangle. It follows that $G$ is minimally saturated. □

**Remark 3.** *If $G \in \mathcal{S}_n \setminus \mathcal{F}_n$, then $G\uparrow$ contains a graph with exactly one triangle more than $G$.*

This follows from the last part of the proof of Theorem 1. By contrast, note that $K_{2,2}$ is triangle-free and $K_{2,2} \in \mathcal{F}_4$, but no graph in $K_{2,2}\uparrow$ has exactly one triangle.

## 4 Duplication and simplification

In $B_r$ (Figure 2) note that $N(v_1) = N(v_2) = \cdots = N(v_r) = \{b, d\}$, so each of $v_2, \ldots, v_r$ duplicates the relationship of $v_1$ to the rest of $B_1$. In general, if $G$ is any graph with two vertices $v$ and $v'$ which have the same nonempty neighbourhood, we shall call $v'$ a *duplicate* of $v$. Let $F$ be the graph resulting from $G$ by deletion of $v'$. We shall say that $G$ results from $F$ by *duplication* of $v$, and write $G := D_vF$. Thus for example, $B_r = D_v^{r-1}B_1$ for $r \geq 1$. Conversely, $F$ results from $G$ by *simplification* of $v$, and $F := D_v^{-1}G$. Often we indicate a duplicate vertex by a dash without further comment. Duplication of vertices has been studied in the context of diameter 2 graphs by Plesnik [8, 9, 10]. Duffus & Hanson [4] also refer to duplicating vertices, in essentially the same context.

More generally, for any graph $G$ let $\mathcal{D}(G)$ be the set of all graphs which are obtainable from $G$ by a finite sequence of duplications and/or simplifications. We call $\mathcal{D}(G)$ the *duplicate class* of $G$.

Duplication of vertices allows us to generate new saturated graphs from old. In fact, it is easy to see

**Remark 4.** *Suppose $F$ and $G$ are graphs such that $G = D_vF$. Then $G$ is triangle-saturated if and only if $F$ is triangle-saturated.*

As a direct result, we have

**Remark 5.** *If a graph $G$ is triangle-saturated, so are all graphs in its duplicate class $\mathcal{D}(G)$.*

How far is it true that duplication of vertices preserves minimality? Not without restriction: for example, $D_bB_1$ is triangle-saturated but not minimally saturated, since it contains $A$ as a spanning proper subgraph (see Figure 1). But suitably restricted duplication does preserve minimality, as we shall show.

For any vertex $a$ of a graph $G$, the *triangular neighbourhood* of $a$ in $G$ is the set $T(a)$ of neighbours $b \in N(a)$ such that the edge $ab$ is triangular. Note that $T(a)$ is nonempty precisely when $a$ is triangular. Extending the terminology introduced in the previous section to characterise minimally triangle-saturated graphs, we say that the vertex $a$ is *selective* if for each $b \in T(a)$ there is an $x \in \tilde{N}(a)$ such that $b$ is the only neighbour of $a$ which is adjacent to $x$. Thus, $a$ is selective if it is selective for every triangular edge with which it is incident. Each selective vertex is at the head of a "lawn rake" subgraph (Figure 3), since the vertex $a$ is selective precisely when, corresponding to its triangular neighbourhood $\{b_1, b_2, \ldots, b_r\} := T(a)$, there is a non-neighbour subset $\{x_1, x_2, \ldots, x_r\} \subseteq \tilde{N}(a)$ such that $N(a) \cap N(x_i) = \{b_i\}$ for $1 \le i \le r$.

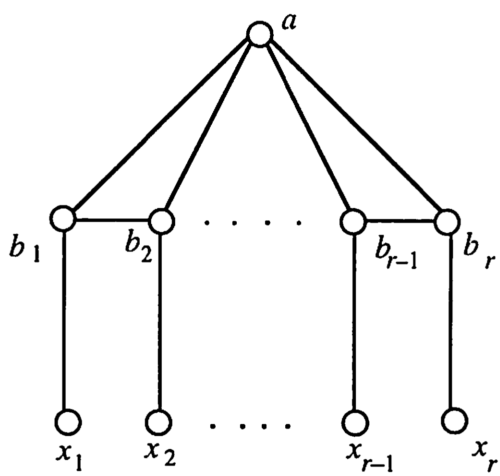


**Figure 3.**
A "lawn rake" subgraph at the selective vertex $a$.

Note that any vertex $a$ which is not triangular is automatically selective because the conditions are then vacuous. Again, any triangular vertex $a$ of degree 2 in a minimally saturated graph must be selective. For if $N(a) := \{b, c\}$, then $abca$ is a triangle. By minimality, the triangular edge $ab$ is selective; but $b$ cannot be selective for $ab$ since $c$ is adjacent to $b$ and therefore no non-neighbour of $b$ is adjacent to $a$. Hence $a$ must be selective for $ab$, and similarly for $ac$, so $a$ is selective.

**Theorem 2.** *Suppose $F$ and $G$ are triangle-saturated graphs such that $G = D_vF$. If $G$ is minimally saturated, so is $F$. Conversely, if $F$ is* *minimally saturated then $G$ is minimally saturated precisely when $v$ is selective.*

**Proof:** Suppose $G$ is minimally saturated. Then $F$ is saturated, by Remark 3, so it suffices to show minimality of $F$. By Theorem 1, in turn, it suffices to show that all triangular edges of $F$ are selective. Any triangular edge $ab$ in $F$ must be triangular in $G$, and therefore selective in $G$. By suitable labelling, in $G$ there is at least one non-neighbour $x$ of $a$ such that $b$ is the only neighbour $x$ has in common with $a$. We may suppose that $x \in F$; for if $v'$ is a suitable candidate for $x$, duplication ensures that $v$ is also. Thus $a$ is selective for $ab$ in $F$, and it follows that $F$ is minimally saturated.

Conversely, suppose $F$ is minimally saturated. Let $ab$ be any triangular edge of $F$. Then $ab$ is triangular in $G$. Also $ab$ is selective in $F$, by Theorem 1, so by suitable labelling there is a vertex $x\tilde{N}_F(a)$ such that $N_F(a) \cap N_F(x) = \{b\}$. This is unchanged by duplication of $v$ unless $v = b$, in which case $N_G(a) \cap N_G(x) = \{b, b'\}$ for every $x \in \tilde{N}_G(a)$. Therefore $a$ is selective for $ab$ in $G$ precisely when $v \neq b$. In the case $v = b$, it follows that $ab$ is selective in $G$ precisely when $b$ is selective for $ab$ in $G$, which holds precisely when $v$ is selective for $av$ in $F$. Therefore all triangular edges of $F$ are selective in $G$ precisely when $v$ is selective for all those with which it is incident, that is, precisely when $v$ is selective. Note that an edge $av'$ is triangular in $G$ if and only if $av$ is triangular in $F$, and $v'$ is selective for $av'$ precisely when $v$ is selective for $av$ in $G$. Therefore all triangular edges in $G$ are selective if $v$ is selective. By Theorem 1, it follows that $G$ is minimally saturated precisely when $v$ is selective. □

Theorem 2 shows that if $G$ is minimally saturated and $F := D_v^{-1}G$, then $F$ is minimally saturated. But $G = D_vF$ is minimally saturated, so $v$ is selective. Hence

**Corollary 2.1.** *If $G$ is a minimally triangle-saturated graph, every duplicate vertex in $G$ is selective.*

For any graph $G$, let the *selective duplicate class* of $G$ be the set $\mathcal{D}^*(G)$ of all graphs which are obtainable from $G$ by a finite sequence of duplications and/or simplifications of selective vertices. Then Theorem 2 implies

**Corollary 2.2.** *If $G$ is a minimally triangle-saturated graph, so are all graphs in its selective duplicate class $\mathcal{D}^*(G)$.*

All vertices of a triangle-free graph $G$ are selective, so $\mathcal{D}^*(G) = \mathcal{D}(G)$. Since $\mathcal{F}_n \subseteq \mathcal{S}_n$, in this case Theorem 2 implies

**Corollary 2.3.** *If $G$ is a maximally triangle-free graph, so are all graphs in its duplicate class $\mathcal{D}(G)$.*

Together these results show that the classes $\mathcal{F}_n$ and $\mathcal{S}_n$ are rather robust under the operation of duplication or simplification of a vertex. In fact,

(1) $G \in \mathcal{S}_n \uparrow \Leftrightarrow D_v G \in \mathcal{S}_{n+1} \uparrow$,

(2a) $G \in \mathcal{S}_n \Rightarrow D_v G \in \mathcal{S}_{n+1}$ provided $v$ is selective,

(2b) $G \in \mathcal{S}_n \Leftarrow D_v G \in \mathcal{S}_{n+1}$,

(3) $G \in \mathcal{F}_n \Leftrightarrow D_v G \in \mathcal{F}_{n+1}$,

(4) $G \in \mathcal{F}_n \downarrow \Leftrightarrow D_v G \in \mathcal{F}_{n+1} \downarrow$.

The last equivalence comes from the observation that duplication of a vertex $v$ in $G$ creates a new triangle only when $v$ belongs to a triangle in $G$.

## 5 Complete bipartite graphs

The graphs of Table 1 include many which are complete bipartite. In fact, every such graph is maximally triangle-free, as is easily proved directly. Alternatively, note that $K_{r,s} = D_1^{r-1} D_2^{s-1} K_{1,1}$ for every $r, s \geq 1$. But $K_{1,1} \in \mathcal{F}_2$, so $K_{r,s} \in \mathcal{F}_{r+s}$, by Corollary 2.3. Hence

**Remark 6.** *The set $\mathcal{F}_n$ contains every complete bipartite graph of order $n$.*

This immediately implies

**Remark 7.** $\mathcal{F}_n \geq \lfloor n/2 \rfloor$ *for* $n \geq 2$.

This accounts for 12 of the maximally triangle-free graphs in Table 1. The bound in Remark 6 is actually achieved when $n \leq 4$ but not for any larger $n$, since there are non-bipartite graphs in $\mathcal{F}_n$ when $n \geq 5$.

The complete bipartite graphs form a kind of backbone for $\mathcal{F}_n$, since they include its smallest and largest members. These facts are special cases of classical theorems of Turán [2, 12, 13] and Erdös, Hajnal and Moon [5]:

**Theorem 3.** (Turán). *The unique graph of largest size in $\mathcal{F}_n \downarrow$ is the complete bipartite graph $K_{r,s}$ with $r := \lfloor n/2 \rfloor$, $s := \lceil n/2 \rceil$.*

**Theorem 4.** (Erdös, Hajnal & Moon). *The unique graph of smallest size in $\mathcal{S}_n \uparrow$ is the complete bipartite graph $K_{1,n-1}$.*

The complete bipartite graphs $K_{r,s}$ with $r + s = n$ all lie in $\mathcal{F}_n$. By Theorems 3 and 4, they include the largest and smallest members of $\mathcal{F}_n$, corresponding to the smallest and largest values of $|r - s|$ respectively. We, and many others before us, have independently conjectured that the graph of largest size in $\mathcal{S}_n$ has $\lfloor \frac{n^2}{4} \rfloor$ edges and that this size is uniquely attained by the Turán graph described in Theorem 3. The conjecture has been attributed to Simon and Murty (see [3]). Recently Füredi [7] has shown it is true for all $n$ larger than some huge but computable number. The conjecture has also been studied by Plesnik [9] and by Fan [6] who has shown that the upper bound of $\lfloor \frac{n^2}{4} \rfloor$ is correct for $n \leq 24$. If the conjecture is true, it would follow that the complete bipartite graphs of order $n$ include the largest and smallest members of $\mathcal{S}_n$, so $\mathcal{F}_n$ would contribute both size extremes to $\mathcal{S}_n$.

## 6 Structure of duplicate classes

In the preamble to Remark 6, we noted that the duplicate class $\mathcal{D}(K_2)$ is the set of all complete bipartite graphs. Call a graph *primitive* if it contains no duplicate vertices: for example, $K_2$ is primitive. For any graph $G$ there is a unique primitive $P \in \mathcal{D}(G)$. This can be seen as follows. Partition the vertices of $G$ into equivalence classes such that two vertices are equivalent precisely if they are duplicates. Call two equivalence classes $A$ and $B$ *adjacent* if there is an edge $ab$ in $G$ with vertices $a \in A$, $b \in B$. Then $a'b'$ is an edge in $G$ for every $a' \in A$, $b' \in B$ since duplication of a ensures that $a'b$ is an edge, then duplication of $b$ ensures that $a'b'$ is an edge. But no vertex is adjacent to any of its duplicates. Thus the subgraph of $G$ induced by any equivalence class has no edges, and the subgraph induced by any two adjacent equivalence classes $A$ and $B$ is the complete bipartite graph with $A$ and $B$ as its parts. The *duplicate quotient* of $G$ is the graph $P$ with one vertex for each equivalence class of vertices in $G$, and such that two vertices of $P$ are adjacent precisely if the corresponding equivalence classes of $G$ are adjacent. No two vertices of $P$ are duplicates, so $P$ is primitive. Thus we are led to

**Theorem 5.** *For any graph $G$, the duplicate quotient $P$ of $G$ is its unique primitive graph, and $\mathcal{D}(G)$ is precisely the set of all graphs having $P$ as duplicate quotient.*

**Proof:** Let $[P]$ be the set of all graphs having $P$ as duplicate quotient. Every $H \in [P]$ is obtainable from $P$ by a suitable sequence of duplications, so $[P] \subseteq \mathcal{D}(P)$. On the other hand, even though duplication and simplification may change the cardinalities of some equivalence classes of vertices, each preserves the set of equivalence classes, and the adjacency relation between them. So every $H \in \mathcal{D}(G)$ has $P$ as its duplicate quotient, and $\mathcal{D}(G) \subseteq [P]$.

Evidently a suitable sequence of duplications from $P$ yields $G$, so $G \in \mathcal{D}(P)$. Thus all graphs obtainable from $G$ by a finite sequence of duplications and/or simplifications are so obtainable from $P$, and conversely, so $\mathcal{D}(P) = \mathcal{D}(G)$. Hence $\mathcal{D}(G) = [P] = \mathcal{D}(P)$. Clearly $P$ is the only graph in $[P]$ which is primitive, so the proof is complete. □

Let $F$ and $G$ be two graphs with the same duplicate quotient $P$. We shall say that $F$ is *nonselectively dominated* by $G$, or $G$ *nonselectively dominates* $F$, if for each nonselective vertex $v$ in $P$, the number of duplicates of $v$ in $G$ is greater than or equal to the number of duplicates of $v$ in $F$. Since $\mathcal{D}^*(G) \subseteq$

$\mathcal{D}(G)$, and duplications in $\mathcal{D}^*(G)$ are restricted to selective vertices, we have

**Corollary 5.1.** *For any graph $G$ with primitive $P$, the selective duplicate class $\mathcal{D}^*(G)$ is the set of all graphs with primitive $P$ which are nonselectively dominated by $G$.*

If $F$ and $G$ are two graphs such that $F \in \mathcal{D}(G)$, then $\mathcal{D}(F) = \mathcal{D}(G)$. But, by contrast, if $F \in \mathcal{D}^*(G)$ then $\mathcal{D}^*(F) \subseteq \mathcal{D}^*(G)$ and equality does not necessarily hold. Corollary 5.1 shows in particular that if $P$ is the duplicate quotient of $G$ then $\mathcal{D}^*(P) \subseteq \mathcal{D}^*(G)$, with equality precisely when each nonselective vertex in $G$ has no duplicate.

What can be said about the minimally saturated graphs in a duplicate class? Note first that the duplicate class of a saturated graph need not contain any minimally saturated graph. For example, $K_3$ is saturated so all members of its duplicate class $\mathcal{D}(K_3)$, comprising all complete tripartite graphs, are saturated; but no graph $G \in \mathcal{D}(K_3)$ is minimally saturated, since every edge in $G$ is triangular and nonselective. However if a given duplicate class $\mathcal{D}(G)$ contains some member $F \in \mathcal{D}(G)$ which is minimally saturated, then all graphs in $\mathcal{D}^*(F)$ are minimally saturated, and no nonselective vertex in $F$ has a duplicate, by Corollaries 2.1 and 2.2. Therefore $\mathcal{D}^*(P) = \mathcal{D}^*(F)$, where $P$ is the duplicate quotient of $F$, by Corollary 5.1. Thus $F \in \mathcal{D}^*(P)$, and all graphs in $\mathcal{D}^*(P)$ are minimally saturated. But $\mathcal{D}(F) = \mathcal{D}(G)$, so $P$ is also the duplicate quotient of $G$, by Theorem 5. In summary, we have

**Corollary 5.2.** *Let $G$ be any graph with duplicate class $\mathcal{D}(G)$ containing at least one minimally triangle-saturated graph. Then the set of all minimally triangle-saturated graphs in $\mathcal{D}(G)$ is precisely the selective duplicate class $\mathcal{D}^*(P)$, where $P$ is the primitive of $G$.*

Having determined the "vertical" structure of duplicate classes and selective duplicate classes, we can now look at their "horizontal cross-sections", that is, the subsets comprising just those graphs of a given order $n$. For any primitive graph $P$, the exact number of graphs of order $n$ in $\mathcal{D}(P)$, or in $\mathcal{D}^*(P)$, depends on the structure of the automorphism group $\mathrm{aut}(P)$. However, we shall content ourselves with lower bounds on these numbers.

**Theorem 6.** *For any primitive graph $P$ of order $k$, the number of nonisomorphic graphs of order $n$ in the duplicate class $\mathcal{D}(P)$ is at least $\frac{1}{c}\binom{n-1}{k-1}$, where $c$ is the order of $\mathrm{aut}(P)$.*

**Proof:** Assign labels $1, 2, \ldots, k$ to the vertices of $P$. For any $k$-sequence $x := x_1, x_2, \ldots, x_k$ of positive integers with sum $n$, the graph

$$D(\mathbf{x})P := D_1^{x_1-1} D_2^{x_2-1} \ldots D_k^{x_k-1} P$$

is an order $n$ member of $\mathcal{D}(P)$, and every order $n$ member of $\mathcal{D}(P)$ is of this form, by Theorem 5. Let $S(k,n)$ be the set of $k$-sequences of positive integers with sum $n$. Call $\mathbf{x}, \mathbf{y} \in S(k,n)$ equivalent $k$-sequences precisely when the graphs $D(\mathbf{x})P$ and $D(\mathbf{y})P$ are isomorphic. The number of $k$-sequences in an equivalence class is at most $c := |\mathrm{aut}(P)|$, so the number of equivalence classes in $S(k,n)$ is at least $|S(k,n)|/c$. Since the number of $k$-sequences of positive integers with sum $n$ is $\binom{n-1}{k-1}$, the theorem now follows. □

Let us define the *selective order* of any graph $G$ to be the number of selective vertices in $G$. Suppose $P$ is a primitive graph of order $k$ and selective order $s$. Only selective vertices of $P$ are duplicated in $\mathcal{D}^*(P)$, so each order $n$ member of $\mathcal{D}^*(P)$ is of the form $D(\mathbf{x})P$, where $\mathbf{x} \in S(k,n)$ and $x_i = 1$ if $i$ labels a nonselective vertex of $P$. The $s$ terms of $\mathbf{x}$ corresponding to the selective vertices of $P$ form a subsequence $\mathbf{x}^* \in S(s, n-k+s)$, so reasoning as for the proof of Theorem 6 yields

**Corollary 6.1.** *For any primitive graph $P$ of order $k$ and selective order $s$, the number of non-isomorphic graphs of order $n$ in the selective duplicate class $\mathcal{D}^*(P)$ is at least $\frac{1}{c}\binom{n-k+s-1}{s-1}$, where $c$ is the order of $\mathrm{aut}(P)$.*

**Corollary 6.2.** *If the primitive graph $P$ of order $k$ and selective order $s$ is minimally triangle-saturated, then the number of graphs in $\mathcal{S}_n$ with duplicate quotient $P$ is at least $\frac{1}{c}\binom{n-k+s+1}{s-1}$, where $c$ is the order of $\mathrm{aut}(P)$.*

**Corollary 6.3.** *If the primitive graph $P$ of order $k$ is maximally triangle-free, then the number of graphs in $\mathcal{F}_n$ with duplicate quotient $P$ is at least $\frac{1}{c}\binom{n-1}{k-1}$, where $c$ is the order of $\mathrm{aut}(P)$.*

When $P = K_2$, Corollary 6.3 shows that $\mathcal{F}_n$ contains at least $(n_1)/2$ complete bipartite graphs. The exact number is $\lfloor n/2 \rfloor$, as reflected in Remark 6.

## 7 Primitive minimally triangle-saturated graphs

The results of the previous section show that duplicate classes of graphs are characterized by the unique primitive graphs which they contain. Again, if a duplicate class contains any minimally saturated graph then the primitive graph of the class is minimally saturated, and its selective duplicate class is precisely the set of all minimally saturated graphs in the class. In this way the determination of all minimally saturated graphs reduces to the determination of all primitive graphs of this type.

For any set of graphs $\mathcal{G}$, let $\mathcal{D}(\mathcal{G})$ denote the union of the duplicate classes $\mathcal{D}(G)$ for all $G \in \mathcal{G}$, and let $\mathcal{D}^*(\mathcal{G})$ denote the corresponding union of selective duplicate classes $\mathcal{D}^*(G)$. Let $\mathcal{PF}_n$ and $\mathcal{PS}_n$ denote the sets of all primitive graphs in $\mathcal{F}_n$ and $\mathcal{S}_n$, respectively. Then

$$\mathcal{PF}_{n+1} = \mathcal{F}_{n+1} \setminus \mathcal{D}(\cup_{r=2}^{n} \mathcal{PF}_r), \tag{1}$$
$$\mathcal{PS}_{n+1} = \mathcal{S}_{n+1} \setminus \mathcal{D}^*(\cup_{r=2}^{n} \mathcal{PS}_r), \tag{2}$$

for every $n \geq 1$. In particular, $\mathcal{PF}_2 = \mathcal{F}_2 = \mathcal{S}_2 = \mathcal{PS}_2 = \{K_2\}$. Since $\mathcal{S}_3$ and $\mathcal{S}_4$ are contained in $\mathcal{D}(K_2)$, there are no primitive graphs in these sets, so $\mathcal{PF}_3 = \mathcal{PS}_3 = \emptyset$ and $\mathcal{PF}_4 = \mathcal{PS}_4 = \emptyset$. The 5-cycle $C_5$ is the sole member of $\mathcal{S}_5$ not contained in $\mathcal{D}(K_2)$, so $\mathcal{PF}_5 = \mathcal{PS}_5 = \{C_5\}$. Then $f_n \geq \frac{1}{10}\binom{n-1}{4}$, since Corollary 6.3 yields this as a lower bound on the number of order $n$ graphs in $\mathcal{D}(C_5)$, superseding Remark 6 for $n \geq 9$. Figure 1 shows four members of $\mathcal{D}(C_5)$ which are in $\mathcal{F}_6$ or $\mathcal{F}_7$: $C_{5,1} = D_1C_5$; $C_{5,2} = D_1^2C_5$; $C_{5,3} = D_1D_3C_5$; $C_{5,4} = D_1D_2C_5$. Hence $\mathcal{F}_6$ and $\mathcal{F}_7$ are contained in $\mathcal{D}(K_2, C_5) := \mathcal{D}(K_2) \cup \mathcal{D}(C_5)$, so $\mathcal{PF}_6 = \emptyset$ and $\mathcal{PF}_7 = \emptyset$.

The graph $B_1$ (Figure 2) is the sole member of $\mathcal{S}_6$ not contained in $\mathcal{D}(K_2, C_5) = \mathcal{D}^*(K_2, C_5)$, so $\mathcal{PS}_6 = \{B_1\}$. Figure 1 shows $\mathcal{D}^*(B_1) \cap \mathcal{S}_7 = \{B_{1,1}, B_2\}$. Then $\mathcal{PS}_7 = \mathcal{S}_7 \setminus \mathcal{D}^*(K_2, C_5, B_1) = \{A, M\}$. In particular, we can now conclude from Corollary 6.2 that $s_n \geq |\mathcal{D}^*(M) \cap \mathcal{S}_n| \geq \frac{1}{6}\binom{n-1}{6}$. Table 2 summarizes these results, with the corresponding bounds derived from Corollary 6.2.

These calculations confirm the internal consistency of Figure 1. We shall now determine deductively the primitive graphs of order $n \leq 7$, thus proving the completeness of the list in Table 2, and hence the completeness of Figure 1. The reasoning occupies the remainder of this section, and in its course yields several other general results.

| order | $P$ | $\lvert\mathcal{D}^*(P) \cap \mathcal{S}_n\rvert$ |
|---|---|---|
| 1 | $K_1$ | 0 when $n \geq 2$ |
| 2 | $K_2$ | $\geq \frac{1}{2}\binom{n-1}{1}$ |
| 5 | $C_5$ | $\geq \frac{1}{10}\binom{n-1}{4}$ |
| 6 | $B_1$ | $\geq \frac{1}{2}\binom{n-4}{2}$ |
| 7 | $M$ | $\geq \frac{1}{6}\binom{n-1}{6}$ |
| 7 | $A$ | $\geq \frac{1}{2}\binom{n-4}{3}$ |

**Table 2.**
Primitive minimally triangle-saturated graphs $P$ of order $\geq 7$.

It turns out to be convenient to fix the minimum degree $\delta$ in the graphs under study. (Duffus & Hanson [4] have established related results about graphs in $\mathcal{F}_n$ with minimum degree $\delta = 2$ or 3.) Any graph in $\mathcal{S}_n$ is connected, so $\delta \geq 1$ when $n \geq 2$. We begin by observing

**Remark 8.** *If $P \in \mathcal{S}_n$ is a primitive graph with minimum degree 1, then $n = 2$ and $P = K_2$. The minimally triangle-saturated graphs with $\delta = 1$ are the stars $D_1^r K_2 = K_{1,r+1}$ with $r \geq 0$.*

Next consider primitive graphs with minimum degree $\delta = 2$. First the triangle-free case:

**Theorem 7.** *If $P \in \mathcal{F}_n$ is a primitive graph with minimum degree 2, then $n = 5$ and $P = C_5$.*

**Proof:** Let $P \in \mathcal{F}_n$ be primitive and have minimum degree 2. Let $a$ be a degree 2 vertex of $P$, and $\{b, c\} := N(a)$. Since $P$ is triangle-free, $b$ and $c$ are not adjacent, $P$ is not complete, and $\operatorname{diam} P = 2$. All remaining vertices are in $N(b) \cup N(c) \setminus \{a\}$. Let $X := N(b) \setminus N(c)$, $Y := N(c) \setminus N(b)$ and $Z := N(b) \cup N(c) \setminus \{a\}$. If $z \in Z$ then $N(z) = \{b, c\}$ because $P$ is triangle-free; but then $z$ is a duplicate of $a$, contradicting primitiveness of $P$. Hence $Z = \emptyset$. But $\deg b \geq 2$, so $X$ is nonempty; similarly $Y$ is nonempty. Any $x \in X$ has $\deg x \geq 2$, so is adjacent to some vertex in $Y$. Indeed, $P$ is maximally triangle-free, so every vertex in $X$ is adjacent to every vertex in $Y$. Then $N(x) = \{b\} \cup Y$ for every $x \in X$, so all vertices in $X$ are duplicates if $|X| > 1$, contradicting primitiveness of $P$. Similarly if $|Y| > 1$. Hence $X = \{x\}$, $Y = \{y\}$, $n = 5$ and $P = C_5$. □

If $G \in \mathcal{F}_n$ but we do not require $G$ to be primitive, the same proof as for Theorem 7 shows that every $z \in Z$ is a duplicate of $a$; also $N(x) = \{b\} \cup Y$ for every $x \in X$, and $N(y) = \{c\} \cup X$ for every $y \in Y$. Now $\delta = 2$ ensures that if either of $X$ and $Y$ is empty, they both are, and then $Z \neq \emptyset$ and $G = D_a^r F$ for some $r \geq 1$, where $F$ is the path $bac$. If $X$ and $Y$ are nonempty then $G = D_a^r D_x^s D_y^t C_5$ for some $r, s, t \geq 0$, where $C_5 = abxyca$. Hence

**Corollary 7.1.** *The maximally triangle-free graphs with minimum degree 2 are $D_1^r D_2 K_2 = K_{2,r+1}$ with $r \geq 1$, and $D_1^r D_3^s D_4^t C_5$ with $r, s, t \geq 0$.*

Continuing with the study of primitive graphs with minimum degree $d = 2$, we next seek those which contain triangles. The vertices of degree 2 may or may not include triangular vertices: the first case is treated in Theorem 8, the second in Theorem 9 and its corollary.

**Theorem 8.** *Any graph in $\mathcal{S}_n$ which has a triangular vertex of degree 2 contains $B_1$ as an induced subgraph.*

**Proof:** Let $G \in \mathcal{S}_n$ have a triangular vertex $a$ with $\{b, c\} := N(a)$. Then $abca$ is a triangle, so its edges must be selective, by Theorem 1. Since $b$ cannot be selective for $ab$, and $c$ cannot be selective for $ac$, it follows that $a$ must be selective. Hence $X := N(b) \setminus N(c)$ and $Y := N(c) \setminus N(b)$ must both be nonempty. All other vertices of $G$ are in $Z := N(b) \cap N(c) \setminus \{a\}$.

Without loss of generality, we may suppose that $c$ is selective for $bc$. Then there is a vertex $x \in X$ such that $N(c) \cap N(x) = \{b\}$, so $x$ is not adjacent to any vertex in $Y \cup Z$. But $\deg x \geq 2$, so there is an adjacent vertex $v \in X$, and $bvxb$ is a triangle. Note that $v$ is not selective for $vx$

because $b \in N(v)$ and $N(x) \subseteq X$, so $\tilde{N}(b) \cap N(x) = Y \cap N(x) = \emptyset$. Hence $x$ must be selective for $vx$, by minimality of $G$: since $b \in N(x)$ there is some vertex $y \in \tilde{N}(b) \cap N(v)$. Then $y \in Y$ and the subgraph of $G$ induced by $\{a, b, c, v, x, y\}$ is $B_1$. □

**Corollary 8.1.** *For $n \leq 7$, if $P \in \mathcal{S}_n$ is a primitive graph with a triangular vertex of degree 2, then either $n = 6$ and $P = B_1$, or $n = 7$ and $P = A$.*

**Proof:** Let $P \in \mathcal{S}_n$ be a primitive graph with a triangular vertex of degree 2. Then $B_1$ is an induced subgraph of $P$ by Theorem 8 so $n \geq 6$, and $P = B_1$ when $n = 6$. Continuing with the notation used in the proof of Theorem 8, let $n = 7$ and let $w$ be the extra vertex. First suppose that $w \in Z$. Then $w$ is a duplicate of $a$ so $D_a B_1 \subseteq P$. Minimality of $P$ implies $P = D_a B_1$; but this contradicts primitiveness, so $w \notin Z$. Now suppose $w \in Y$. If $w$ is adjacent to $v$ then $D_y B_1 \subseteq P$. This contradicts minimality and primitiveness of $P$, so $w$ is not adjacent to $v$. Then $w$ is adjacent to $x$, since $\text{diam} P = 2$. This contradicts $N(x) \cap Y = \emptyset$, so $w \notin Y$. Hence $w \in X$. If $w$ is adjacent to $v$ then $D_x B_1 \subseteq P$. This again contradicts minimality and primitiveness of $P$, so $w$ is not adjacent to $v$. But $\text{diam} P = 2$, so $w$ is adjacent to $y$. Then $A \subseteq P$, and minimality requires $P = A$. □

To complement Theorem 8, now consider graphs in $\mathcal{S}_n \setminus \mathcal{F}_n$ containing a non-triangular vertex of degree 2. The graph $M$ (Figure 1) plays an important role. In Theorem 9 we restrict attention to graphs containing neither $B_0$ nor $W$ as an induced subgraph (Figure 4). In the proof of Corollary 9.1 we cover the remaining cases.

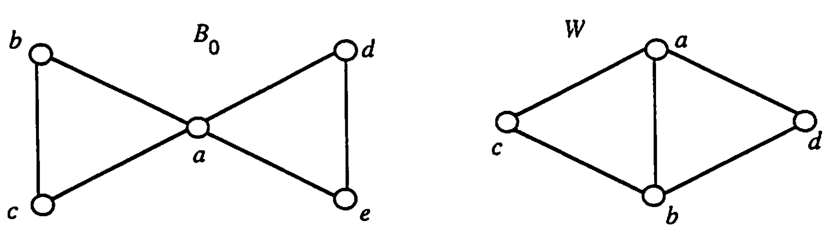


**Figure 4.** The graphs $B_0$ and $W$.

**Theorem 9.** *Suppose $G \in \mathcal{S}_n \setminus \mathcal{F}_n$ has minimum degree 2. If $G$ contains no induced subgraph isomorphic to $B_0$ or $W$, then $G$ contains $M$ as an induced subgraph.*

**Proof:** Let $a$ be a vertex of degree 2 in $G$ with neighbours $b$ and $c$. By Theorem 8, $b$ is not adjacent to $c$. As before, let $X = N(b) \setminus N(c)$, $Y = N(b) \setminus N(c)$, and $Z = N(b) \cap N(c) \setminus \{a\}$. Since $G$ contains a triangle, there is a triangle with $b$ or $c$ as one of its vertices. For suppose there is a triangle entirely in $X \cup Y \cup Z$: at least two of its vertices are adjacent either to $b$ or

$c$, forming a triangle of the type claimed, so we may assume $G$ contains a triangle $bxzb$. Now $x$ and $z$ can't both lie in $Z$, since the induced subgraph on $\{b, c, x, z\}$ would be $W$. Thus we assume $x \in X$ and $z \in X \cup Z$. Either some triangle on $b$ has a vertex $z$ in $Z$, or each triangle on $b$ has its other vertices lying in $X$.

Suppose there is a triangle $bxzb$ where $x \in X$ and $z \in Z$. Now $N(z) \cap Y = \emptyset$, since otherwise $G$ would contain one of the induced subgraphs $B_0$ or $W$. This implies that $b$ is not selective for the edge $bz$, so $z$ must be selective for $bz$. Thus there is a vertex $w \in X$ not adjacent to $z$ or any neighbour of $z$ other than $b$. There can be no neighbour of $w$ in $X \cup Z$ for otherwise there would be an induced subgraph $B_0$ or $W$ in $X \cup \{b, z\}$. But $\deg w \geq 2$, so $w$ is adjacent to some $y \in Y$. Now $z$ is not selective for the edge $xz$ since all neighbours of $x$ other than $b$ are adjacent to $b$ or $c$, which are neighbours of $z$. Thus $x$ is selective for $xz$, so there is a neighbour of $z$ in $Y \cup \{c\}$ which is not adjacent to any neighbour of $x$. But $N(z) \cap Y = \emptyset$, so $c$ is not adjacent to any neighbour of $x$. Therefore $N(x) \subseteq X \cup \{b, z\}$. Consequently $x$ and $y$ are not adjacent. However, $\text{diam} G = 2$ implies that $x$ and $y$ have a common neighbour $u$, and the above reasoning shows that $x \in X$. Now the subgraph induced by $\{b, u, x, z\}$ cannot be $W$, so $u$ must be adjacent to $z$. But then the edge $uz$ is not selective, and this contradiction shows there cannot be a triangle $bxzb$ with $z \in Z$.

So we may suppose that any triangle on $b$ has its other two vertices in $X$. Let $bxvb$ be such a triangle. We note that $x$ and $v$ cannot have a common neighbour $w$ in $Y$, otherwise the induced subgraph on $\{b, v, w, x\}$ would be $W$. Without loss of generality, assume $x$ is selective for the edge $xv$. Then there is a $y \in N(v) \setminus \{b\}$ which is not adjacent to any neighbour of $x$. Thus $y \in Y$. Since $\text{diam} G = 2$, $x$ and $c$ must have a common neighbour $u \in Y$, and $y$ and $u$ are not adjacent. Hence the subgraph induced on $\{a, b, c, u, v, x, y\}$ is $M$. □

**Corollary 9.1.** *For $n \leq 7$, if $P \in \mathcal{S}_n \setminus \mathcal{F}_n$ is a primitive graph with minimum degree 2 but no triangular vertex of degree 2, then $n = 7$ and $P = M$.*

**Proof:** We show that if $P$ is a graph satisfying the stated hypotheses, then $P$ contains neither $B_0$ nor $W$ as an induced subgraph. The result then follows from Theorem 9.

First suppose that $P$ contains an induced subgraph $B_0$, labelled as in Figure 4. Since $P$ is minimally saturated, every triangular edge in $P$ is selective. Without loss of generality, $b$ is selective for the edge $bc$, so there is a vertex $v \in \tilde{N}(b)$ such that $N(v) \cap N(b) = \{c\}$. Now $\deg b \geq 3$ because $b$ is a triangular vertex, so $b$ has at least one more neighbour, say $u$. Then $n = 7$. Also $u$ has at least one more neighbour. But $N(v) \cap N(b) = \{c\}$, so $u$ is not adjacent to $v$. But $\text{diam} P = 2$, so $u$ and $v$ must have a common neighbour. If either $d$ or $e$ is a common neighbour of $u$ and $v$, we may assume $d \in N(u) \cap N(v)$ without loss of generality. But then $P$ properly contains a spanning subgraph $M$, with $abca$ as its triangle and $d$ as the vertex at distance 2 from the triangle. This contradicts minimality of $P$, so neither $d$ nor $e$ is a common neighbour of $u$ and $v$. Therefore $c$ must be the common neighbour of $u$ and $v$. Then $u$ is a triangular vertex, as are $d$ and $e$, so each has degree at least 3; also $\deg v = 2$. Hence each of $d$, $e$, $u$, $v$ has at least one more adjacency and without loss of generality we can assume that $du$ and $ev$ are edges. Then $P$ again properly contains a spanning subgraph $M$, with $adea$ as its triangle and $c$ as the vertex at distance 2 from the triangle. This contradicts minimality of $P$, so it follows that $P$ cannot contain an induced subgraph $B_0$.

Now suppose that $P$ contains an induced subgraph $W$, labelled as in Figure 4. All triangular edges in $P$ must be selective, so without loss of generality we can assume that $a$ is selective for the edge $ab$, and there is a vertex $v \in \tilde{N}(a)$ such that $N(v) \cap N(a) = \{b\}$. Now $b$ is the only neighbour of $v$ in $W$, and $n \leq 7$, so $\deg v \leq 3$. Suppose $\deg v = 3$, say $N(v) := \{b, u, w\}$. Then $n = 7$. As $P$ has a nontriangular vertex of degree 2, without loss of generality $u$ is nontriangular and $\deg u = 2$. Now $u$ is not adjacent to $a$, since $N(v) \cap N(a) = \{b\}$, so $u$ is adjacent to $c$ without loss of generality. But $\mathrm{diam} P = 2$, so $u$ and $d$ must have a common neighbour. But $W$ is induced, so $d$ is not adjacent to $c$; and $N(v) \cap N(a) = \{b\}$, so $d$ is not adjacent to $v$. This contradiction shows that $\deg v \neq 3$. Hence $\deg v = 2$, say $N(v) := \{b, u\}$, and $v$ is not triangular, so $u$ is not adjacent to $b$. Also $u$ is not adjacent to $a$, since $N(v) \cap N(a) = \{b\}$. Now $c$ and $d$ each have at least one more neighbour, and they cannot be duplicates. Since $n \leq 7$, it follows that one of them is adjacent to $u$, one of them is adjacent to another vertex $w$, and they are not both adjacent to $u$ and $w$. Without loss of generality we can suppose $cu$ and $dw$ are edges. Now $u$ is not adjacent to $w$, for otherwise $P$ would properly contain a spanning subgraph $M$, with $abda$ as its triangle and $u$ as the vertex at distance 2 from the triangle. But $\mathrm{diam} P = 2$: then $v$ and $w$ must have a common neighbour, so $w$ is adjacent to $b$.

There is no vertex which is a common neighbour of $d$ and $u$, and $\mathrm{diam} P = 2$, so $d$ must be adjacent to $u$. Then $c$ is adjacent to $w$, since $c$ and $d$ cannot be duplicates. But $\deg w \geq 3$, since $w$ is triangular. Hence $w$ is adjacent to $a$. Now $P$ properly contains a spanning subgraph $M$, with $abwa$ as its triangle and $u$ as the vertex at distance 2 from the triangle. This contradiction shows that $P$ cannot contain an induced subgraph $W$, and the corollary now follows. □

It would be interesting to know if Theorem 9 holds with weaker hypotheses. Note that the contradictions achieved in proving Corollary 9.1 were based on the discovery of spanning proper subgraphs isomorphic to

$M$. This suggests that something approaching the conclusion of Theorem 9 may hold even if $B_0$ and $W$ are not excluded.

**Theorem 10.** *If $n \le 7$, there are no primitive graphs in $\mathcal{F}_n$ with minimum degree $\delta \ge 3$.*

**Proof:** Suppose $P \in \mathcal{F}_n$ is a primitive graph with minimum degree $\delta \ge 3$, and $n \le 7$. Let $a$ be a vertex of degree $\delta$ in $P$ with neighbours which include $b$, $c$ and $d$, no two of which are adjacent. Then $b$ has at least two more neighbours, say $u$ and $v$, and these are not adjacent. Also $b$ and $c$ are not duplicates, so without loss of generality we can assume that $c$ has a neighbour $w$ which is not adjacent to $b$. Then $n = 7$, and without loss of generality $c$ is adjacent to $u$. No two of $u$, $v$ and $w$ are adjacent, and $u$ must have at least one more neighbour, so $u$ is adjacent to $d$. But now $u$ duplicates $a$, contradicting the requirement that $P$ is primitive. Consequently $P$ does not exist. □

**Theorem 11.** *If $n \le 7$, there are no primitive graphs in $\mathcal{S}_n \setminus \mathcal{F}_n$ with minimum degree $\delta \ge 3$.*

**Proof:** We begin by showing that $P \in \mathcal{S}_n \setminus \mathcal{F}_n$ does not contain a subgraph isomorphic to $W$ (not necessarily induced). Suppose $P$ has a subgraph $W$, labelled as in Figure 4. We can assume $a$ to be selective for the edge $ab$, so there is a vertex $v \in \tilde{N}(a)$ such that $N(v) \cap N(a) = \{b\}$. Now $\deg v \ge 3$, so $v$ has neighbours $u$ and $w$, not adjacent to $a$. Thus $n = 7$. Now $c$ is not selective for $ac$, so $a$ must be. Therefore $c$ has a neighbour in $\tilde{N}(a)$. So we can suppose $u$ is adjacent to $c$ and then $u$ is not adjacent to $b$ or $d$; but $u$ must be adjacent to $w$, since $\deg u \ge 3$. Now the edge $bc$ is not selective, contradicting minimality of $P$, so $P$ does not have a subgraph $W$.

Let $abca$ be a triangle in $P$. Now $a$ has a neighbour $v$, since $\deg a \ge 3$. But no subgraph $W$ is present in $P$, so $v$ is not adjacent to $b$ or $c$. Similarly $b$ and $c$ have distinct neighbours $u$ and $w$, while $u$ is not adjacent to $a$ or $c$ and $w$ is not adjacent to $a$ or $b$. Without loss of generality $c$ is selective for the edge $bc$, so we can assume that $N(u) \cap N(c) = \{b\}$. But $\deg u \ge 3$, so $u$ is adjacent to $v$ and there is another vertex $x \in P$ adjacent to $u$; then $x \in \tilde{N}(c)$. Thus $n = 7$. Since $\deg w \ge 3$, $w$ must be adjacent to both $x$ and $v$. But neither $a$ nor $c$ can be selective for the edge $ac$, regardless of whether additional edges are present. Since $ac$ is not selective, $P$ is not minimally saturated, so does not exist. □

It follows from Remark 7, Theorems 7, 10 and 11, and Corollaries 8.1 and 9.1, that the list of primitive graphs of order $n \le 7$ in Table 2 is complete.

## 8 Remarks

A number of unsolved problems suggest themselves in this paper. The following seem particularly interesting. Corollary 1.1 suggests

**Problem 1.** *What conditions on $G \in \mathcal{F}_n$ ensure that $G\uparrow$ contains a graph with exactly one triangle?*

Note that the graph obtained by duplicating each vertex of $C_5$ once does not have this property. (We are indebted to a referee for this example.) Again, Theorem 9 suggests

**Problem 2.** *If $G \in \mathcal{S}_n \setminus \mathcal{F}_n$ has minimum degree 2 but no triangular vertex of degree 2, is it true that $G$ contains $M$ as a subgraph?*

We conjecture that the answer to Problem 2 is "yes". Theorem 9 notwithstanding, it is not true that $M$ must appear as an induced subgraph when the conditions of this conjecture hold. The graph in Figure 5 is in $\mathcal{S}_8 \setminus \mathcal{F}_8$, has minimum degree 2 and no triangular vertex of degree 2: it does contain $M$ as a subgraph, but not as an induced subgraph.

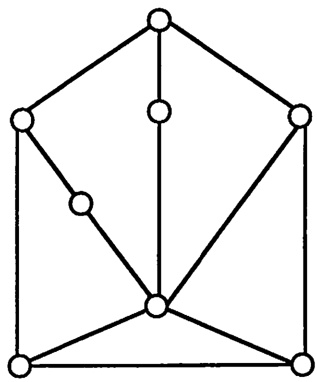

**Figure 5.** A graph with $M$ as a non-induced subgraph.

We would also like to repeat the conjecture, attributed to Simon and Murty, that the answer to the following problem is "yes":

**Problem 3.** *Is the Turán graph the unique graph of largest size in $\mathcal{S}_n$ for every $n$?*

In follow-up papers we plan to further the study of triangle-free and triangle-saturated graphs with a generalization of vertex duplication, the description of several infinite families of primitive minimally saturated graphs (including triangle-free families), and the determination of all maximally unsaturated graphs.

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