Complete closure and regular factors

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Abstract

The complete stability $cs(P_k)$, where P_k denotes the property of having a k-factor, satisfies $cs(P_k) = n + k - 2$, if $1 \leq k \leq 3$, and $n+k-2 \leq cs(P_k) \leq n+k-1$, if $k \geq 4$. A similar result for bipartite graphs with complete biclosure is proved also.

1 Introduction

We consider only undirected graphs with no loops or multiple edges. Let G be a graph with vertex set $V(G)$, edge set $E(G)$ and order $n = |V(G)|$. By $d_G(u)$ we denote the degree of a vertex $u \in V(G)$ in G and $\delta(G)$ denotes the minimum degree of G. A k-regular spanning subgraph of G is called a k-factor of G. The union and the join of two graphs G and H are denoted by $G \cup H$ and $G + H$, respectively.

In [3], Bondy and Chvátal introduced the closure of a graph and the stability of a property. For a non-negative integer l, the *l-closure* $C_l(G)$ of a graph G is the graph obtained from G by recursively joining pairs of non-adjacent vertices whose degree sum is at least l, until no such pair remains. The l-closure is independent of the order of adjunction of the edges and any graph G of order n satisfies

$$G = C_{2n-3}(G) \subseteq C_{2n-4}(G) \subseteq \ldots \subseteq C_0(G) = K_n,$$

where K_n denotes the complete graph of order n. Let P be a property defined on all graphs of order n and let l be a non-negative integer. Then P is said to be *l-stable* if for any graph G of order n and any two non-adjacent vertices u und v of G such that $d_G(u) + d_G(v) \geq l$ and $G + uv$ has property P, then G itself has property P. Every property is $(2n-3)$-stable and every l-stable property is $(l+1)$-stable. The *stability* $s(P)$ of P is the smallest integer l such that P is l-stable. This number usually depends on n and is at most $2n-3$.

If P is a l-stable property and if $C_l(G)$ satisfies P, then G satisfies P. Clearly, it is not always easier to check a property in $C_l(G)$ than in G and so the implication above is often used in the weaker form, when $C_l(G)$ is complete. This led Faudree, Favaron, Flandrin and Li [6] to introduce the

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complete stability $cs(P)$ of a property P, defined on all graphs of order n and satisfied by K_n, to be the smallest integer l such that any graph G of order n satisfies P if $C_l(G)$ is complete. This number usually depends on n and satisfies $cs(P) \leq s(P)$. In [6] it was shown that for some properties equality holds. But the interest in $cs(P_k)$ is motivated by some properties P for which $cs(P)$ is much smaller than $s(P)$.

For a positive integer k we denote the property of having a k-factor by P_k. Note that a graph of odd order n cannot have a k-factor, when k is odd. Thus we may restrict our attention only to situations with kn even. Bondy and Chvátal [3] investigated the stability of P_k. The case $k = 1$ appeared as a part $(2p = n)$ of

Theorem 1 [3] *Let p be an integer with $2 \leq 2p \leq n$. Then for the property P: "G contains p independent edges" holds $s(P) = 2p - 1$.*

Faudree et al. [6] observed that the examples given in [3] showing that $2p-1$ is best possible have their $(2p-2)$-closure complete. Therefore we have $cs(P) = s(P)$ and especially $cs(P_1) = n - 1$.

For $k \geq 2$ Bondy and Chvátal proved

Theorem 2 [3] *Let $2 \leq k \leq n-1$ and kn be even. Then $s(P_k) \leq n+2k-4$ and equality holds if $n \geq 3k+3$.*

Clearly, Theorem 2 and $cs(P_k) \leq s(P_k)$ yield an upper bound for $cs(P_k)$. A lower bound was obtained in [6] by considering the graph $K_{k-1} + (K_{n-k} \cup K_1)$. This graph has its $(n+k-3)$-closure complete and contains no k-factor, since its minimum degree is $k-1$.

Theorem 3 [6] *Let $2 \leq k \leq n-1$ and kn be even. Then $n+k-2 \leq cs(P_k) \leq n+2k-4$.*

In particular, Theorem 3 yields $cs(P_2) = s(P_2) = n$.

The aim of this paper is to improve the upper bound for $cs(P_k)$ from Theorem 3 for $k \geq 3$. Our main result, including the lower bounds and the exact values from above, is

Theorem 4 *Let $1 \leq k \leq n-1$ and kn be even. If $1 \leq k \leq 3$, then $cs(P_k) = n+k-2$ and if $k \geq 4$, then $n+k-2 \leq cs(P_k) \leq n+k-1$.*

So, for $k \geq 4$, there are still two values for $cs(P_k)$ possible. We make the following

Conjecture 1 *Let $1 \leq k \leq n-1$ and kn be even. Then $cs(P_k) = n+k-2$.*

Theorem 4 will be proved in section 3. Thereafter we will briefly discuss what possibly can be done to prove the conjecture by extending the present method. There we will show also that in general it is impossible to decrease the lower bound from Theorem 3 further, even when the problem is restricted to graphs G satisfying the necessary minimum degree condition $\delta(G) \geq k$.

In a final section we will give a complete solution to the corresponding problem for bipartite graphs with complete biclosure. This problem has its origin from Amar, Favaron, Mago and Ordaz [1].

2 Preliminary results

Let G be a graph and let $U, W \subseteq V(G)$. By $e_G(U, V)$ we denote the number of edges joining a vertex in U with a vertex in W. For abbreviation we let $d_G(U) = \sum_{u \in U} d_G(u)$ and $N_G(U) = \bigcup_{u \in U} N_G(u)$, where $N_G(u)$ denotes the neighborhood of the vertex u in G. By $< U >_G$ we denote the subgraph, which is induced by U in G. The number of components of G is $\omega(G)$.

Theorem 5 [3] *Let l be a positive integer and let P be the property "G is l-connected". Then $s(P) = n + l - 2$.*

Theorem 6 [4] *Let n and l be integers with $0 \leq l \leq 2n-3$. If G is a graph of order n with $C_l(G) = K_n$, then $|E(G)| \geq \lfloor \frac{1}{8}(l+2)^2 \rfloor$. Moreover, there exists a graph $H(n,l)$ of order n with $C_l(H(n,l)) = K_n$ and $|E(H(n,l))| = \lfloor \frac{1}{8}(l+2)^2 \rfloor$.*

Proposition 1 *Let G be a graph of order n. Let $D \subset V(G)$ and let l be an integer with $l \geq 2|D|$. If $C_l(G) = K_n$, then $C_{l-2|D|}(G-D) = K_{n-|D|}$.*

Proof. The non-adjacent vertices of $G - D$ can be joined in the same order as in G. ∎

The next result is a special case of *Tutte's f-factor Theorem* [12], which characterizes those graphs that do not have a k-factor and it was first proved by Belck [2].

Let k be a non-negative integer and let D, S be disjoint subsets of $V(G)$. We call a component of $G - (D \cup S)$ an *odd component* (of G with respect to (D, S, k)), if $k|V(C)| + e_G(C, S)$ is odd, and by $q_G(D, S, k)$ we denote the number of odd components. Let $h_G(D, S, k) = k|D| - k|S| + d_{G-D}(S) - q_G(D, S, k)$.

Theorem 7 *Let G be a graph of order n and let k be a non-negative integer with kn even. Then the following statements hold.*
(i) [12] *$h_G(D, S, K)$ is even for any disjoint sets $D, S \subseteq V(G)$;*
(ii) [2], [12] *G does not have a k-factor if and only if G has a k-*Tutte-pair, *that is a pair of disjoint subsets (D, S) of $V(G)$ with $h_G(D, S, k) \leq -2$.*

The next lemma was first observed in [13].

Lemma 1 *Let G be a graph and let u and v be two non-adjacent vertices of G. Then for every non-negative integer k and every pair (D,S) of disjoint subsets of $V(G)$*

$$0 \leq h_{G+uv}(D,S,k) - h_G(D,S,k) \leq 2.$$

A k-Tutte-pair (D,S) of a graph G is called *tight*, if $h_G(D,S,k) = -2$. Furthermore, we call a graph G *k-maximal*, if G has no k-factor and G is edge-maximal with this property. Clearly, every graph without a k-factor is a factor of a k-maximal graph.

From Lemma 1 and Theorem 7 it is easy to derive

Lemma 2 *Let G be a graph of order n and let k be an integer with $1 \leq k \leq n-1$ and kn even. If G is k-maximal, then every k-Tutte-pair of G is tight.*

The usual way to prove the existence of a k-factor in a graph G by Theorem 7 is to assume, contrarily, that G has a k-Tutte-pair. However, it is not easy to obtain a contradiction from the inequality of Theorem 7 (ii). So there have been several attempts to overcome some of the difficulties. In particular, the following two approaches have been used recently in several occasions. Katerinis and Woodall [7] chose a k-Tutte-pair (D,S) maximal with respect to $|D \cup S|$. Thereby they obtained information on the vertices of $V(G) - (D \cup S)$. Enomoto, Jackson, Katerinis and Saito [5] chose a k-Tutte-pair (D,S') minimal with respect to $|S'|$. Thereby they obtained information on the vertices in S'. In [10] we observed that it is possible to combine these ideas by choosing first a k-Tutte-pair (D,S), which is maximal with respect to $|D \cup S|$ and then a k-Tutte-pair (D,S'), where $S' \subseteq S$ and $|S'|$ is minimal. By the following theorem we will extend this idea. The proof of our main result heavily depends on the statements listed therein. Some more statements as well as a generalization to the f-factor problem and further applications appeared in the author's doctoral thesis [9].

Theorem 8 *Let G be a graph of order n and let k be an integer with $1 \leq k \leq n-1$ and kn even. If G is k-maximal with $\delta(G) \geq k$, then there exist tight k-Tutte-pairs (D,S) and (D,S') of G with $S' \subseteq S$ such that the following statements hold.*

C1 $d_{G-D}(x) \geq k+1$ *for every vertex* $x \in V(G) - (D \cup S)$;

C2 $e_G(x,S) \leq k-1$ *for every vertex* $x \in V(G) - (D \cup S)$;

C3 $|V(C)| \geq \max\{3, k+2-|S|\}$ *for every component C of* $G-(D \cup S)$;

C4 *$d_{G-D}(X) \leq k|X| - 2 + c(X) \leq k|X| - 2 + q_G(D, S', k)$ for every $\emptyset \neq X \subseteq S'$, where $c(X)$ denotes the number of odd components C of G with respect to (D, S', k) with $N_G(X) \cap V(C) \neq \emptyset$;*

C5 *the subgraph induced by S' in G has maximum degree at most $k-2$;*

C6 *$d_G(y) = n - 1$ for every vertex $y \in D$;*

C7 *every component of $G - (D \cup S)$ or $G - (D \cup S')$ is complete;*

C8 *every component of $G - (D \cup S)$ or $G - (D \cup S')$ is an odd component of G with respect to (D, S, k) or (D, S', k), respectively;*

C9 *$k - 1 \leq d_{G-D}(x) \leq k$ for every vertex $x \in S - S'$;*

C10 *for every component C' of $G - (D \cup S')$ holds either $V(C') = V(C) \cup M$, where C is a component of $G - (D \cup S)$ and $M \subseteq \{x \in S - S' \mid d_{G-D}(x) = k\}$, or $V(C') = \{y\}$, where $y \in S - S'$ with $d_{G-D}(y) = k - 1$;*

C11 *$q_G(D, S', k) = q_G(D, S, k) + |\{x \in S - S' \mid d_{G-D}(x) = k - 1\}|$.*

Proof. Since G is k-maximal, there exists a k-Tutte-pair (D, S') of G. Let (D, S') be chosen such that $|D|$ is maximal under all k-Tutte-pairs of G and $|S'|$ is minimal under all k-Tutte-pairs of the form (D, S^*). We next choose a finite sequence of k-Tutte-pairs (D, S_i), $i = 1, 2, \ldots, p$, with the following three properties: $S' = S_1$; $S_{i+1} = S_i \cup \{x_i\}$ for a vertex $x_i \in V(G) - (D \cup S_i)$ for $i = 1, 2, \ldots, p-1$; $(D, S_p \cup \{x\})$ is no k-Tutte-pair for every $x \in V(G) - (D \cup S_p)$.

To see that such a sequence exists, start with $(D, S') = (D, S_1)$. Suppose then that we have $(D, S_1), (D, S_2), \ldots, (D, S_r)$ already found. If there exists a vertex $x_r \in V(G) - (D \cup S_r)$ such that $(D, S_r \cup \{x_r\})$ is a k-Tutte-pair, then let $S_{r+1} = S_r \cup \{x_r\}$. Otherwise we are done and $p = r$.

Let now $S = S_p$. Then (D, S) and (D, S') are k-Tutte-pairs of G, which are tight by Lemma 2. We now verify the statements of the theorem.
(C1) and (C2) are proved in [7] in the situation, where (D, S) is a k-Tutte-pair, which is maximal with respect to $|D \cup S|$. But the proof needs only that for every vertex $x \in V(G) - (D \cup S)$ the pairs $(D \cup \{x\}, S)$ and $(D, S \cup \{x\})$ are not k-Tutte-pairs and this is here satisfied by the choice of D and S.

(C3) is an immediate consequence of (C1) and (C2).
To see (C4) let $\emptyset \neq X \subseteq S'$. Then $(D, S' - X)$ is not a k-Tutte-pair by the choice of S' and so

$$\begin{aligned} 0 &\leq h_G(D, S' - X, k) \\ &= k|D| - k(|S'| - |X|) + d_{G-D}(S' - X) - q_G(D, S' - X, k) \end{aligned}$$

$$\begin{aligned} &= h_G(D,S',k) + k|X| - d_{G-D}(X) + q_G(D,S',k) - q_G(D,S'-X,k) \\ &\leq -2 + k|X| - d_{G-D}(X) + c(X) \\ &\leq -2 + k|X| - d_{G-D}(X) + q_G(D,S',k). \end{aligned}$$

(C5) follows from (C4), since for $x \in S'$ it holds $d_{<S'>_G}(x) \leq d_{G-D}(x) - c(\{x\}) \leq k-2$.
Suppose next that there exists $y \in D$ with $d_{G-D}(y) < n-1$. Then there exists a vertex u, which is not joined to y. Now $h_{G+yu}(D,S,k) = h_G(D,S,k) = -2$ contradicts the edge-maximality of G. So (C6) is proved. By the same argument we can prove that every component of $G-(D\cup S_i)$, $i = 1,2,\ldots,p$, is complete and so (C7) is verified.
To verify (C8), suppose that there exists a component C of $G-(D\cup S_i)$ for some $i \in \{1,2,\ldots,p\}$ such that $k|V(C)| + e_G(C,S_i)$ is even. Then there exists no further component C^* of $G-(D\cup S_i)$, since otherwise we could add edges joining C and C^* with $h(D,S_i,k)$ remaining unchanged. Thereby we have especially that $q_G(D,S_i,k) = 0$. Moreover, every vertex $y \in V(C)$ is joined to every vertex $x \in S_i$, since otherwise

$$\begin{aligned} h_{G+yx}(D,S_i,k) &= k|D| - k|S_i| + d_{G-D}(S_i) + 1 - q_{G+yx}(D,S_i,k) \\ &= h_G(D,S_i,k) = -2, \end{aligned}$$

contradicting the edge-maximality of G again. Thus we have $G = K_{n-|S_i|} + < S_i >_G$ with (C6) and (C7). The remainder is done by induction as follows. If $i = p$, then $G = K_{n-|S|} + < S >_G$. Since every vertex $x \in V(C)$ is joined with all vertices of S, we obtain with (C2)

$$|S| = e_G(x,S) \leq k-1.$$

On the other hand we have by $\delta(G) \geq k$

$$\begin{aligned} -2 &= h_G(D,S,k) = k|D| - k|S| + d_{G-D}(S) \\ &\geq k|D| - k|S| + |S|(k-|D|) = k|D| - |S||D|. \end{aligned}$$

Therefore $|D| > 0$ and so

$$|S| \geq \frac{k|D|+2}{|D|} > k,$$

a contradiction.

Let now $1 \leq i \leq p-1$ and suppose by the induction hypothesis that $q_G(D,S_{i+1},k) = \omega(G-(D\cup S_{i+1}))$. By Lemma 2 we have

$$\begin{aligned} 0 &= h_G(D,S_{i+1},k) - h_G(D,S_i,k) \\ &= -k + d_{G-D}(x_i) - q_G(D,S_{i+1},k). \end{aligned} \tag{1}$$

By the structure of G there are only two values for $q_G(D, S_{i+1}, k)$ possible.
CASE 1. $q_G(D, S_{i+1}, k) = 0$.
Then $V(G)$ is the disjoint union of D, S_i and $\{x_i\}$ and by (1) we have $d_{G-D}(x_i) = k$. Therefore $|S_i| = d_{G-D}(x_i) = k$ and $|D| = n - |S_i| - 1 = n - k - 1$. Thus with $\delta(G) \geq k$ we obtain

$$\begin{aligned} -2 &= h_G(D, S_i, k) = k(n-k-1) - k^2 + d_{G-D}(S_i) \\ &\geq k(n-2k-1) + k(k-(n-k-1)) = 0, \end{aligned}$$

a contradiction.
CASE 2. $q_G(D, S_{i+1}, k) = 1$.
With (1) we get here $d_{G-D}(x_i) = k+1$, and then $k+1 = d_{G-D}(x_i) = n - 1 - |D|$. Thus $|D| = n-k-2$. Moreover, since $q_G(D, S_{i+1}, k) = 1$, we know that $D \cup S_i \cup \{x_i\} \neq V(G)$ and so $|S_i| \leq n - |D| - 2 = k$. Now we obtain again with $\delta(G) \geq k$

$$\begin{aligned} -2 &= h_G(D, S_i, k) = k(n-k-2) - k|S_i| + d_{G-D}(S_i) \\ &\geq k(n-k-2) - |S_i|(n-k-2) = (k - |S_i|)(n-k-2). \end{aligned}$$

Since $|S_i| \leq k$, we have $n-k-2 \leq -1$ or, equivalently, $n - 1 \leq k$. By our hypothesis $n-1 \geq k$ we get $n-1 = k$. Now $\delta(G) \geq k$ implies $G = K_n$, but the complete graph is not $(n-1)$-maximal. This contradiction completes the proof of (C8).

Let now $x \in S - S'$. Then there exists an index i such that $x = x_i$ and $S_{i+1} = S_i \cup \{x_i\}$. By Lemma 2 we have

$$\begin{aligned} 0 &= h_G(D, S_{i+1}, k) - h_G(D, S_i, k) \\ &= -k + d_{G-D}(x) - q_G(D, S_{i+1}, k) + q_G(D, S_i, k) \end{aligned}$$

and thus $d_{G-D}(x) = k + q_G(D, S_{i+1}, k) - q_G(D, S_i, k)$. Since the components of $G - (D \cup S_i)$ are complete, x is adjacent to at most one component of $G - (D \cup S_{i+1})$. If there exists such a component, then we have by (C8)

$$q_G(D, S_{i+1}, k) = \omega(G - (D \cup S_{i+1})) = \omega(G - (D \cup S_i)) = q_G(D, S_i, k)$$

and therefore $d_{G-D}(x) = k$.

Otherwise $\{x\}$ is a component of $G - (D \cup S_i)$ and with (C8) we obtain

$$q_G(D, S_{i+1}, k) = \omega(G - (D \cup S_{i+1})) = \omega(G - (D \cup S_i)) - 1 = q_G(D, S_i, k) - 1,$$

and so $d_{G-D}(x) = k - 1$. Thereby, we have already verified (C9). Moreover we have seen that $d_{G-D}(x_i) = k - 1$ if and only if $\{x_i\}$ is a component of $G - (D \cup S_i)$ and $d_{G-D}(x_i) = k$ if and only if x_i is adjacent to one component of $G - (D \cup S_{i+1})$. This implies that for every component C'

of $G-(D\cup S')$ there exists a set $M\subseteq\{x\in S-S'|d_{G-D}(x)=k\}$ such that either $V(C')=V(C)\cup M$, where C is a component of $G-(D\cup S)$, or $V(C')=\{y\}\cup M$, where $y\in S-S'$ with $d_{G-D}(y)=k-1$. So we can verify (C10) by showing $M=\emptyset$ in the latter case. Therefore we suppose that there exists a component C' with $V(C')=\{y\}\cup M$ and $M\neq\emptyset$. Let $j:=\max\{i|x_i\in M\}$. We consider now the pair $(D\cup\{x_j\},S_j\cup\{y\})$. Note that $y\notin S_j$, since otherwise x_j forms a component of $G-(D\cup S_j)$. It holds

$$\begin{aligned} d_{G-(D\cup\{x_j\})}(S_j\cup\{y\}) &= d_{G-D}(S_j)+d_{G-D}(y)-e_G(x_j,S_j\cup\{y\}) \\ &= d_{G-D}(S_j)-1, \end{aligned}$$

and thus

$$\begin{aligned} & h_G(D\cup\{x_j\},S_j\cup\{y\},k) \\ = & \ k|D|-k|S_j|+d_{G-(D\cup\{x_j\})}(S_j\cup\{y\})-q_G(D\cup\{x_j\},S_j\cup\{y\},k) \\ = & \ k|D|-k|S_j|+d_{G-D}(S_j)-1-(q_G(D,S_j,k)-1) \\ = & \ h_G(D,S_j,k) \quad = \quad -2, \end{aligned}$$

contradicting the choice of $|D|$. Finally, (C11) follows immediatly from (C10). ∎

3 Proof of Theorem 4

Since $cs(P_k)\geq n+k-2$ by Theorem 3, we need only to prove the upper bound. The proof is by contradiction. Suppose that there exists a graph G contradicting the theorem. We choose G with minimum order n under all counterexamples. Let $\epsilon_k=1$, if $0\leq k\leq 3$, and $\epsilon_k=0$, if $k\geq 4$. Then G is a graph without a k-factor having its $(n+k-1-\epsilon_k)$-closure complete, and so the $(n+k-2)$-closure of G is complete, which will suffice for major parts of the proof. By Theorem 1 we have $cs(P_1)\leq s(P_1)\leq n-1$ and so $k\geq 2$. Furthermore, we may assume that G is chosen edge-maximal without a k-factor. So G is k-maximal. Moreover, by Theorem 5 we know that G is k-connected and in particular $\delta(G)\geq k$. Therefore, G satisfies the hypotheses of Theorem 8 and so there exist tight k-Tutte-pairs (D,S) and (D,S') of G with $S'\subseteq S$ such that the statements (C1)-(C11) hold. For abbreviation we let $q=q_G(D,S,k)$, $q'=q_G(D,S',k)$, $W=V(G)-(D\cup S)$ and $q_a=|\{x\in S-S'\mid d_{G-D}(x)=k-1\}$. Note that (C11) becomes $q'=q+q_a$.

Claim 1. $|S'|\geq|D|+k-1$.
Let first $|D|\geq 2$. Since G was choosen with minimum order under all counterexamples, there does not exist a graph of order $n-2$ without a $(k-2)$-factor having its $((n-2)+(k-2)-1-\epsilon_{k-2})$-closure complete

(the case $k = 2$ can be included in this formulation, since every graph has a 0-factor). Choose now two vertices $v, w \in D$ and consider the graph $G^* = G - \{v, w\}$. By Propositon 1 and $\epsilon_{k-2} \geq \epsilon_k$ it follows that G^* has its $((n-2) + (k-2) - 1 - \epsilon_{k-2})$-closure complete and so G^* has a $(k-2)$-factor. Therefore $(D - \{v, w\}, S')$ is no $(k-2)$-Tutte pair of G^*, that is $h_{G^*}(D - \{v, w\}, S', k-2) \geq 0$. This yields

$$\begin{aligned} -2 &\geq h_G(D, S', k) - h_{G^*}(D - \{v, w\}, S', k-2) \\ &= k|D| - k|S'| + d_{G-D}(S') - q' - \Big((k-2)(|D|-2) - (k-2)|S'| \\ &\quad + d_{G^*-(D-\{v,w\})}(S') - q_{G^*}(D - \{v, w\}, S', k)\Big) \\ &= 2|D| - 2|S'| + 2k - 4, \end{aligned}$$

or, equivalently, $|S'| \geq |D| + k - 1$ as required.

Let now $|D| \leq 1$. Suppose that $|S'| \leq |D| + k - 2$. If $|D| = 0$, then $h_G(D, S', k) = -2$ and $\delta(G) \geq k$ imply $q' = -k|S'| + d_G(S') + 2 \geq 2$. Thus S' is a cutset of G with $|S'| \leq k - 2$, contradicting the k-connectedness of G. If $|D| = 1$, then $|S'| \leq k - 1$. With $h_G(D, S', k) = -2$ and $\delta(G) \geq k$ we obtain here

$$\begin{aligned} q' &= k - k|S'| + d_{G-D}(S') + 2 \\ &\geq k - k|S'| + |S'|(k-1) + 2 = k - |S'| + 2 \geq 3. \end{aligned}$$

Thus $D \cup S'$ is a cutset of G and so $|S'| = k - 1$, since G is k-connected. Therefore, G is a subgraph of $K_k + (K_a \cup K_b \cup K_c)$ for suitable integers $a \geq b \geq c \geq 1$ with $n = a + b + c + k$. Since $C_{n+k-2}(G) = K_n$, we get $C_{n+k-2}(K_k + (K_a \cup K_b \cup K_c)) = K_n$. Therefore $(a + k - 1) + (b + k - 1) \geq n + k - 2$, but the left-hand-side of this inequality is equal to $n + k - 2 - c$, contradicting $c \geq 1$.

Claim 2. $W \neq \emptyset$ and $q \geq 1$.
By (C8) it suffices to show that $W \neq \emptyset$. Suppose, therefore, that $W = \emptyset$, that is $D \cup S = V(G)$. We have

$$\begin{aligned} -2 &= h_G(D, S, k) = k|D| - k|S| + d_{G-D}(S) \\ &= 2k|D| - kn + 2|E(G - D)|, \end{aligned}$$

and so

$$2|E(G - D)| = kn - 2k|D| - 2. \tag{2}$$

By Proposition 1 we have $C_{n+k-2-2|D|}(G - D) = K_{n-|D|}$ and thus by Theorem 6

$$2|E(G-D)| \geq 2\Big\lfloor \frac{1}{8}(n + k - 2|D|)^2 \Big\rfloor > 2\Big(\frac{1}{8}(n + k - 2|D|)^2 - 1\Big).$$

With (2) we obtain

$$0 < kn - 2k|D| - \frac{1}{4}(n + k - 2|D|)^2 = -\left(\frac{n}{2} - \frac{k}{2} - |D|\right)^2,$$

a contradiction.

Since G has its $(n + k - 1 - \epsilon_k)$-closure complete and since G is not complete itself, there exist two non-adjacent vertices $x, y \in V(G)$ with $d_G(x) + d_G(y) \geq n + k - 1 - \epsilon_k$. By (C6) we have $x, y \notin D$ and also $d_G(x) = d_{G-D}(x) + |D|$ and $d_G(y) = d_{G-D}(y) + |D|$. Thus with Claim 1

$$\begin{aligned} d_{G-D}(x) + d_{G-D}(y) &\geq n + k - 1 - \epsilon_k - 2|D| \\ &= |D| + |S'| + |S - S'| + |W| + k - 1 - \epsilon_k - 2|D| \\ &\geq |W| + |S - S'| + 2k - 2 - \epsilon_k \end{aligned}$$

and so

$$d_{G-D}(x) + d_{G-D}(y) \geq |W| + |S - S'| + 2k - 3, \tag{3}$$

where the inequality is strict for $k \geq 4$. Next we will use (3) in six cases depending on to which of the sets W, $S - S'$ and S' the vertices x and y belong. First we consider the three cases in which x and y belong to the same set.

Case 1. $x, y \in W$.
By (3) and (C2) we have

$$\begin{aligned} |W| + |S - S'| + 2k - 3 &\leq d_{G-D}(x) + d_{G-D}(y) \\ &= |(N(x) \cup N(y)) \cap W| + e_G(x, S) + e_G(y, S) \\ &\leq |W - \{x, y\}| + e_G(x, S) + e_G(y, S) \\ &\leq |W| - 2 + 2(k - 1) = |W| + 2k - 4, \end{aligned}$$

a contradiction.

Case 2. $x, y \in S - S'$.
By (3) and (C9) we have

$$\begin{aligned} |W| + 2k - 1 &\leq |W| + |S - S'| + 2k - 3 \\ &\leq d_{G-D}(x) + d_{G-D}(y) \leq 2k, \end{aligned}$$

and so $|W| \leq 1$. Since $|W| \geq 3q$ by (C3), we obtain even $|W| = 0$, contradicting Claim 2.

Case 3. $x, y \in S'$.
Here we have with (3), (C4) and (C11)

$$\begin{aligned} |W| + q_a + 2k - 3 &\leq |W| + |S - S'| + 2k - 3 \\ &\leq d_{G-D}(x) + d_{G-D}(y) \\ &\leq 2k - 2 + q' = 2k - 2 + q + q_a, \end{aligned}$$

and so $|W| \leq q-1$. Together with $|W| \geq 3q$ by (C3), we obtain $2q \leq 1$, contradicting Claim 2.

Next we consider the cases, where x and y belong to different sets. Without loss of generality we may consider only the following three cases.

CASE 4. $x \in S - S'$ and $y \in W$.
Note that $|N(y) \cap W| \leq |W| - 1 - 3(q-1)$ by (C3). Together with (3), (C9) and (C2) we get

$$\begin{aligned} |W| + 2k - 2 &\leq |W| + |S - S'| + 2k - 3 \leq d_{G-D}(x) + d_{G-D}(y) \\ &\leq k + |N(y) \cap W| + e_G(y, S) \\ &\leq k + |W| - 1 - 3(q-1) + k - 1, \end{aligned}$$

and therefore $q \leq 1$. Thus $q = 1$ by Claim 2. Moreover, equality holds in all estimations above. This yields $d_{G-D}(x) = k$ and $S = S' \cup \{x\}$. By (C11) we have therefore $q' = 1$. But now (C7) implies that x is adjacent to all vertices of W, in particular x is adjacent to y, a contradiction.

CASE 5. $x \in S'$ and $y \in S - S'$.
With (3), (C4) and (C9) we obtain

$$\begin{aligned} |W| + q_a + 2k - 3 &\leq |W| + |S - S'| + 2k - 3 \leq d_{G-D}(x) + d_{G-D}(y) \\ &\leq k - 2 + q' + k = 2k - 2 + q', \end{aligned}$$

and thus $|W| - 1 \leq q' - q_a = q$ by (C11). But, since $|W| \geq 3q$ by (C3), we have $|W| = 0$, contradicting Claim 2.

CASE 6. $x \in S'$ and $y \in W$.
With (3), (C4), (C2) and (C11) we get

$$\begin{aligned} |W| + q_a + 2k - 3 &\leq |W| + |S - S'| + 2k - 3 \leq d_{G-D}(x) + d_{G-D}(y) \\ &\leq k - 2 + q' + |N(y) \cap W| + e_G(y, S) \\ &\leq k - 2 + q' + |W| - 1 - 3(q-1) + k - 1 \\ &= |W| + 2k - 1 - 2q + q_a, \end{aligned}$$

and thus $q \leq 1$. For $k \geq 4$, we have strict inequality in (3) and therefore even $q < 1$, already contradicting Claim 2. Let now $k \leq 3$. Then $q = 1$ by Claim 2 and equality holds in all estimations above. This yields $|S'| = |D| + k - 1$ (from equality in (3)), $q_a = |S - S'|$, $d_{G-D}(x) = k - 2 + c(\{x\}) = k - 2 + q'$ and $e_G(y, S) = k - 1$.

Now we get

$$\begin{aligned} -2 = h_G(D, S', k) &= k|D| - k|S'| + d_{G-D}(S') - q' \\ &= k|D| - k(|D| + k - 1) + d_{G-D}(S' - \{x\}) + (k - 2 + q') - q', \end{aligned}$$

and so $d_{G-D}(S' - \{x\}) = k^2 - 2k$.

Let first $k = 2$. Then $d_{G-D}(S' - \{x\}) = 0$. But since $1 = e_G(y, S) = e_G(y, S')$ by (C7), this implies that x and y are adjacent, a contradiction.

Finally, let $k = 3$. Then $d_{G-D}(S' - \{x\}) = 3$. By $q_a = |S - S'|$ we know that every vertex of $S - S'$ has degree $k - 1 = 2$ and is a component of $G - (D \cup S')$. Furthermore, $c(\{x\}) = q'$ implies that x is adjacent to every vertex of $S - S'$. Moreover we have with (C7) $e_G(y, S') = e_G(y, S) = k - 1 = 2$. Thus we obtain $3 = d_{G-D}(S' - \{x\}) \geq q_a + e(y, S') = q_a + 2$, that is $q_a \leq 1$.

For $q_a = 0$ the information implies that $G - D$ is either the graph G_1 in Figure 1 with an additional edge joining S' and W, which is not incident with x and y, or the graph G_2 in Figure 1 with an additional edge joining x and a vertex in $S' - \{x\}$.

For $q_a = 1$, the information implies that $G - D$ is the graph sketched in Figure 2 with an additional edge joining the vertex in $S - S'$ to a vertex in $S' - \{x\}$.

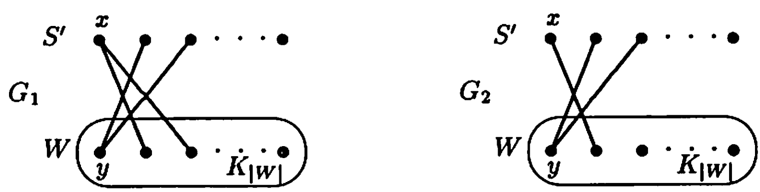

Figure 1

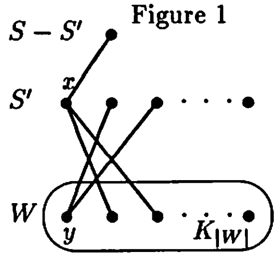

Figure 2

In all cases it is easy to verify that $G - D$ does not have its $(|W| + 3 + q_a)$-closure complete. With Proposition 1 it follows that G does not have its $(|W| + 3 + q_a + 2|D|)$-closure complete. Now this is the final contradiction, since $|W| + 3 + q_a + 2|D| = |W| + |D| + |S'| + |S - S'| + 1 = n + 1$. ∎

We would like to note that with the exception of Case 6 the proof of Theorem 4 works under the weaker hypothesis $C_{n+k-2}(G) = K_n$. Therefore, a proof of Conjecture 1 can possibly be done by finding the missing argument in this case.

Next we present graphs of order $n \geq 3k+1$ with minimum degree at least k and $C_{n+k-3}(G) = K_n$, which do not have a k-factor. Therefore we use the graphs from Theorem 6. For nk even let

$$G(n,k) = \begin{cases} K_{\frac{1}{2}(n-k-1)} + H\left(\frac{1}{2}(n+k+1), 2k-2\right) & \text{if } n-k \text{ is odd} \\ K_{\frac{1}{2}(n-k)} + H\left(\frac{1}{2}(n+k), 2k-3\right) & \text{if } n-k \text{ is even.} \end{cases}$$

We check only the case '$n-k$ is odd'. The case 'n-k is even' can be treated analogously. We have $\delta(G(n,k)) \geq \frac{1}{2}(n-k-1) \geq \frac{1}{2}(3k+1-k-1) = k$. Moreover $G(n,k)$ has its $(n+k-3)$-closure complete, since $H(\frac{1}{2}(n+k+1), 2k-2)$ has its $(2k-2)$-closure complete. To see that $G(n,k)$ does not have a k-factor, choose $D = V(K_{\frac{1}{2}(n-k-1)})$ and $S = V(G) - D$. Then

$$\begin{aligned} h_G(D,S,k) &= -k^2 - k + 2|E(H(\frac{1}{2}(n+k+1), 2k-2))| \\ &= -k^2 - k + 2\left\lfloor \frac{k^2}{2} \right\rfloor \leq -k. \end{aligned}$$

4 Complete biclosure and regular factors

Let G be a bipartite graph with partition (A,B). For an integer l, $0 \leq l \leq |A| + |B| - 1$, the *$l$-biclosure* $BC_l(G,A,B)$ of G (with respect to (A,B)) is the graph obtained from G by recursively joining non-adjacent vertices belonging to different partition sets and having degree sum at least l, until no such pair remains. The l-biclosure is independent of the order of adjunction of the edges, but it may be possible that it depends on the partition, if G is not connected. It holds

$$G = BC_{|A|+|B|-1}(G,A,B) \subseteq \ldots \subseteq BC_0(G,A,B) = K_{|A|,|B|},$$

where $K_{|A|,|B|}$ denotes the complete bipartite graph with partition sets of size $|A|$ and $|B|$.

Besides many other results the following theorem is proved in Amar, Favaron, Mago and Ordaz [1].

Theorem 9 *Let G be a bipartite graph with partition (A,B) and $|A| = |B| = n$. Furthermore, let k be an positive integer and let $u \in A$ and $v \in B$ be non-adjacent vertices with $d_G(u) + d_G(v) \geq n$, if $k = 1$, or $d_G(u) + d_G(v) \geq n + 2k - 3$, if $k \geq 2$. Then G has a k-factor if $G + uv$ has a k-factor. Moreover, the degree bounds are best possible for $n \geq 3k$.*

From Theorem 9 it follows that a bipartite graph with partition sets of size n has a k-factor ($k \geq 2$), if its $(n+2k-3)$-biclosure is complete (this is already mentioned in [1]). In this section we prove

Theorem 10 *Let G be a bipartite graph with partition (A,B) and $|A| = |B| = n$. Furthermore, let $k \leq n$ be a positive integer such that $BC_{n+k-1}(G,A,B) = K_{n,n}$. Then G has a k-factor.*

Theorem 10 is best possible in the sense that $BC_{n+k-1}(G,A,B) = K_{n,n}$ implies $\delta(G) \geq k$, but $BC_{n+k-2}(G,A,B) = K_{n,n}$ does not.

To prove Theorem 10 we need the following results.

Theorem 11 *Let G be a bipartite graph with partition (A,B) and $|A| = |B|$. Then G has no k-factor, if and only if there exist sets $X \subseteq A$ and $Y \subseteq B$ with*

$$e_G(X, B-Y) \leq k|X| - k|Y| - 1,$$

The f-factor version of Theorem 11 is sometimes called *Ore's Theorem*, since it can be derived from results on directed graphs in [11]. In [13] Tutte gave a direct proof and in [8] Lovász and Plummer derived it from the *Max-Flow Min-Cut Theorem.*

Lemma 3 *Let G be a bipartite graph with partition (A,B) and $|A| \leq |B|$. Furthermore, let l be an integer with $0 \leq l \leq |A|+|B|-1$. If $BC_l(G,A,B) = K_{|A|,|B|}$, then*

$$|E(G)| \geq \begin{cases} \left\lfloor \frac{1}{4}(l+1)^2 \right\rfloor & \text{if } 0 \leq l \leq 2|A|-1 \\ |A|(l+1-|A|) & \text{if } l \geq 2|A|. \end{cases}$$

Proof. Let $p = \min\{|A|, \lfloor \frac{l}{2} \rfloor + 1\}$. Starting with $H_0 = G$ we define graphs $H_1, H_2, \ldots, H_p$ recursively as follows. Let $0 \leq i \leq p-1$. If $H_i \neq K_{|A|-i,|B|-i}$, then choose non-adjacent vertices $x \in V(H_i) \cap A$ and $y \in V(H_i) \cap B$ with $d_{H_i}(x) + d_{H_i}(y) \geq l - 2i$. If $H_i = K_{|A|-i,|B|-i}$, then choose adjacent vertices $x \in V(H_i) \cap A$ and $y \in V(H_i) \cap B$ with $d_{H_i}(x) + d_{H_i}(y) \geq l - 2i + 1$. Then let $H_{i+1} = H_i - \{x, y\}$.

To see that $H_1, H_2, \ldots, H_p$ exist we argue as follows. $G = H_0$ has its l-biclosure complete and so, if $H_0 \neq K_{|A|,|B|}$, then there exist nonadjacent vertices $x \in A$ and $y \in B$ with $d_{H_0}(x) + d_{H_0}(y) \geq l$ and thus H_1 can be defined. Moreover, H_1 has its $(l-2)$-biclosure with respect to $(A - \{x\}, B - \{y\})$ complete, since missing edges can be added in the same order as in H_0. By repeating this argument we see that H_{i+1} can be defined, if $H_i \neq K_{|A|-i,|B|-i}$. If $H_i = K_{|A|-i,|B|-i}$, then choose arbitrary vertices $x \in V(H_i) \cap A$ and $y \in V(H_i) \cap B$. Then we have $d_{H_i}(x) + d_{H_i}(y) = |A| + |B| - 2i \geq l + 1 - 2i$ and thus we can define H_{i+1}. Moreover, $H_{i+1} = K_{|A|-(i+1),|B|-(i+1)}$. By repeating the argumentation we see that also $H_{i+2}, \ldots, H_p$ can be defined.

From the definition of $H_0, H_1, \ldots, H_p$ it follows

$$|E(H_i))| \geq l - 2i + |E(H_{i+1})|$$

for $i = 0, \ldots, p-1$. Thus

$$\begin{aligned}|E(G)| &= |E(H_0)| \geq \sum_{i=0}^{p-1}(l-2i) + |E(H_p)| \geq \sum_{i=0}^{p-1}(l-2i) \\ &= p(l-p+1) = \begin{cases} \lfloor \frac{1}{4}(l+1)^2 \rfloor & \text{if } 0 \leq l \leq 2|A|-1 \\ |A|(l+1-|A|) & \text{if } l \geq 2|A|. \end{cases}\end{aligned}$$ ∎

Proof of Theorem 10. The proof is by contradiction. Let G be a graph satisfying the hypotheses of the theorem and having no k-factor. By Theorem 11 there exist $X \subseteq A$ and $Y \subseteq B$ with

$$e_G(X, B-Y) \leq k|X| - k|Y| - 1. \tag{4}$$

We consider the graph $H = G - ((A - X) \cup Y)$. H has its $(n+k-1-(|A-X|+|Y|))$-biclosure with respect to $(X, B-Y)$ complete, since the missing edges can be added in the same order as in G. Therefore $BC_{|X|-|Y|+k-1}(H, X, B-Y) = K_{|X|, n-|Y|}$, because $n+k-1-(|A-X|+|Y|) = |X|-|Y|+k-1$. Note that (4) implies $|X| > |Y|$ and thus $|X|-|Y|+k-1 \geq 0$. Moreover, we have $|X|-|Y|+k-1 \leq |X|+|B-Y|-1$, since $k \leq n$. Thus we may apply Lemma 3 to H and $l = |X|-|Y|+k-1$.

Case 1. $|X|-|Y|+k-1 \leq 2\min\{|X|, |B-Y|\}-1$.

Here we have with (4) and Lemma 3

$$\begin{aligned}k|X|-k|Y|-1 &\geq e_G(X, B-Y) = |E(H)| \\ &\geq \left\lfloor \frac{1}{4}(|X|-|Y|+k)^2 \right\rfloor > \frac{1}{4}(|X|-|Y|+k)^2 - 1,\end{aligned}$$

and thus

$$0 < k|X| - k|Y| - \frac{1}{4}(|X|-|Y|+k)^2 = -\frac{1}{4}(|X|-|Y|-k)^2,$$

a contradiction.

Case 2. $|X|-|Y|+k-1 \geq 2\min\{|X|, |B-Y|\}$.

Subcase 1. $|X| \leq |B-Y|$.

Then we have $|X|-|Y|+k-1 \geq 2|X|$, and therefore $|X| \leq k-1-|Y| < k$. On the other hand, (4) and Lemma 3 imply

$$\begin{aligned}k|X|-k|Y| &> e_G(X, B-Y) = |E(H)| \\ &\geq |X|((|X|-|Y|+k-1)+1-|X|) = k|X|-|X||Y|,\end{aligned}$$

that is $|X| > k$, a contradiction.

SUBCASE 2. $|X| > |B - Y|$.

Then $|X|-|Y|+k-1 \geq 2|B-Y| = 2n-2|Y|$, and so $|Y| \geq 2n-|X|-k+1 \geq n-k+1$. On the other hand, (4) and Lemma 3 imply

$$\begin{aligned} k|X| - k|Y| &> e_G(X, B-Y) = |E(H)| \\ &\geq (n-|Y|)((|X|-|Y|+k-1)+1-(n-|Y|)) \\ &= (n-|Y|)(|X|+k-n), \end{aligned}$$

equivalent to $k(n-|X|) < (n-|Y|)(n-|X|)$ and thus $|Y| < n-k$. This contradiction completes the proof of the theorem. ∎

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