The Forcing Domination Number of a Graph

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ABSTRACT. A vertex of a graph G dominates itself and its neighbors. A set S of vertices of G is a dominating set if each vertex of G is dominated by some vertex of S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A minimum dominating set is one of cardinality $\gamma(G)$. A subset T of a minimum dominating set S is a forcing subset for S if S is the unique minimum dominating set containing T. The forcing domination number $f(S,\gamma)$ of S is the minimum cardinality among the forcing subsets of S, and the forcing domination number $f(G,\gamma)$ of G is the minimum forcing domination number among the minimum dominating sets of G. For every graph G, $f(G,\gamma) \leq \gamma(G)$. It is shown that for integers a, b with b positive and $0 \leq a \leq b$, there exists a graph G such that $f(G,\gamma) = a$ and $\gamma(G) = b$. The forcing domination numbers of several classes of graphs are determined, including complete multipartite graphs, paths, cycles, ladders, and prisms. The forcing domination number of the cartesian product G of k copies of the cycle C_{2k+1} is studied. Viewing the graph G as a Cayley graph, we consider the algebraic aspects of minimum dominating sets in G and forcing subsets.

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1 Introduction

A vertex v in a graph G is said to *dominate* all the vertices in its closed neighborhood $N[v]$. A subset S of $V(G)$ is a *dominating set* of G if $\cup_{v \in S} N[v] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality among the dominating sets of G. A *minimum dominating set* of G is a dominating set of cardinality $\gamma(G)$. The book by Haynes, Hedetniemi, and Slater [6] is devoted entirely to domination in graphs. For graph theory in general, we follow the notation and terminology of [1,3].

Let S be a minimum dominating set of a graph G. A subset T of S such that S is the unique minimum dominating set containing T is called a *forcing subset* for S. The *forcing domination number* $f(S, \gamma)$ of S is the minimum cardinality of a forcing subset for S. The *forcing domination number* $f(G, \gamma)$ of G is the smallest forcing number of a minimum dominating set of G. Hence, if G is a graph with $f(G, \gamma) = a$ and $\gamma(G) = b$, then $0 \leq a \leq b$ and there exists a minimum dominating set S (of cardinality b) containing a forcing subset T of cardinality a. Forcing concepts have been studied for a variety of subjects in graph theory, including such diverse parameters as the chromatic number [2] and the graph reconstruction number [5]. A survey of graphical forcing parameters is discussed in [4].

For the graph G of Figure 1, $\gamma(G) = 2$. For example, the sets $S_1 = \{t, x\}$ and $S_2 = \{v, x\}$ are minimum dominating sets. All other minimum dominating sets of G are similar to S_1 or S_2. Since S_1 is the unique minimum dominating set containing $\{t\}$, it follows that $f(S_1, \gamma) = 1$. On the other hand, S_2 is not the unique minimum dominating set containing $\{v\}$ or $\{x\}$, so $f(S_2, \gamma) = 2$. Consequently, $f(G, \gamma) = 1$.

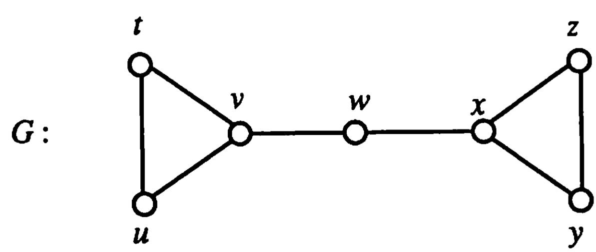

Figure 1

As another example, consider the nontrivial star graph $G = K_{1,n}$. Then G has exactly one minimum dominating set consisting of the unique vertex of degree n in G. Thus $f(G, \gamma) = 0$. In fact, for any graph G, the domination number $\gamma(G) = 1$ if and only if G has a spanning star, i.e., the radius of G is 1. Therefore, if rad $G = 1$, then $f(G, \gamma) \leq 1$ and furthermore, if G has a unique vertex of eccentricity 1, then $f(G, \gamma) = 0$. The following observation will be useful.

Lemma 1. *For a graph G, the forcing domination number $f(G,\gamma) = 0$ if and only if G has a unique minimum dominating set. Moreover, $f(G,\gamma) = 1$ if and only if G does not have a unique minimum dominating set but some vertex of G belongs to exactly one minimum dominating set.*

Proof: The first equivalence is immediate. To prove the second, assume that G does not have a unique minimum dominating set but that some vertex v of G belongs to exactly one minimum dominating set, say S. Then $f(S,\gamma) = 1$, so $f(G,\gamma) = 1$. On the other hand, if G is a graph such that $f(G,\gamma) = 1$, then there is a minimum dominating set S' such that $f(S',\gamma) = 1$. Consequently, S' contains a vertex u such that S' is the unique minimum dominating set containing u. □

Next, we describe a class of graphs for which the domination number is considerably larger than the forcing domination number. For each positive integer b, let P be a path of order $3b$, say $P\colon v_1, v_2, \ldots, v_{3b}$, and let S be a minimum dominating set for P. Since $\gamma(P) = b$ and P has maximum degree 2, each vertex v of S dominates three vertices in P and every vertex of P is dominated exactly once. Thus S contains neither v_1 nor v_{3b}. Since v_1 must be adjacent to a vertex of S, it follows that $v_2 \in S$. Hence S must contain $v_5, v_8, \ldots, v_{3b-1}$. Therefore S is uniquely determined and $f(P,\gamma) = 0$. Thus for the path of order $3b$, the domination number is b while the forcing domination number is 0.

The following result is a direct consequence of Lemma 1.

Corollary 2. *For a graph G, the forcing domination number $f(G,\gamma) > 1$ if and only if every vertex of each minimum dominating set belongs to at least two minimum dominating sets.*

We have already noted that if G is a graph with $f(G,\gamma) = a$ and $\gamma(G) = b$, then $0 \le a \le b$. We now show the corresponding realization result: For every pair a, b of integers, with b positive and $0 \le a \le b$, there exists a graph G such that $f(G,\gamma) = a$ and $\gamma(G) = b$.

Theorem 3. *Every pair a, b of integers, with b postive and $0 \le a \le b$, can be realized as the forcing domination number and domination number, respectively, of some graph.*

Proof: We have already seen that when $G = P_{3b}$, we have $f(G,\gamma) = 0$ and $\gamma(G) = b$. Thus, we assume that $0 < a \le b$. Let $P\colon v_1, v_2, \ldots, v_{3b}$ be a path of order $3b$. Recall that the unique minimum dominating set for P is $S = \{v_2, v_5, \ldots, v_{3b-1}\}$. To obtain a graph G with the desired property, we add a new vertices $v_2', v_5', \ldots, v_{3a-1}'$ to P and for $1 \le i \le a$, we join v_{3i-1}' to both v_{3i-1} and the neighbors of v_{3i-1} in P. Hence $\gamma(G) = b$. An example is shown in Figure 2 in the case where $a = 2$ and $b = 4$.

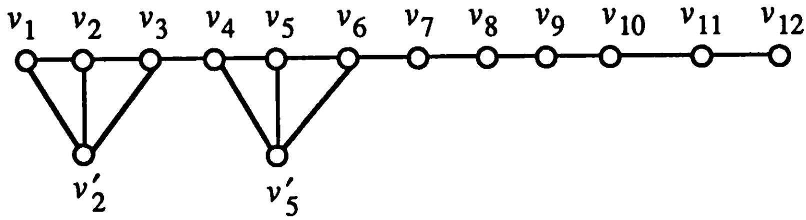

Figure 2

Clearly the set S is a minimum dominating set for G, and the subset $T = \{v_{3i-1} \mid i = 1, 2, \ldots, a\}$ is a forcing subset for S. So $f(G, \gamma) \leq a$. Let S' be a minimum dominating set for G. Since S' is a minimum dominating set for a subgraph P_{3b} as well, and P_{3b} has a unique minimum dominating set, it follows that S' must contain v_{3i-1} or v'_{3i-1} for $i = 1, 2, \ldots, a$. Let T' be a forcing subset for S'. To prove that T' contains v_{3i-1} or v'_{3i-1} for each i ($1 \leq i \leq a$), we suppose, to the contrary, that for some j ($1 \leq j \leq a$) neither v_{3j-1} nor v'_{3j-1} is in T'. Then, $S' - \{v_{3j-1}, v'_{3j-1}\} \cup \{v_{3j-1}\}$ and $S' - \{v_{3j-1}, v'_{3j-1}\} \cup \{v'_{3j-1}\}$ are two minimum dominating sets for G containing T', which contradicts the fact that T' is a forcing subset for S'. Thus, $|T'| \geq a$, and hence $f(S'\gamma) \geq a$. Therefore, $f(G, \gamma) \geq a$ and so $f(G, \gamma) = a$. □

P:

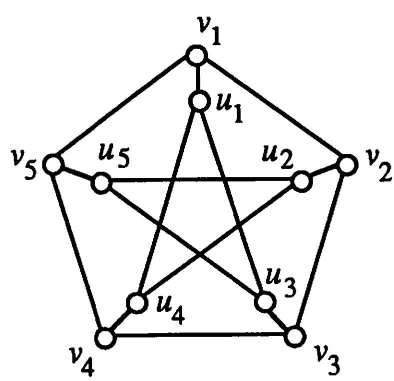

Figure 3

For a graph G and a subset S of vertices of G, the *closed neighborhood* $N[S]$ of S, is the union of the closed neighborhoods of the vertices of S, that is, $N[S] = \cup_{v \in S} N[v]$. For vertices $v_1, v_2, \ldots, v_n$ of G, we write $N[v_1, v_2, \ldots, v_n]$ for $N[\{v_1, v_2, \ldots, v_n\}]$. We now determine the forcing domination numbers for several well-known graphs or classes of graphs. We begin with the famous Petersen graph P. It is well-known that $\gamma(P) = 3$. Consider the labeling of the Petersen graph P shown in Figure 3. First, $\{v_1, v_4, u_5\}$ and $\{v_1, u_3, u_4\}$ are two minimum dominating sets for P, both containing the vertex v_1. Since P is vertex-transitive, *every* vertex belongs to at least two minimum dominating sets and hence $f(P,\gamma) \geq 2$. Next, we show that $f(P,\gamma) \leq 2$. Let $S = \{v_1, v_4, u_5\}$. Now $P - N[v_4, u_5]$ is the path u_1, v_1, v_2, whose vertices are uniquely dominated in P by v_1. Thus $\{v_4, u_5\}$ is a forcing subset for S and hence $f(S,\gamma) = 2$. Therefore, $f(P,\gamma) \leq 2$ and, consequently, $f(P,\gamma) = 2$.

Next, we determine the forcing domination number of complete multipartite graphs. Let $p_1, p_2, \ldots, p_k$ be $k \geq 2$ positive integers with $p_1 \leq p_2 \leq \cdots \leq p_k$ and $p_1 + p_2 + \cdots + p_k = n$, and let $G = K(p_1, p_2, \ldots, p_k)$. For $i = 1, 2, \ldots, k$, denote the p_i vertices of G in the ith partite set by $v_{i,1}, v_{i,2}, \ldots, v_{i,p_i}$. First, suppose that $p_1 = 1$ and $p_2 > 1$. Then $v_{1,1}$ is the unique vertex of G of degree $n-1$ and hence $\{v_{1,1}\}$ is the unique minimum dominating set for G. Therefore, $f(G,\gamma) = 0$.

Next, suppose that $p_1 = p_2 = 1$. Then both $\{v_{1,1}\}$ and $\{v_{2,1}\}$ are minimum dominating sets for G and hence $f(G,\gamma) = 1$. Finally, suppose that $p_1 > 1$. Since any two adjacent vertices form a minimum dominating set for G, every vertex belongs to at least two minimum dominating sets and hence $1 < f(G,\gamma) \leq \gamma(G) = 2$. Therefore, $f(G,\gamma) = 2$. In summary,

$$f(K(p_1,p_2,\ldots,p_k),\gamma) = \begin{cases} 0 & \text{if } p_1 = 1 \text{ and } p_2 > 1 \\ 1 & \text{if } p_1 = p_2 = 1 \\ 2 & \text{if } p_1 > 1 \end{cases}$$

Since K_n is the complete n-partite graph $K(1,1,\ldots,1)$, it follows that for every integer $n \geq 2$, the forcing domination number $f(K_n,\gamma) = 1$.

The *corona* G° of a graph G of order n is that graph obtained from G by joining one new vertex to each vertex of G. Thus the order of G° is $2n$. For each end-vertex v of G°, every minimum dominating set for G° must contain v or its neighbor, and hence $\gamma(G^\circ) = n$. We now show that the forcing domination number of the corona of a graph of order n is n.

Let G be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and let v_i' be the new vertex joined to v_i in G°. Then each minimum dominating set S must contain v_i or v_i' for $i = 1, 2, \ldots, n$. Therefore, each subset T of S of order $n-1$ cannot dominate some vertex v_j'. Thus $T \cup \{v_j'\}$ and $T \cup \{v_j\}$ are two minimum dominating sets for G° so that $f(G^\circ,\gamma) > n-1$. Since $\gamma(G^\circ) = n$, it follows that $f(G^\circ,\gamma) = n$.

2 Forcing Domination Numbers of Paths and Cycles

We begin with the forcing domination number of paths. Since $f(P_n,\gamma) = 1$ for $n = 2$, 3, we consider paths of order at least 4.

Theorem 4. *For the path P_n of order $n \geq 4$,*

$$f(P_n, \gamma) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 3 \\ 1 & \text{if } n \equiv 2 \pmod 3 \\ 2 & \text{if } n \equiv 1 \pmod 3 \end{cases}$$

Proof: Let P be a path of order n, say $P: v_1, v_2, \ldots, v_n$. If $n \equiv 0 \pmod 3$, then we have already seen that $f(P, \gamma) = 0$. Assume then that $n \equiv 2 \pmod 3$. Thus $n = 3j + 2$ for some positive integer j and $\gamma(P) = j + 1$. Consider the minimum dominating set $S = \{v_1, v_4, v_7, \ldots, v_{3j+1}\}$ for P. We show that $\{v_1\}$ is a forcing subset for S. Since the $3j$ vertices of the path $P - N[v_1]$ are uniquely dominated by the j vertices of $S - \{v_1\}$, it follows that $f(S, \gamma) \leq 1$. Since S and $S - \{v_1\} \cup \{v_2\}$ are two distinct minimum dominating sets for P, we have that $f(P, \gamma) > 0$. Therefore $f(P, \gamma) = 1$.

Finally, assume that $n \equiv 1 \pmod 3$. Then $n = 3j + 1$ for some positive integer j, and $\gamma(P) = j + 1$. Let $S = \{v_1, v_3, v_6, v_9, \ldots, v_{3j}\}$, a minimum dominating set for P. We begin by showing $\{v_1, v_3\}$ is a forcing subset of S. Since the path $P - N[v_1, v_3]$ of order $3(j - 1)$ is uniquely dominated by the $j - 1$ vertices of $S - \{v_1, v_3\}$, it follows that $f(P, \gamma) \leq 2$. To show that $f(P, \gamma) \geq 2$, we verify that every vertex of P belongs to two minimum dominating sets. Let $S_1 = \{v_1, v_3, v_6, \ldots, v_{3j}\}$, $S_2 = \{v_2, v_3, v_6, \ldots, v_{3j}\}$, $S_3 = \{v_2, v_4, v_7, \ldots, v_{3j+1}\}$, $S_4 = \{v_1, v_4, v_7, \ldots, v_{3j+1}\}$, $S_5 = \{v_2, v_5, \ldots, v_{3j-1}, v_{3j+1}\}$ and $S_6 = \{v_2, v_5, \ldots, v_{3j-1}, v_{3j}\}$. Then for each $i = 1, 2, \ldots, 6$, the set S_i is a miminum dominating set for P and furthermore every vertex of P belongs to at least two of these sets. Hence $f(P, \gamma) \geq 2$, and so $f(P, \gamma) = 2$. □

Next, we present an upper bound for the forcing domination number of every minimum dominating set of a path. The proof is tedious and is therefore omitted.

Theorem 5. *For every minimum dominating set S of P_n, where $n \geq 2$, the forcing domination number $f(S, \gamma) \leq 4$.*

Next, we determine $f(C_n, \gamma)$ for all cycles. Since $f(C_3, \gamma) = 1$ and $f(C_n, \gamma) = 2$ when n is 4 or 5, we consider cycles of order at least 6.

Theorem 6. *For the cycle C_n of order $n \geq 6$,*

$$f(C_n, \gamma) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3 \\ 2 & \text{otherwise.} \end{cases}$$

Proof: Let C be a cycle of order n, say $C: v_1, v_2, \ldots, v_n, v_1$. Suppose first that $n \equiv 0 \pmod 3$, say that $n = 3k$ for some positive integer k. Let $S = \{v_2, v_5, \ldots, v_{3k-1}\}$. Then S is a minimum dominating set for C.

We show that $f(S,\gamma) = 1$. The path $P = C - N[v_2]$ is a path of order $3(k-1)$ which has a unique minimum dominating set by Theorem 4, namely $S - \{v_2\}$. Thus $f(S,\gamma) \leq 1$. Since C has other minimum dominating sets, $f(C,\gamma) = 1$.

Next suppose that $n \equiv 1 \pmod 3$, say that $n = 3\ell + 1$ for some positive integer ℓ. For each $i = 1, 2, \ldots, n$, the path $C - N[v_i]$ is of order $3(\ell - 1) + 1$ and is not dominated uniquely, say S_1 and S_2 are two minimum dominating sets for $C - N[v_i]$. Thus $S_1 \cup \{v_i\}$ and $S_2 \cup \{v_i\}$ are two minimum dominating sets for C and hence every vertex of C belongs to at least two minimum dominating sets. Therefore $f(C,\gamma) > 1$. The set $S = \{v_2, v_3, v_6, \ldots, v_{3\ell}\}$ is a minimum dominating set for C. Now since $\gamma(C) = \ell + 1$ and $C - N[v_2, v_3]$ is a path of order $3(\ell - 1)$ that is dominated uniquely by $\ell - 1$ vertices, each minimum dominating set containing v_2 and v_3 must also contain $v_6, v_9, \ldots, v_{3\ell}$. So the subset $\{v_2, v_3\}$ is a forcing subset for S. Thus $f(S,\gamma) = 2$ and hence $f(C,\gamma) = 2$.

Finally, let $n \equiv 2 \pmod 3$ say that $n = 3m + 2$ for some positive integer m. Then for each $i = 1, 2, \ldots, n$, the path $C - N[v_i]$ is of order $3(m-1)+2$ and is not dominated uniquely. Therefore, as before, every vertex of C belongs to at least two minimum dominating sets and thus $f(C,\gamma) > 1$. To see that $f(C,\gamma) = 2$, we let $S = \{v_2, v_4, v_7, \ldots, v_{3m+1}\}$. Then S is a minimum dominating set for C and $C - N[v_2, v_4]$ is a path of order $3(m-1)$. As before, since $\gamma(C) = m + 1$ and $C - N[v_2, v_4]$is dominated uniquely by $m-1$ vertices, each minimum dominating set containing v_2 and v_4 must also contain $v_7, v_{10}, \ldots, v_{3m+1}$. Thus $\{v_2, v_4\}$ is a forcing subset for S. Hence $f(S,\gamma) = 2$ so that $f(C,\gamma) = 2$. □

3 Forcing Domination Numbers of Ladders $P_n \times K_2$ and Prisms $C_n \times K_2$

We now consider the ladders $P_n \times K_2$ for $n \geq 2$. Now when $n = 2$, the ladder $P_2 \times K_2 = C_4$ so that $f(P_2 \times K_2, \gamma) = 2$. In the following theorem, we use the facts that $\gamma(P_{2k-1} \times K_2) = k$ and $\gamma(P_{2k} \times K_2) = k + 1$.

Theorem 7. *For every integer $k \geq 2$, the forcing domination number $f(P_{2k-1} \times K_2, \gamma) = 1$.*

Proof: Let $G = P_{2k-1} \times K_2$, where the vertices of G are labeled as in Figure 4.

First, suppose that $k = 2$. Then $S_1 = \{u_1, v_3\}$, $S_2 = \{u_2, v_2\}$, and $S_3 = \{u_3, v_1\}$ are the only minimum dominating sets for $P_3 \times K_2$. Since each set S_i $(i = 1, 2, 3)$ is the unique minimum dominating set containing the vertex u_i, it follows that $f(S_i,\gamma) = 1$ and hence $f(P_3 \times K_2, \gamma) = 1$. Thus assume that $k \geq 3$. We begin by showing that every minimum dominating set contains either u_1 or v_1. Suppose, to the contrary, that some minimum dominating set S' of G contains neither u_1 nor v_1. Then both u_2 and v_2 must be in S'. Then $G - N[v_2, u_2] = P_{2(k-2)} \times K_2$, whose vertices must be dominated by $k-1$ vertices, producing a contradiction. Therefore, as claimed, either u_1 or v_1 is in every minimum dominating set, and so too is u_{2k-1} or v_{2k-1}.

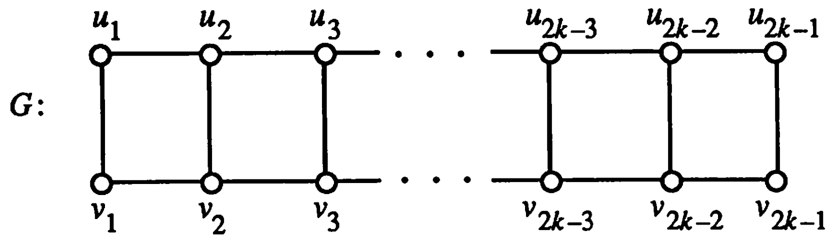

Figure 4

Now let S be a minimum dominating set of G. Then S contains u_1 or v_1, say u_1. Similarly, S must contain either u_{2k-1} or v_{2k-1}. Thus u_1 and either u_{2k-1} or v_{2k-1} dominate a total of six vertices of G. The remaining $4(k-2)$ vertices of G must be dominated by $k-2$ vertices of S, each of which has degree 3. Hence the closed neighborhoods of each pair of distinct vertices in S must be pairwise disjoint. Since $u_1 \in S$ and v_2 is dominated by some vertex (of degree 3) in G, it follows that $v_3 \in S$. Since the closed neighborhoods of the vertices in S are pairwise disjoint, the remaining vertices of S are uniquely determined, where $u_{2k-1} \in S$ if k is odd and $v_{2k-1} \in S$ if k is even. Thus $f(S, \gamma) \leq 1$, where S consists of the vertices $u_1, v_3, u_5, v_7, \ldots$, and either u_{2k-1} (if k is odd) or v_{2k-1} (if k is even). However, then the set S' that consists of the vertices $v_1, u_3, v_5, u_7, \ldots$ and v_{2k-1} if k is odd or u_{2k-1} if k is even, is another minimum dominating set for G. Hence $f(G, \gamma) > 0$, so $f(P_{2k-1} \times P_2, \gamma) = 1$ for $k \geq 2$. □

For each integer $k \geq 2$, let G_k be the graph obtained from $P_{2k-1} \times K_2$, whose vertices are labeled as in Figure 4, by joining a new vertex w to one of the vertices of degree 2 in $P_{2k-1} \times K_2$, say u. The graphs G_2 and G_3 are shown in Figure 5. Then any minimum dominating set S for G_k must include w or u and since $\gamma(G_k) = k$, it follows that $u \in S$. Combining these observations with Theorem 11, we have the following corollary.

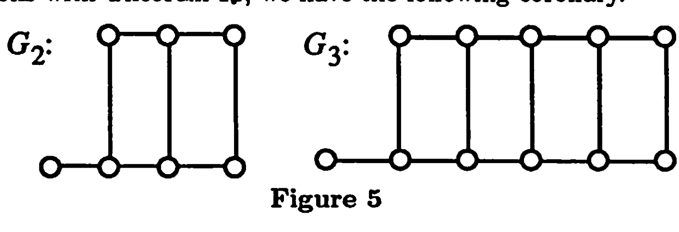

Figure 5

Corollary 8. *For every integer $k \geq 2$, the forcing domination number $f(G_k, \gamma) = 0$.*

Next, for each integer $k \geq 2$, let H_k be the graph obtained from $P_{2k-1} \times K_2$, whose vertices are labeled as in Figure 4, by adding two vertices w_1 and w_2 to $P_{2k-1} \times K_2$ along with the edges w_1v_1 and either w_2u_{2k-1} if k is even or w_2v_{2k-1} if k is odd. The graphs H_2 and H_3 are shown in Figure 6. Now $\gamma(H_k) = k$ and any minimum dominating set for H_k must contain v_1 and either u_{2k-1} or v_{2k-1}. Thus we have the following corollary.

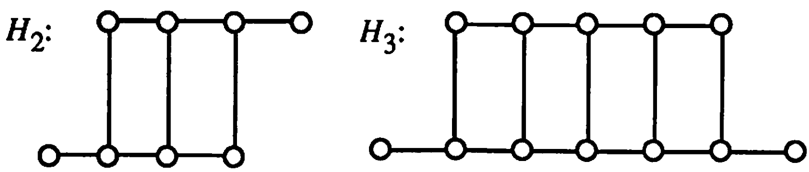

Figure 6

Corollary 9. *For every integer $k \geq 2$, the forcing domination number $f(H_k, \gamma) = 0$.*

We now determine the forcing domination number of the ladder $P_{2k} \times K_2$ for each positive integer k.

Theorem 10. *For $k \geq 1$, the forcing domination number $f(P_{2k} \times K_2, \gamma) = 2$.*

Proof: Let $G = P_{2k} \times K_2$, where the vertices of G are labeled as in Figure 7.

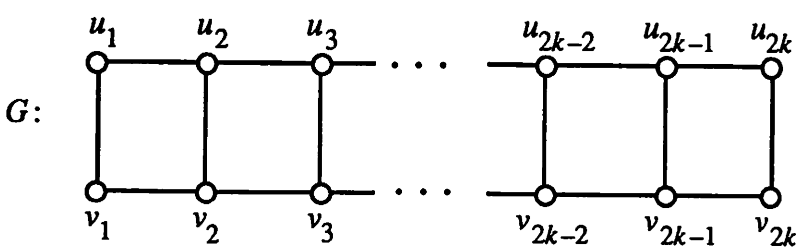

Figure 7

Let S consist of the vertices $u_1, v_2, u_4, v_6, u_8, v_{10}, u_{12}, \ldots$ and u_{2k} if k is even or v_{2k} if k is odd. Then S is a minimum dominating set for G. We show that $\{u_1, v_2\}$ is a forcing subset for S. We begin by showing that any minimum dominating set containing u_1 and v_2 cannot also contain u_2 or v_3. Let S' be a minimum dominating set for G containing u_1 and v_2. If $u_2 \in S'$, then the subgraph $H = G - N[u_1, u_2, v_2]$ is isomorphic to $P_{2(k-1)-1} \times K_2$ and the vertices of H must be dominated by the remaining $k - 2$ vertices of S'. Since $\gamma(P_{2(k-1)-1} \times K_2) = k - 1$, this is impossible. Thus $u_2 \notin S'$. Finally, if $v_3 \in S'$, then the subgraph $H = G - N[u_1, v_2, v_3]$ of G has a subgraph isomorphic to $P_{2(k-2)} \times K_2$ and H must be dominated by the remaining $k-2$ vertices of S'. As before, since $\gamma(P_{2(k-2)} \times K_2) = k-1$, this is impossible. Thus neither u_2 nor v_3 belong to S'. Now $G - N[u_1, v_2] \cong G_{k-1}$ and by Corollary 12, the forcing domination number $f(G_{k-1}, \gamma) = 0$. Thus $\{u_1, v_2\}$ is a forcing subset for S and $f(G, \gamma) \leq 2$. Next, we will show $f(G, \gamma) > 1$. First, $S' = \{v_1, v_2, u_4, v_6, u_8, v_{10}, u_{12}, \ldots, w\}$ where $w = u_{2k}$ if k is even or $w = v_{2k}$ if k is odd is a minimum dominating set for G. Thus v_i and u_j for $1 \leq i, j \leq 2_k$ with $i \equiv 2 \pmod 4$ and $j \equiv 0 \pmod 4$ belong to two distinct minimum dominating sets. By using the automorphisms σ and τ of G where $\sigma(v_i) = u_i$ and $\tau(u_1) = u_{2k}$, we see that each vertex in G lies in two distinct minimum dominating sets, and so $f(G, \gamma) \geq 2$. Combining the two inequalities, we get $f(P_{2k} \times K_2, \gamma) = 2$. □

Combining Theorems 7 and 10, we have the following result for the ladder $P_n \times K_2$.

Corollary 11. *For every integer $n \geq 2$,*

$$f(P_n \times K_2, \gamma) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

In a manner similar to the proof of Theorem 10, we can use Corollary 9 to establish the following result about prisms.

Theorem 12. *For every integer $n \geq 3$,*

$$f(C_n \times K_2, \gamma) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4 \\ 2 & \text{if } n \equiv 1 \pmod 4 \\ 3 & \text{if } n \equiv 2 \pmod 4 \\ 2 & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

Proof: Let $G = C_n \times K_2$, where the vertices of G are labeled as in Figure 8.

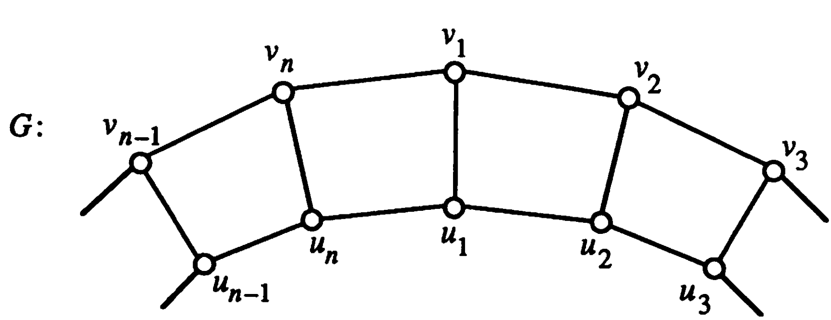

Figure 8

Assume first that $n \equiv 0 \pmod 4$, so $n = 4k$ for some positive integer k. Since $\gamma(G) = 2k$ and G is 3-regular, every vertex in G is dominated exactly once by a vertex of each minimum dominating set. Let $S = \{v_1, u_3, v_5, u_7, \ldots, v_{n-3}, u_{n-1}\}$. We show that $\{v_1\}$ is a forcing subset for S. Let S' be a minimum dominating set for G containing v_1. Since every vertex of G is dominated exactly once by vertices of S', it follows that $u_2 \notin S'$. Next, some neighbor of u_2 must belong to S'. Since $v_1 \in S'$, we have that $u_1, v_2 \notin S'$. Thus $u_3 \in S'$. Continuing in this manner, we see that $S = S'$. Hence $f(S,\gamma) \leq 1$ and so $f(G,\gamma) \leq 1$. Since $\{v_2, u_4, v_6, u_8, \ldots, v_{n-2}, u_n\}$ is another minimum dominating set for G, by Lemma 1, $f(G,\gamma) > 0$. Therefore, $f(G,\gamma) = 1$.

Next suppose that $n \equiv 1 \pmod 4$, so $n = 4k+1$ for some positive integer k. Now $\gamma(G) = 2k+1$ and $S = \{v_1, v_2, u_4, v_6, u_8, \ldots, u_{4k-4}, v_{4k-2}, u_{4k}\}$ is a minimum dominating set for G. We will show that the subset $T = \{v_1, v_2\}$ is a forcing subset for S. Now $G - N[T] = H_{2k-1}$, and from Corollary 9, $G - N[T]$ must be dominated by $2k-1$ vertices of G. Because of degree and order conditions, none of the vertices in $N[T]$ may be used to complete a minimum dominating set of G, so by Corollary 9 and Lemma 1, we have that S is the unique minimum dominating set containing T. Thus $f(S,\gamma) \leq 2$, so $f(G,\gamma) \leq 2$. We now show that every vertex of G belongs to at least two minimum dominating sets. Let w be a vertex of G. Since G is vertex-transitive, there exist automorphisms σ and τ of G such that $\sigma(v_1) = w$ and $\tau(v_4) = w$. Next $\sigma(S)$ and $\tau(S)$ are two minimum dominating sets for G containing w. Furthermore, $\sigma(S) \cap N(w) \neq \emptyset$ while $\tau(S) \cap N(w) = \emptyset$; so $\sigma(S) \neq \tau(S)$. Therefore, $f(G,\gamma) > 1$ and hence $f(G,\gamma) = 2$.

Next assume that $n \equiv 2 \pmod 4$. Thus $n = 4k+2$ for some positive integer k. Then $\gamma(G) = 2k+2$ and $S_1 = \{v_1, v_2, v_3, u_5, v_7, u_9, v_{11}, \ldots, u_{4k-3}, v_{4k-1}, u_{4k+1}\}$ and $S_2 = \{v_1, u_2, v_3, u_5, v_7, u_9, v_{11}, \ldots, u_{4k-3}, v_{4k-1}, u_{4k+1}\}$ are two minimum dominating sets for G. We now show that $T_1 = \{v_1, v_2, v_3\}$ and $T_2 = \{v_1, u_2, v_3\}$ are forcing subsets for S_1 and S_2, respectively. Let $U = N[T_1] = N[T_2]$. Then $G - U = H_{2n-1}$, and from Corollary 13, $G - U$ must be dominated by $2n-1$ vertices in G. Because of degree and order conditions, none of the vertices in U may be used to complete a minimum dominating set for G; so by Corollary 13 and Lemma 1, we have that S_1 and S_2 are the unique minimum dominating sets containing T_1 and T_2, respectively. Thus $f(S_i,\gamma) \leq 3$ for $i = 1,2$ and hence $f(G,\gamma) \leq 3$. It remains to show that each pair of vertices in G belongs to at least two distinct minimum dominating sets. Let u and v be two distinct vertices of G. We consider two cases, depending on whether u and v belong to a common n-cycle induced by $\{u_i \mid i = 1,2,\ldots,n\}$ or $\{v_i \mid i = 1,2,\ldots,n\}$.

Case 1. *Suppose that u and v belong to the n-cycle induced by $\{u_i \mid i = 1,2,\ldots,n\}$ or $\{v_i \mid i = 1,2,\ldots,n\}$.*

If $d(u,v) \equiv 0 \pmod 4$, say $d(u,v) = 4m$, then the automorphism σ of

G such that $\sigma(v_3) = u$ and $\sigma(v_{4m+3}) = v$ gives two minimum dominating sets, namely $\sigma(S_1)$ and $\sigma(S_2)$, both containing u and v. Now let $d(u,v) \equiv 1 \pmod 4$, say $d(u,v) = 4m+1$. Then the automorphisms σ and τ of G such that $\sigma(v_2) = u$, $\sigma(v_{4m+3}) = v$, $\tau(v_2) = v$, and $\tau(v_{4m+3}) = u$ give the two minimum dominating sets $\sigma(S_1)$ and $\tau(S_1)$, both containing u and v. Next, if $d(u,v) \equiv 2 \pmod 4$, say $d(u,v) = 4m+2$, then the automorphism σ of G such that $\sigma(v_1) = u$ and $\sigma(v_{4m+3}) = v$ give two minimum dominating sets $\sigma(S_1)$ and $\sigma(S_2)$, each of which contains both u and v. Finally, if $d(u,v) \equiv 3 \pmod 4$, say $d(u,v) = 4m+3$, then the automorphisms σ and τ of G such that $\sigma(u_2) = u$, $\sigma(u_{4(m+1)+1}) = v$, $\tau(u_2) = v$, and $\tau(u_{4(m+1)+1}) = u$ give two minimum dominating sets $\sigma(S_2)$ and $\tau(S_2)$, both containing u and v.

Case 2. *Suppose that u and v do not both belong to the same n-cycle, induced by $\{u_i \mid i = 1, 2, \ldots, n\}$ or $\{v_i \mid i = 1, 2, \ldots, n\}$.*

First, if $d(u,v) \equiv 0 \pmod 4$, say $d(u,v) = 4m$, then the automorphisms σ and τ of G such that $\sigma(v_2) = u$, $\sigma(u_{4m+1}) = v$, $\tau(v_2) = v$, and $\tau(u_{4m+1}) = u$ give the two minimum dominating sets $\sigma(S_1)$ and $\tau(S_1)$, both containing u and v. Next, let $d(u,v) \equiv 1 \pmod 4$, say $d(u,v) = 4m+1$. If $m = 0$, then $G - N[u,v] = P_{4k-1} \times K_2$, which must be dominated by $2k$ vertices. By Theorem 11, there are two minimum dominating sets S and S' for $G - N[u,v]$ and hence $S \cup \{u,v\}$ and $S' \cup \{u,v\}$ are two minimum dominating sets for G containing u and v. If $d(u,v) \equiv 2 \pmod 4$, say $d(u,v) = 4m+2$, then the automorphisms σ and τ of G such that $\sigma(u_2) = u$, $\sigma(v_{4m+3}) = v$, $\tau(u_2) = v$, and $\tau(v_{4m+3}) = u$ give the two minimum dominating sets $\sigma(S_2)$ and $\tau(S_2)$, both containing u and v. Finally if $d(u,v) \equiv 3 \pmod 4$, say $d(u,v) = 4m+3$, then the automorphisms σ and τ of G such that $\sigma(v_3) = u$, $\sigma(u_{4(m+1)+1}) = v$, $\tau(v_3) = v$, and $\tau(u_{4(m+1)+1}) = u$ give the two minimum dominating sets $\sigma(S_1)$ and $\tau(S_1)$, both containing u and v. Thus $f(C_n \times K_2, \gamma) \geq 3$ and hence $f(C_n \times K_2, \gamma) = 3$, when $n \equiv 2 \pmod 4$.

Finally, assume that $n \equiv 3 \pmod 4$, so that $n = 4p+3$ for some integer p. Then $\gamma(G) = 2p+2$ and $S = \{v_1, u_2, v_4, u_6, v_8, \ldots, u_{4p-2}, v_{4p}, u_{4p+2}\}$ is a minimum dominating set for G. Let $T = \{v_1, u_2\}$. Then $G - N[T] = H_{2p}$, and from Corollary 13, $G - N[T]$ must be dominated by $2p$ vertices of G. Because of degree and order conditions, none of the vertices in $N[T]$ may be used to complete a minimum dominating set of G, so by Corollary 13 and Lemma 1, we have that S is the unique minimum dominating set containing T. So T is a forcing set for S, and hence $f(G,\gamma) \leq 2$. Since G is vertex-transitive, each vertex of G can be put into a minimum dominating set similar to S, either as a member of the forcing subset, or as a vertex of the forced set so that by Corollary 2, $f(C_n \times K_2, \gamma) = 2$. □

Since the domination numbers for $P_n \times P_k$ and $C_n \times P_k$ are not known in general, it is impossible to generalize entirely the results on ladders and prisms. However, we close with a discussion of $f(G,\gamma)$, where $G = \prod_{i=1}^{k} C_{2k+1}$. Let G be a nontrivial finite group. The group Γ is said to be *generated* by the nonidentity elements $h_1, h_2, \ldots, h_k$ (called *generators*) if every element of Γ can be expressed as a finite product of generators. For a generating set Δ for Γ, the *Cayley color graph of* Γ *with respect to* Δ, denoted $D_\Delta(\Gamma)$, has as its vertex set the group elements of Γ. Each generator $h \in \Delta$ is regarded as a color and for $g_1, g_2 \in \Gamma$, there exists an arc (g_1, g_2) colored h in $D_\Delta(\Gamma)$ if and only if $g_2 = g_1 h$. Let $G_\Delta(\Gamma)$ denote the underlying graph of $D_\Delta(\Gamma)$.

In [7, p. 35], it is shown that $G = \prod_{i=1}^{k} C_{2k+1}$ is the underlying graph of $D_\Delta(\Gamma)$ for $\Gamma \cong \prod_{i=1}^{k} \mathbb{Z}_{2k+1}$ and $\Delta = \{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,\ldots,0,1)\}$, where $\mathbb{Z}_{2k+1}$ denotes the cyclic group of order $2k+1$. The graph G is $2k$-regular and contains $(2k+1)^k$ vertices. Thus any minimum dominating set S for G contains at least $(2k+1)^{k-1}$ vertices.

Let S be the subgroup of Γ generated by $\Delta' = \{(2,1,0,\ldots,0),(3,0,1,0,\ldots,0),(4,0,0,1,\ldots,0),\ldots,(k,0,0,\ldots,0,1)\}$. Each element of Δ' has order $2k+1$ and since Δ' is linearly independent, $|S| = (2k+1)^{k-1}$. Consider S as a subset of $V(G)$. We show that S is a minimum dominating set for G, where $G = G_\Delta(\Gamma)$. Let v be a vertex of G. If $v \in S$, then v is dominated. Suppose then that $v \notin S$. Now v belongs to one of the cosets of Γ/S, and Δ is a list of coset representatives for Γ/S, so $v \in \beta + S$ for some $\beta \in \Delta$. Thus $v = \beta + s$ for some $s \in S$. Therefore v is adjacent to s in G and hence v is dominated. Consequently S is a minimum dominating set for G.

When $k = 1$, then $G = C_3$ and, by Theorem 10, the forcing dominating number $f(G,\gamma) = 1$. When $k = 2$, then $G = C_5 \times C_5$. Viewing G as the underlying graph of $D_\Delta(\Gamma)$ for $\Gamma = \mathbb{Z}_5 \times \mathbb{Z}_5$ and $\Delta = \{(1,0),(0,1)\}$, we consider the subgroup S as described above; that is, S is the subgroup of $\mathbb{Z}_5 \times \mathbb{Z}_5$ generated by the element $(2,1)$. So $S = \{(0,0),(2,1),(4,2),(1,3),(3,4)\}$ is a minimum dominating set for G. Let $T = \{(0,0),(2,1)\}$. We show that T is a forcing subset for S. The graph G, drawn on the torus, is shown in Figure 9.

Consider the vertex $(3,0)$. Every vertex of G is dominated by exactly one vertex of each minimum dominating set and all the neighbors of $(3,0)$, except $(3,4)$, are dominated by $(0,0)$ or $(2,1)$. Thus, if S' is a minimum dominating set for G containing T, then $(3,4) \in S'$. Similarly, since all the neighbors of $(3,2)$, except $(4,2)$, are dominated by a vertex of T or $(3,4)$, it must be that $(4,2) \in S'$. Finally $(1,3)$ must also belong to S'. Hence $S = S'$ and T is a forcing subset for S. Therefore $f(S,\gamma) = 2$, so $f(G,\gamma) \le 2$. The set $\{(0,0),(1,2),(2,4),(3,1),(4,3)\}$ is another minimum dominating set for G containing $(0,0)$. Thus, since G is vertex-transitive, $f(G,\gamma) > 1$. Therefore $f(G,\gamma) = 2$. Based on this information, we close with the following conjecture.

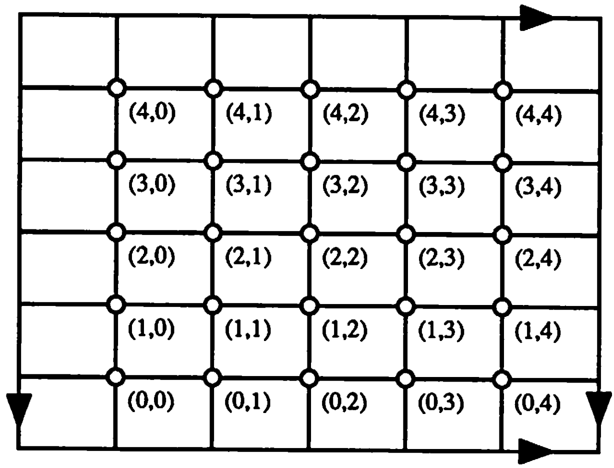

Figure 9

Conjecture. For $G = \prod_{i=1}^{k} C_{2k+1}$, the forcing dominating number $f(G,\gamma) = k$

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