What Makes An Irredundant Set Maximal?

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ABSTRACT. Two closely related types of vertex subsets of a graph, namely external redundant sets and weak external redundant sets, together with associated parameters are discussed. Both types may be used to characterize those irredundant subsets of a graph which are maximal.

1 Introduction

The well-known definitions of dominating sets, independent sets of graphs and the associated parameters lower and upper domination numbers ($\gamma(G)$,

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$\Gamma(G)$) and independence numbers ($i(G), \beta(G)$) for a graph $G = (V, E)$ may be found (if necessary) in [9].

The *closed neighborhood* of a set X of vertices of a graph is denoted by $N[X]$ and is defined by

$$N[X] = X \cup \{v \in V \mid v \text{ is adjacent to some } x \in X\}.$$

The notation $N[x]$ for the closed neighborhood of the single vertex x, is abbreviated to $N[x]$. For $x \in X \subseteq V$, *the private neighborhood of x in X*, denoted by $PN(x, X)$, is defined by

$$PN(x, X) = N[x] - N[X - \{x\}].$$

The set X is an *irredundant set* of G if for all $x \in X$, $PN(x, X) \neq \phi$. The property of irredundance (originally considered in [6]) is a hereditary property. The *lower and upper irredundance numbers of G*, denoted by $ir(G)$ and $IR(G)$, are the smallest and largest cardinalities of maximal redundant sets of G.

The reader is referred to [4] for a bibliography (circa 60 papers) of work concerning irredundant sets and these two parameters. The work of Berge [1] and Cockayne, Hedetniemi and Miller [6] established the following implications for vertex subsets.

$$\begin{array}{ccccc} \text{maximal} & & \text{minimal} & & \\ \text{independent} & (I_1) & \text{dominating} & (I_2) & \text{maximal} \\ \Updownarrow & \Rightarrow & \Updownarrow & \Rightarrow & \text{irredundant} \\ \text{independent} & & \text{dominating} & & \\ \text{and dominating} & & \text{and irredundant} & & \end{array} \qquad (1)$$

The implications I_1, I_2 of (1) imply the well-studied chain of inequalities for any graph G:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G). \qquad (2)$$

In this paper we show that maximal irredundant sets may be characterized in terms of external redundant sets (Section 2) and weak external redundant sets (Section 4). This enables the extensions of the implication scheme (1) and the inequality chain (2) in two ways. Properties of graph parameters involving these two types of vertex subsets are also established (Sections 3 and 4). The reader is referred to [8,9] for bibliographies of recent work on the six parameters which appear in (2).

2 External redundant sets

The subset S of the vertex set V of graph G is an *external redundant set* (abbreviated *er-set*) if for all $v \in V - S$, there exists $w \in S \cup \{v\}$ such that

$PN(w, S \cup \{v\}) = \phi$ and if $w \in S$, then $PN(w, S) \neq \phi$. This concept was defined and generalized in [5]. The authors proved that external redundance characterizes those irredundant sets of a graph which are maximal and that any maximal irredundant set is also minimal external redundant. We now give an alternative definition of external redundant sets which will clarify the connection between *er*-sets and weak external redundant sets which will be discussed in Section 4. This definition will be used in alternative proofs of the results mentioned in the preceding paragraph. We will require the following preliminary result concerning private neighborhoods. The proof is easy and omitted.

Lemma 1. *Let $s \in S \subseteq V$ and $v \in V - S$.*

(i) $PN(s, S \cup \{v\}) = PN(s, S) - N[v]$

(ii) $PN(v, S \cup \{v\}) = N[v] - N[S]$.

Let $R = V - N[S]$, i.e. R is the set of vertices of G which are undominated by S.

Theorem 2. *S is an external redundant set if and only if for all $v \in N[R]$, there exists $s_v \in S$ such that $\phi \neq PN(s_v, S) \subseteq N[v]$.*

Proof: Suppose S is an external redundant set and suppose that $v \in N[R]$. Then $v \in V - S$ and (by the definition of external redundant sets) there exists $w \in S \cup \{v\}$ such that

$$PN(w, S \cup \{v\}) = \phi \text{ and } w \in S \Rightarrow PN(w, S) \neq \phi. \tag{3}$$

If $w = v$, then $PN(v, S \cup \{v\}) = \phi$. By Lemma 1(ii), $N[v] - N[S] = \phi$, i.e., $N[v] \subseteq N[S]$. However, this is impossible since $v \in N[R]$. Hence $w \in S$ and from Lemma 1(i) and (3), we deduce that

$$\phi = PN(w, S \cup \{v\}) = PN(w, S) - N[v]$$

Therefore

$$\phi \neq PN(w, S) \subseteq N[v].$$

The required condition is satisfied by $s_v = w$.

Conversely, suppose that S satisfies the condition of the statement and let $v \in V - S$. If $v \in N[R]$, then there exists $s_v = w \in S$ such that $\phi \neq PN(w, S) \subseteq N[v]$. Also, $v \in N[R]$ implies $w \neq v$. By Lemma 1(i), $PN(w, S \cup \{v\}) = PN(w, S) - N[v] = \phi$.

If $v \notin N[R]$, then $N[v] \subseteq N[S]$ and $PN(v, S \cup \{v\}) = \phi$. It now follows that S is external redundant, as required. □

Corollary 3. *If S is a dominating set of G, then S is external redundant.*

Proof: For S dominating, $R = N[R] = \phi$ and G is external redundant by Theorem 2. □

The following example shows that the property of external redundance is not superhereditary. Let G have $V = \{1, \ldots, 6\}$ and $E = \{12, 13, 15, 23, 24, 34, 35, 45, 46\}$. For $S = \{1, 2, 5\}$ we observe that $R = \{6\}$. However, for each $s \in S$, $PN(s, S) = \phi$ and by Theorem 2, S is not an *er*-set. Now, let $S' = \{1, 2\}$. For each $v \in N[R'] = \{4, 6\}$ let $s_v = 2 \in S'$. Then $\phi \neq \{4\} = PN(s_v, S') \subseteq N[v]$. By Theorem 2, S' is an *er*-set and so the *er*-sets of G are not superhereditary.

Theorem 4. *An irredundant set S of G is maximal irredundant if and only if S is an er-set.*

Proof: Suppose that S is maximal irredundant in G and, contrary to the statement, S is not an *er*-set. Then by Theorem 2, there exists $v \in N[R]$ such that for each $s \in S$, $PN(s, S) \not\subseteq N[v]$, i.e., there exists $x_s \in PN(s, S)$ and $x_s \notin N[v]$. By Lemma 1(i), $x_s \in PN(s, S \cup \{v\})$. Furthermore there exists $r \in R \cap N[v]$, i.e. $r \notin N[S]$ and by Lemma 1(ii), $r \in PN(v, S \cup \{v\})$. We have proved that $S \cup \{v\}$ is irredundant, contradicting the maximality of S.

Conversely, suppose that S is an irredundant *er*-set and consider any $v \in V - S$. If $v \in N[R]$, then by Theorem 2 there exists $s_v \in S$ such that $PN(s_v, S) \subseteq N[v]$. By Lemma 1(i), $PN(s_v, S \cup \{v\}) = \phi$. Otherwise, if $v \notin N[R]$, then $N[v] \subseteq N[S]$ and by Lemma 1(ii), $PN(v, S \cup \{v\}) = \phi$. In either case $S \cup \{v\}$ is not irredundant and so S is maximal irredundant. □

Theorem 4 and the following simple result concerning hereditary and superhereditary classes of subsets of a set will enable us to extend the scheme of implications (1).

Proposition 5. *Let $\mathcal{S}, \mathcal{T}$ be families of subsets of a set V. Suppose that $\mathcal{S}$ is hereditary (resp. superhereditary) and that $S \in \mathcal{S}$ is maximal (resp. minimal) if and only if $S \in \mathcal{T}$. Then S is minimal (resp. maximal) in $\mathcal{T}$.*

Proof: Let S be maximal in $\mathcal{S}$. Then, by hypothesis, $S \in \mathcal{T}$. Suppose $S' \subset S$ were in $\mathcal{T}$. Then since $\mathcal{T}$ is hereditary, $S' \in \mathcal{S}$. Thus S' is in $\mathcal{S}$ and $\mathcal{T}$, hence, by hypothesis, is maximal in $\mathcal{S}$, a contradiction. The superhereditary case is similar. □

Corollary 6. *If S is a maximal irredundant set of G, then S is a minimal er-set.*

Proof: Apply Proposition 5 to $\mathcal{S}$, the family of irredundant sets of G and $\mathcal{T}$, the family of *er*-sets of G. By Theorem 4, the hypothesis of Proposition 5 is satisfied. □

The implications I_1 and I_2 of the scheme (1) are also special cases of Proposition 5. Theorem 4 and Corollary 6 permit the extension of this scheme as follows.

$$\begin{array}{c} \text{maximal} \\ \text{irredundant} \\ \Updownarrow \\ \text{irredundant and} \\ \text{external redundant} \end{array} \quad \begin{array}{c} (I_3) \\ \Rightarrow \end{array} \quad \begin{array}{c} \text{minimal external} \\ \text{redundant} \end{array} \tag{4}$$

The next result asserts that induced P_4's are present when vertices undominated by external redundant sets exist.

Proposition 7. *Suppose that vertex r is not dominated by the external redundant set S and that s_r is a vertex whose existence is asserted by Theorem 2 (i.e. satisfying $\phi \neq PN(s_r, S) \subseteq N[r]$). Then there exists $s'_r \in S - \{s_r\}$ such that for all $t \in PN(s_r, S)$, $G[\{r, t, s_r, s'_r\}]$ is isomorphic to P_4.*

Proof: Since $PN(s_r, S) \subseteq N[r]$, $s_r \notin PN(s_r, S)$ and hence s_r is adjacent to some $s'_r \in S$. Let $t \in PN(s_r, S) \subseteq N[r]$. Then tr, ts_r and $s_r s'_r$ are edges of G. The set S does not dominate r, hence rs_r and rs'_r are not in G. By the private neighbor property, ts'_r is not in G. Hence $G[\{r, t, s_r, s'_r\}]$ is isomorphic to P_4. □

Note that Proposition 7 is a generalization of Corollary 1 of [3] which establishes the same result for maximal irredundant sets S.

3 The parameters $er(G)$ and $ER(G)$

For any graph G let $er(G)$ and $ER(G)$ be the smallest and largest cardinalities of minimal external redundant sets of G. These parameters (together with generalizations) were defined in [5]. The implication I_3 of (4) facilitates the extension of the inequality chain (2) since it immediately follows that

$$er(G) \leq ir(G) \quad \text{and} \quad ER(G) \geq IR(G). \tag{5}$$

Examples are given in [5] to show that each of these inequalities may be strict.

A corollary to the next result improves the inequality $\gamma(G) \leq 2ir(G) - 1$, which was established independently in [1,3]. For the external redundant set S, as above, let $R = V - N[S]$ and for $r \in R$ define

$$S_r = \{s \in S \mid \phi \neq PN(s, S) \subseteq N[r]\}.$$

Observe that for each $r \in R$, $S_r \neq \phi$ (Theorem 2).

Theorem 8. *Let S be external redundant such that $R \neq \phi$ and let $M(S)$ be a subset of S of smallest cardinality $m(S)$, such that $S_r \cap M(S) \neq \phi$ for each $r \in R$. Then $\gamma(G) \leq |S| + m(S) - 1$.*

Proof: Label the vertices of S so that $S = \{s_1, \ldots, s_t\}$ and $M(S) = \{s_1, \ldots, s_m\}$, where $m = m(S)$ (> 0). By the definition of $M(S)$, for each $i = 1, \ldots, m$, $s_i \in S_r$ for some $r \in R$, i.e. $\phi \neq PN(s_i, S) \subseteq N[r]$. Now S does not dominate r and therefore $s_i \notin N[r]$ and so $s_i \notin PN(s_i, S)$. But $PN(s_i, S) \neq \phi$, hence there exists $s' \in PN(s_i, S) \cap (V - S)$. Further, by definition of private neighborhoods, if $i \neq j$, then $s'_i \neq s'_j$. Let $D = S \cup \{s'_1, \ldots, s'_m\}$. For $r \in R$, the definition of $M(S)$ asserts the existence of $s_i \in S_r \cap M(S)$. It follows that $s'_i \in PN(s_i, S) \subseteq N[r]$ and hence $r \in N(s'_i)$. Thus $\{s'_1, \ldots, s'_m\}$ dominates R, S dominates $N[S]$ and we conclude that D is a dominating set of G. Suppose D is minimal dominating. Then the implications I_2 and I_3 of (1) and (4) assert that D is minimal external redundant, a contradiction since $S \subset D$. Thus D is a non-minimal dominating set so that $\gamma(G) \leq |D| - 1 = |S| + m(S) - 1$ as required. □

Corollary 9. *For any graph G, $\gamma(G) \leq 2er(G) - 1$.*

Proof: Let S be an er-set of minimum cardinality $er(G)$. If S is dominating, then $\gamma(G) \leq |S| = er(G)$ and thus $\gamma(G) = er(G) \leq 2er(G) - 1$. Otherwise $R \neq \phi$ and by Theorem 8

$$\begin{aligned}\gamma(G) &\leq |S| + m(S) - 1\\ &= er(G) + m(S) - 1\\ &\leq 2er(G) - 1.\end{aligned}$$

□

The proofs of Theorem 8 and Corollary 9 are almost identical to those of [3, Theorem 3 and Corollary 2] which establish similar results concerning maximal irredundant sets. Arguments like those used to establish Theorem 4 and Corollary 3 of [3], enable us to generalize those results also to external redundance. We state these generalizations and omit the proofs.

Theorem 10. *If $\gamma(G) = er(G) + k$, $(k \geq 1)$, then G has $k+1$ induced subgraphs isomorphic to P_4 with vertex sequences (a_i, b_i, c_i, d_i), $i = 1, \ldots, k+1$, where $\cup_{i=1}^{k+1}\{b_i, c_i, d_i\}$ is a set of $3k+3$ vertices, i.e., no duplication occurs among the b_i, c_i and d_i, and for each $j = 1, \ldots, k+1$, $a_j \notin \cup_{i=1}^{k+1}\{c_i, d_i\}$.*

Corollary 11. *If G does not have two induced subgraphs isomorphic to P_4 with vertex sequences (a_i, b_i, c_i, d_i), $i = 1, 2$, where $b_1, b_2, c_1, c_2, d_1, d_2$ are distinct and for $i = 1, 2$, $a_i \notin \{c_1, c_2, d_1, d_2\}$, then $er(G) = \gamma(G)$.*

In [7] Cockayne and Mynhardt proved that for any graph G having n vertices and maximum degree $\Delta \geq 2$, $ir(G) \geq 2n/3\Delta$. This result may also be improved by replacing $ir(G)$ with $er(G)$. The proof is very similar to that of [7] and for brevity we will omit parts of the argument which may be found there.

Theorem 12. *For any graph G with n vertices and maximum degree $\Delta \geq 2$, $er(G) \geq 2n/3\Delta$.*

Proof: Let X be an external redundant set of G and for $x \in X$, let $B_x = PN(x, X) \cap (V - X)$ and $|B_x| = b_x$. Notice that $x_1 \neq x_2$ implies $B_{x_1} \cap B_{x_2} = \phi$. Partition X as follows. Let

$$Z = \{z \mid z \text{ is isolated in } G[X]\}$$
$$Y = \{y \in X - Z \mid B_y = PN(y, X) \neq \phi\}$$

and

$$W = X - (Y \cup Z).$$

Note that $Z = \{z \in X \mid z \in PN(z, X)\}$ and $W = \{w \in X \mid PN(w, X) = \phi\}$, which implies $B_w = \phi$ for $w \in W$. Denote $B = \cup_{x \in X} B_x$ (disjoint union), let C be the set of vertices of $V - X$ which are adjacent to at least two vertices of X and $R = V - N[X] = V - (X \cup B \cup C)$. For $y \in Y$, a vertex w *annihilates* y (or w *is an annihilator of* y) if $B_y \subseteq N[w]$. The external redundance of X implies the following two facts:

Fact F1 For each $v \in R$, there exists $y \in Y$ such that v annihilates y.

Fact F2 If $v \in B_{y'}$ where $y' \in Y$ and v is adjacent to some $r \in R$, then there exists $y \in Y$ (possibly $y = y'$) such that v annihilates y.

We now establish these facts. Let v be a vertex mentioned in the hypothesis of F1 or F2. Then $v \in N[R]$ and hence (by external redundance and Theorem 2), there exists $y \in X$ such that $\phi \neq PN(y, X) \subseteq N[v]$. The definition of W implies $y \notin W$ and for all $x \in Z$, $z \in PN(z, X) - N[v]$, i.e. $PN(x, Z) \not\subseteq N[v]$. We conclude $y \notin Z$. It now follows that $y \in Y$. Hence $PN(y, X) = B_y$ and so $B_y \subseteq N[v]$, i.e. v annihilates y. Thus F1 and F2 are true.

The proof from this point is very similar to that of [7]. (In [7], $W = \phi$.) We need further definitions. For $y \in Y$, let $R_y = \{r \in R \mid r \text{ annihilates } y\}$ and $|R_y| = r_y$. It is possible that $R_y = \phi$ for some $y \in Y$; however, F1 implies that

$$R = \cup_{y \in Y} R_y. \tag{6}$$

Recall that no $y \in Y$ is isolated in $G[X]$. Let

$$Y_1 = \{y \in Y \mid |N(y) \cap X| = 1\}$$

and

$$Y_2 = \{y \in Y \mid |N(Y) \cap X| \geq 2\}.$$

To obtain an upper bound for $|C|$, observe that the number of edges from C to X is at least $2|C|$, while the numbers of edges joining C to Z, Y_1, Y_2,

W are at most $\sum_{z\in Z}(\Delta - b_z)$, $\sum_{y\in Y_1}(\Delta - 1 - b_y)$, $\sum_{y\in Y_2}(\Delta - 2 - b_y)$, and $\sum_{w\in W}(\Delta - 1)$ respectively (recall $B_w = \phi$ for $w \in W$). Therefore

$$2|C| \leq \Delta|Z| + (\Delta-1)|Y_1| + (\Delta-2)|Y_2| + (\Delta-1)|W| - |B|. \quad (7)$$

From (6) we have

$$|R| \leq \sum_{y\in Y} r_y. \quad (8)$$

Since

$$n = |Z| + |Y_1| + |Y_2| + |W| + |B| + |C| + |R|,$$

(7) and (8) yield

$$\begin{aligned} n &\leq \frac{1}{2}[(\Delta+2)|Z| + (\Delta+1)|Y_1| + \Delta|Y_2| + (\Delta+1)|W| + |B| \\ &\quad + 2\sum_{y\in Y} r_y] \\ &= \frac{1}{2}[(\Delta+2)|Z| + (\Delta+1)|Y_1| + \Delta|Y_2| + (\Delta+1)|W| + \sum_{z\in Z} b_z \\ &\quad + \sum_{y\in Y}(b_y + 2r_y)]. \end{aligned} \quad (9)$$

Now $b_z \leq \Delta$ for each $z \in Z$ and it is shown in [7, Theorem 2.1] that

$$\sum_{y\in Y}(b_y + 2r_y) \leq (2\Delta - 1)(|Y_1| + |Y_2|).$$

Using these in (9), we obtain

$$\begin{aligned} n &\leq (\Delta+1)|Z| + \frac{3}{2}\Delta|Y_1| + (\frac{3\Delta-1}{2})|Y_2| + (\Delta+1)|W| \\ &\leq \frac{3}{2}\Delta(|Z| + |Y_1| + |Y_2| + |W|), \end{aligned}$$

provided that $\Delta \geq 2$. Hence $n \leq 3\Delta|X|/2$ as required. □

It is clear from the proof of Theorem 12 that it could be further improved. Let $X \subseteq V$ and Y, R, B be defined from X as in the proof of Theorem 12. Call X an *F12-set* if it satisfies Facts F1 and F2. We have shown that any external redundant set is an F12-set. The proof of Theorem 12 establishes that any F12-set X in an n-vertex graph with maximum degree Δ (≥ 2) satisfies $|X| \geq 2n/3\Delta$.

4 Weak external redundant sets

We define a new type of vertex subset by removing the non-empty condition in the characterization of external redundance given in Theorem 2. The vertex subset S is a *weak external redundant set* (abbreviated *wer-set*) if for each $v \in N[R]$ there exists $s_v \in S$ such that $PN(s_v, S) \subseteq N[v]$. It is clear that any *er*-set is also a *wer*-set and that the properties are not equivalent. For example, let G have $V = \{1,2,3,4\}$ and $E = \{12, 23, 31, 34\}$. The set $S = \{1,2\}$ is a *wer*-set ($PN(1,S) = PN(2,S) = \phi$) but S is not an *er*-set (defining condition is not satisfied for vertex 3 of $V - S$).

Using arguments almost identical to proofs of Theorem 4 and Corollary 6, the implication scheme (1) may be extended by

$$\begin{array}{ccc} \text{maximal} & & \\ \text{irredundant} & (I_3') & \text{minimal weak} \\ \Updownarrow & \Rightarrow & \text{external redundant} \\ \text{irredundant and weak} & & \\ \text{external redundant} & & \end{array} \qquad (10)$$

Further, the inequality chain (2) may be augmented with

$$wer(G) \leq ir(G) \quad \text{and} \quad WER(G) \geq IR(G), \qquad (11)$$

where $wer(G)$ and $WER(G)$ are the smallest and largest cardinalities of minimal *wer*-sets.

Proposition 13. *The class of wer-sets of any graph G is superhereditary.*

Proof: Let S be a *wer*-set of G and let $S' \supset S$. Let R, R' ($R' \subseteq R$) be the sets of vertices which are undominated by S, S' respectively. If $v \in N[R']$, then $v \in N[R]$ and since S is a *wer*-set, there exists $s_v \in S$ such that $PN(s_v, S) \subseteq N[v]$. Moreover, $s_v \in S'$ and $PN(s_v, S') \subseteq PN(s_v, S)$. Hence $s_v \in S'$ satisfies $PN(s_v, S') \subseteq N[v]$. Therefore S' is a *wer*-set as required. □

On the one hand, the superhereditary property makes the *wer*-sets more appealing than *er*-sets since the four properties involved in the combined implication schemes of (1) and (10) i.e. independent sets, dominating sets, irredundant sets and *wer*-sets are alternately hereditary and superhereditary. However, the following simple characterization perhaps lessens the appeal of *wer*-sets.

Proposition 14. *A set S is weak external redundant if and only if it is maximal irredundant or not irredundant.*

Proof: Let S be a *wer*-set. If S is irredundant, then by (10), S is maximal irredundant. Otherwise S is not irredundant as required. Conversely, any maximal irredundant set is a *wer*-set by (10) and if S is not irredundant, then there exists $s \in S$ for which $\phi = PN(s, S) \subseteq N[v]$ for any $v \in N[R]$. Thus S is a *wer*-set. □

Also lessening the appeal of *wer*-sets is the final result which shows that the parameter $WER(G)$ is equal to the upper redundance number $IR(G)$ for all graphs G.

Theorem 15. *For any graph G, $WER(G) = IR(G)$.*

Proof: Suppose that S is a minimal *wer*-set of G having largest cardinality $WER(G)$ and let S' be a subset of S of maximum cardinality which is irredundant in G. If $S' = S$, then $IR(G) \geq |S'| = |S| = WER(G)$. Otherwise there exists $v \in S - S'$ and by choice of S', $S' \cup \{v\}$ is not irredundant. By Proposition 14, $S \cup \{v\}$ is a *wer*-set and the minimality of S implies that $S' \cup \{v\} = S$. Now S' is not maximal irredundant (otherwise, using (10), S' is a *wer*-set which is contrary to the minimality of S). Hence

$$IR(G) \geq |S'| + 1 = |S| = WER(G).$$

In each case $IR(G) \geq WER(G)$ and the result follows from (11). □

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