Some Ramsey Numbers of Graphs with Bridge

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ABSTRACT. It is shown that if H is a connected graph obtained from H_1 and H_2 by joining them with a bridge, then $r(K_k, H) \leq r(K_k, H_1) + r(K_k, H_2) + k - 2$. We give some applications of this result.

1 Introduction

In this paper, a two-colored complete graph K_p means that each edge of K_p is colored either red or blue. Let G and H be two graphs. The Ramsey number $r(G, H)$ is the smallest positive integer p such that for any two-colored K_p there is either a monochromatic red G or a monochromatic blue H. The chromatic surplus $s(G)$ is the minimum number of vertices in the smallest color class taken over all proper vertex colorings of G by $\chi(G)$ colors.

Theorem 1.1 [1] *If H is a connected graph of order $n \geq s(G)$, then*

$$r(G, H) \geq (\chi(G) - 1)(n - 1) + s(G).$$

Reference [1] defined a graph H of order $n \geq s(G)$ to be $G - good$ if

$$r(G, H) = (\chi(G) - 1)(n - 1) + s(G).$$

For the case $G = K_k$, H is also called $k-$good.

Theorem 1.2 [3] *All trees are $k-$good for any k, i.e.,*

$$r(K_k, T_n) = (k - 1)(n - 1) + 1,$$

where T_n is a tree of order n.

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Note that each edge of a tree is a bridge. In section 2, we prove that if H has a bridge e, then $r(K_k, H)$ has approximate summability relative to the Ramsey numbers of components of $H - e$.

In section 3, we give some applications of this summability.

2 H contains a bridge

Before strictly proving the theorem, we give a lemma which will be used in the proof.

Lemma 2.1 *If H is a connected graph of order m, then*

$$r(K_{k-i}, H) + im - k + 1 \leq r(K_k, H)$$

for $i = 1, 2, \ldots, k-1$.

Proof: Consider a two-colored K_p, where $p = r(K_{k-i}, H) - 1$, such that there is no red K_{k-i}, and no blue H. Then consider this two-colored K_p and i disjoint blue copies of K_{m-1}, and join these graphs completely in red. We obtain a two-colored K_r, where

$$r = p + i(m-1) = r(K_{k-i}, H) - 1 + i(m-1).$$

It is easy to see that there is no red K_k in this two-colored K_r, and also no blue H. So we have

$$r(K_{k-i}, H) + i(m-1) \leq r(K_k, H).$$

Now for $i = 1, 2, \ldots, k-1$,

$$r(K_{k-i}, H) + im - k + 1 \leq r(K_{k-i}, H) + i(m-1) \leq r(K_k, H).$$

Thus we prove the lemma.

Theorem 2.1 *Let H be a connected graph obtained from H_1 and H_2 by joining them with a bridge. Then*

$$r(K_k, H) \leq r(K_k, H_1) + r(K_k, H_2) + k - 2.$$

Proof: Suppose H_1 and H_2 have order m and n respectively. Consider two-colored K_p, where

$$p = r(K_k, H_1) + r(K_k, H_2) + k - 2.$$

If there is no red K_k and no blue H, we will derive a contradiction.

It is trivial for the case $k = 1$ or $k = 2$, so we suppose $k \geq 3$.

Assume that e is the bridge in H which joins H_1 and H_2, and $e = uv$, where $u \epsilon V(H_1)$ and $v \epsilon V(H_2)$. We will find i vertex disjoint blue copies of H_1: $H_1^{(1)}, H_1^{(2)}, \dots, H_1^{(i)}$, with $u_1, u_2, \dots, u_i$ corresponding to u respectively, such that $u_1, u_2, \dots, u_i$ span a complete red K_i for $i = 1, 2, \dots, k-1$.

For $i = 1$, by Theorem 1.1 $r(K_k, H_2) + k - 2 \geq (k-1)n > 0$, so $p > r(K_k, H_1)$ and there is no red K_k, thus there exists a blue $H_1^{(1)}$.

Suppose that we have blue $H_1^{(1)}, H_1^{(2)}, \dots, H_1^{(i)}$ satisfing the properties we mentioned previously, $1 \leq i \leq k-2$. Outside these $H_1^{(1)}, H_1^{(2)}, \dots, H_1^{(i)}$ there are

$$p - im = r(K_k, H_1) + r(K_k, H_2) + k - 2 - im$$

vertices remaining.

By Lemma 2.1,

$$r(K_k, H_1) - im + k - 2 \geq r(K_{k-i}, H_1) - 1 \geq 0,$$

so $p - im \geq r(K_k, H_2)$. Then there is a blue H_2 outside $H_1^{(1)}, H_1^{(2)}, \dots, H_1^{(i)}$. Denote this blue copy of H_2 by $H_2^{(1)}$ with v_1 corresponding to v. The edge $u_j v_1$ is red for $j = 1, 2, \dots, i$, since otherwise we get a blue H.

If we delete v_1, we still can get another blue copy of H_2 with v_2 corresponding to v. In this way we obtain s blue copies of H_2 : denoted as $H_2^{(1)}, H_2^{(2)}, \dots, H_2^{(s)}$ outside $H_1^{(1)}, H_1^{(2)}, \dots, H_1^{(i)}$ with distinct $v_1, v_2, \dots, v_s$ corresponding to v, where

$$s = r(K_k, H_1) + k - 1 - im.$$

Also any u_j is joined to any v_l for $j = 1, 2, \dots, i$, and $l = 1, 2, \dots, s$ in red.

By Lemma 2.1 again,

$$s = r(K_k, H_1) + k - 1 - im \geq r(K_{k-i}, H_1).$$

So the subgraph spaned by $\{v_1, v_2, \dots, v_s\}$ contains a red K_{k-i} or a blue H_1.

A red K_{k-i} together with $\{u_1, u_2, \dots, u_i\}$ will give us a red K_k therefore it is not possible. Thus we get another blue copy of H_1 outside $H_1^{(1)}, H_1^{(2)}, \dots, H_1^{(i)}$. Denote this copy by $H_1^{(i+1)}$, with u_{i+1} corresponding u. Since $H_1^{(i+1)}$ is completely joined to $u_1, u_2, \dots, u_i$ in red, then $u_1, u_2, \dots, u_i, u_{i+1}$ are completely joined in red. Then by induction we get $k-1$ blue copies of H_1 :

$$H_1^{(1)}, H_1^{(2)}, \dots, H_1^{(k-1)},$$

with $u_1, u_2, \ldots, u_{k-1}$ coresponding to u. Also $\{u_1, u_2, \ldots, u_{k-1}\}$ span a red K_{k-1}.

Outside $H_1^{(1)}, H_1^{(2)} \ldots, H_1^{(k-1)}$ there are $p-(k-1)m$ vertices remaining. By Theorem 1.1 in section 1,

$$\begin{aligned} p-(k-1)m &= r(K_k, H_1) + r(K_k, H_2) + k - 2 - (k-1)m \\ &\geq (k-1)(m-1) + 1 + r(K_k, H_2) + k - 2 - (k-1)m \\ &= r(K_k, H_2). \end{aligned}$$

Since we supposed that there is no red K_k, there is a blue H_2 which is disjoint from $H_1^{(1)}, H_1^{(2)}, \ldots, H_1^{(k-1)}$ with vertex v^* corresponding to v.

If for some i $u_i v^*$ is blue ($i = 1, 2, \ldots, k-1$), we get a blue H, so this is not possible. Thus for each $i = 1, 2, \ldots, k-1$, $u_i v^*$ is red,thus we get a red K_k with vertex set $\{u_1, u_2, \ldots, u_{k-1}, v^*\}$. This contradiction completes the proof of Theorem 2.1.

3 Applications

In this section we give some applications of the result in the previous section.

Corollary 3.1 [2] *If both connected graphs H_1 and H_2 are $k-$good, then the graph H obtained from H_1 and H_2 by joining them with a bridge is $k-$good.*

Proof: Suppose that the orders of H_1 and H_2 are m and n respectively. The order of H is $m+n$. By theorems 1.1 and 2.1 we have:

$$\begin{aligned} (k-1)(m+n-1)+1 &\leq r(K_k, H) \\ &\leq r(K_k, H_1) + r(K_k, H_2) + k - 2 \\ &= (k-1)(m-1) + 1 + (k-1)(n-1) + 1 + k - 2 \\ &= (k-1)(m+n-1) + 1. \end{aligned}$$

So $r(K_k, H) = (k-1)(m+n-1)+1$, thus H is $k-$good.

Here we see that when both H_1 and H_2 are $k-$good, the inquality in Theorem 2.1 becomes an equality, so the result in the theorem is sharp.

Corollary 3.2 *Let H be a connected graph obtained from nontrivial graphs H_1 and H_2 by joining them with a bridge. If $r(K_k, H_2) = o(r(K_k, H_1))$, then $r(K_k, H) \sim r(K_k, H_1)$ as $k \to \infty$.*

Proof: Since H_2 contains at least one edge, we have $r(K_k, H_2) \geq r(K_k, K_2)$ $= k$. Then by the condition we have and Theorem 1.2, H_1 is not a tree.

Suppose that H_1 contains a cycle C_m. By a result in [4], there is a constant $c > 0$ such that

$$r(K_k, H_1) \geq r(K_k, C_m) \geq c\left(\frac{k}{\log k}\right)^{(m-1)/(m-2)}.$$

Therefore $k = o(r(K_k, H_1))$. But by Theorem 2.1,

$$r(K_k, H) \leq r(K_k, H_1) + r(K_k, H_2) + k - 2,$$

thus we finish the proof.

Corollary 3.3 *Let H be a graph, and let S_H be the subgraph of H by recursively deleting all end vertices (vertices of degree 1). If S_H is nontrivial, then $r(K_k, H) \sim r(K_k, S_H)$ as $k \to \infty$.*

As a special case, let H be a unicyclic graph containing cycle C_m, then $r(K_k, H) \sim r(K_k, C_m)$ as $k \to \infty$.

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