# Coverings of the complete directed graph with $k$-circuits

Qing-De Kang
Mathematics Department
Hebei Normal College
Shijiazhuang 050091
P.R. China

Zhi-He Liang
Mathematics Department
Hebei Educational College
Shijiazhuang 050091
P.R. China

ABSTRACT. A covering of the complete directed symmetric graph $DK_v$ by $m$-circuits, denoted by $(v,m)-DCC$, is a family of $m$-circuits in $DK_v$ whose union is $DK_v$. The covering number $C(v,m)$ is the minimum number of $m$-circuits in such a covering. The covering problem is to determine the value $C(v,m)$ for every integer $v \geq m$. In this paper, the problem is reduced to the case $m+5 \leq v \leq 2m-4$, for any fixed even integer $m \geq 4$. In particular, the values of $C(v,m)$ are completely determined for $m = 12$, 14 and 16. As well as, a directed construction of optimal $(6k+11, 4k+6)-DCC$ is given.

## 1 Introduction

Let $DK_v$ denote the complete directed symmetric graph with $v$ vertices, where any two distinct vertices $x$ and $y$ are joined by exactly two arcs $(x,y)$ and $(y,x)$. A *covering* of $DK_v$ by $m$-circuits, briefly $(v,m)-DCC$, is defined to be a collection of $m$-circuits whose union is $DK_v$. A $(v,m)-DCC$ is said to be *minimum* if there is no $(v,m)-DCC$ with fewer circuits. The number of circuits in a minimum covering is called the *covering number*, denoted by $C(v,m)$. It is easy to see that

$$C(v,m) \geq T(v,m) = \lceil \frac{v(v-1)}{m} \rceil,$$

JCMCC 25 (1997), pp. 231-253

where $\lceil x \rceil$ is the least integer $y$ satisfying $y \geq x$. If $C(v,m) = T(v,m)$ for given $v$ and $m$, then a minimum $(v,m) - DCC$ is said to be *optimal*.

F.E. Bennett and J.X. Yin [1] determined the value of $C(v,m)$ for $m = 3, 4$:

$$C(v,3) = \begin{cases} T(v,3) & v \equiv 0,1 \pmod 3, v \neq 6 \\ T(v,3)+1 & v \equiv 2 \pmod 3 \end{cases};$$

$$C(v,4) = \begin{cases} T(v,4) & v \neq 4 \\ T(v,4)+1 & v = 4 \end{cases}.$$

Furthermore, for any fixed even integer $m$, J.X. Yin [2] reduced the determination of the value of $C(v,m)$ to the case where $m+3 \leq v \leq 2m-2$, And, he proved that $C(v,m) = T(v,m)$ for $m = 6$, 8 and 10 except $C(6,6) = T(6,6)+1$.

In this paper, we are interested in determining $C(v,m)$ for more even $m$. Firstly, directed constructions of optimal $(2k+3,2k)-DCC$, $(2k+4,2k)-DCC$, $(4k-2,2k)-DCC$, $(4k-3,2k)-DCC$ and $(6k+11,4k+6)-DCC$ are given. Furthermore, $C(v,12)$, $C(v,14)$ and $C(v,16)$ are determined for any $v$.

## 2 Reduction

J.X. Yin have proved the following result in [2].

**Lemma 1.** *Let $m \geq 4$ be an even integer. If $C(v,m)$ is determined for any $m+3 \leq v \leq 2m-2$, then $C(v,m)$ is determined for any $v \geq m$.*

Now, we will furthermore reduce the range to calculate the value of $C(v,m)$. To simplify our statement, we will use the following notations (where $Z$ is the integer ring, $a, b, k \in Z$ and $k > 1$):

$$[a,b] = \{x \in Z; a \leq x \leq b\},$$
$$[a,b]_k = \{x \in Z; a \leq x \leq b \text{ and } x \equiv a \pmod k\} \text{ where } a \equiv b \pmod k.$$

As well as, the symbol $\lfloor x \rfloor$ represents the greatest integer $y$ satisfying $y \leq x$.

**Lemma 2.** *There exists an optimal $(m+3,m)-DCC$ for any even integer $m \geq 10$.*

**Construction.**

Let the vertex set of $DK_{m+3}$ be $Z_{m+3}$. Define three $m$-circuits on $Z_{m+3}$ by

$$M = \langle 0, 1, 2, \ldots, m-1 \rangle,$$
$$N = \langle 1, 0, m+2, m+1, \ldots, 6, 5 \rangle,$$
$$P = \langle m-1, m, m+1, m+2, 0, m-5, m-6, \ldots, 2, 1 \rangle.$$

Furthermore, construct an $m$-circuit $A = \langle a_0, a_1, \ldots, a_{m-1}\rangle$ on $Z_{m+3}$ as follows:

when $m \equiv 0 \pmod 4$,

$$a_{2i} = i+1, (0 \le i \le \frac{m}{2} - 1),$$

$$a_{2i+1} = \begin{cases} m+2-i(0 \le i \le \frac{m}{4} - 1) \\ m+1-i(\frac{m}{4} \le i \le \frac{m}{2} - 1) \end{cases};$$

when $m \equiv 2 \pmod 4$,

$$a_{2i} = \begin{cases} i+1(0 \le i \le \frac{m-2}{4}) \\ i+2(\frac{m+2}{4} \le i \le \frac{m}{2} - 1) \end{cases},$$

$$a_{2i+1} = m+2-i, (0 \le i \le \frac{m}{2} - 1).$$

Developing the $m$-circuit $A$ module $m+3$, we obtain $m+3$ $m$-circuits. The $m+6$ $m$-circuits form a desired $(m+3, m) - DCC$.

**Proof:** Firstly, $T(m+3,m) = \lceil \frac{(m+3)(m+2)}{m} \rceil = \lceil m+5+\frac{6}{m} \rceil = m+6$ when $m \ge 6$. All pairs $(x, x+1)$ are contained in $M$ or $P$. When $m-5 \ge 5$ (i.e. $m \ge 10$), all pairs $(x, x-1)$ are contained in $N$ or $P$. As for the circuit $A$, we have the following observation:

(1) All $a_i$ are pairwise distinct:
when $m \equiv 0 \pmod 4$, $\{a_{2i}\}_i = [1, \frac{m}{2}]$ and $\{a_{2i+1}\}_i = [\frac{3m}{4}+3, m+2] \cup [\frac{m}{2}+2, \frac{3}{4}m+1]$,
when $m \equiv 2 \pmod 4$, $\{a_{2i}\}_i = [1, \frac{m+2}{4}] \cup [\frac{m+10}{4}, \frac{m}{2}+1]$ and $\{a_{2i+1}\}_i = [\frac{m}{2}+3, m+2]$.

(2) All differences $a_{i+1} - a_i$ run over $Z^*_{m+3} \setminus \{1, -1\}$:
when $m \equiv 0 \pmod 4$, $\{a_{2i+1} - a_{2i}\}_i = [\frac{m}{2}+3, m+1]_2 \cup [2, \frac{m}{2}]_2$, $\{a_{2i+2} - a_{2i+1}\}_i = [3, \frac{m}{2}+1]_2 \cup [\frac{m}{2}+4, m]_2$ and $a_0 - a_{m-1} = \frac{m}{2}+2$;
when $m \equiv 2 \pmod 4$, $\{a_{2i+1} - a_{2i}\}_i = [\frac{m}{2}+2, m+1]_2 \cup [2, \frac{m}{2}]_2$, $\{a_{2i+2} - a_{2i+1}\}_i = [3, \frac{m}{2}]_2 \cup [\frac{m}{2}+3, m]_2$ and $a_0 - a_{m-1} = \frac{m}{2}+1$.

Therefore, the construction is an optimal $(m+3, m) - DCC$. □

**Lemma 3.** *There exists an optimal $(m+4, m) - DCC$ for any even integer $m \ge 14$ and $m \ne 16$.*

**Construction.**

Let the vertex set of $DK_{m+4}$ be $Z_{m+4}$. Construct $m+8$ $m$-circuits on $Z_{m+4}$

$$\begin{aligned} A &= \langle a_0, a_1, \ldots, a_{m-1}\rangle \text{ develop } m+4,\\ R &= \langle r_0, r_1, \ldots, r_{m-1}\rangle,\\ N &= \langle n_0, n_1, \ldots, n_{m-1}\rangle,\\ P &= \langle p_0, p_1, \ldots, p_{m-1}\rangle,\\ Q &= \langle q_0, q_1, \ldots, q_{m-1}\rangle. \end{aligned}$$

These parameters $a_i$, $r_i$, $n_i$, $p_i$ and $q_i$ are defined as follows.

**Case 1:** $m = 4t-2$, $t \geq 4$.

$a_{2i+1} = 4t+1-i$ $(0 \leq i \leq 2t-2)$, $a_{2i} = \begin{cases} i+1 & (0 \leq i \leq t-1) \\ i+3 & (t \leq i \leq 2t-2) \end{cases}$;

$r_{4i} = 2t+2i+1$, $r_{4i+1} = 2t+2i+2$, $r_{4i+2} = 2i+1$, $r_{4i+3} = 2i+2$ $(0 \leq i \leq t-3)$, $r_{4t-8+i} = 4t-3+i$ $(0 \leq i \leq 5)$;

$n_{4i} = 2i$, $n_{4i+1} = 2i+1$, $n_{4i+2} = 2t+2i+2$, $n_{4i+3} = 2t+2i+3$ $(0 \leq i \leq t-3)$, $n_{4t-8+i} = 2t-4+i$ $(0 \leq i \leq 5)$;

$p_{4i} = 2t-2i+1$, $p_{4i+1} = 2t-2i$, $p_{4i+2} = 4t-2i+1$, $p_{4i+3} = 4t-2i$ $(0 \leq i \leq t-3)$, $p_{4t-8+i} = 5-i$ $(0 \leq i \leq 5)$;

$q_{4i} = 4t-2i+2$, $q_{4i+1} = 4t-2i+1$, $q_{4i+2} = 2t-2i$, $q_{4i+3} = 2t-2i-1$ $(0 \leq i \leq t-3)$, $q_{4t-8+i} = 2t+6-i$ $(0 \leq i \leq 5)$.

**Case 2:** $m = 8t$, $t \geq 3$.

$a_{2i} = i+1$ $(0 \leq i \leq 4t-1)$, $a_{2i+1} = \begin{cases} 8t+3-i & (0 \leq i \leq 2t-1) \\ 8t+1-i & (2t \leq i \leq 4t-1) \end{cases}$;

$r_{4i} = 2i+1$, $r_{4i+1} = 4t+2i+3$, $r_{4i+2} = 4t+2i+4$, $r_{4i+3} = 2i+2$ $(0 \leq i \leq t)$,

$r_{4t+4+i} = 2t+3+i$ $(0 \leq i \leq 2t-1)$, $r_{6t+4+i} = 6t+9+i$ $(0 \leq i \leq 2t-5)$;

$n_{4i} = 4t+3+2i$, $n_{4i+1} = 2i+1$, $n_{4i+2} = 2i+2$, $n_{4i+3} = 4t+2i+4$ $(0 \leq i \leq t)$,

$n_{4t+4+i} = 6t+5+i$ $(0 \leq i \leq 2t-1)$, $n_{6t+4+i} = 2t+7+i$ $(0 \leq i \leq 2t-5)$;

$p_{4i} = 8t-2i+4$, $p_{4i+1} = 4t-2i+2$, $p_{4i+2} = 4t-2i+1$, $p_{4i+3} = 8t-2i+3$ $(0 \leq i \leq t-1)$, $p_{4t+i} = 6t+4-i$ $(0 \leq i \leq 2t+1)$, $p_{6t+2+i} = 2t-2-i$ $(0 \leq i \leq 2t-3)$;

$q_{4i} = 4t-2i+2$, $q_{4i+1} = 8t-2i+4$, $q_{4i+2} = 8t-2i+3$, $q_{4i+3} = 4t-2i+1$ $(0 \leq i \leq t-1)$, $q_{4t+i} = 2t+2-i$ $(0 \leq i \leq 2t+1)$, $q_{6t+2+i} = 6t-i$ $(0 \leq i \leq 2t-3)$.

**Case 3:** $m = 8t+4$, $t \geq 2$.

$$a_{2i} = i+1 \ (0 \leq i \leq 4t+1) \ a_{2i+1} = \begin{cases} 8t+7-i & (0 \leq i \leq 2t) \\ 8t+5-i & (2t+1 \leq i \leq 4t+1) \end{cases};$$

$r_{4i} = 2i+1$, $r_{4i+1} = 4t+2i+5$, $r_{4i+2} = 4t+2i+6$, $r_{4i+3} = 2i+2$ $(0 \leq i \leq t)$, $r_{4t+4+i} = 2t+3+i$ $(0 \leq i \leq 2t+1)$, $r_{6t+6+i} = 6t+11+i$ $(0 \leq i \leq 2t-3)$;

$n_{4i} = 4t+5+2i$, $n_{4i+1} = 2i+1$, $n_{4i+2} = 2i+2$, $n_{4i+3} = 4t+2i+6$ $(0 \leq i \leq t)$, $n_{4t+4+i} = 6t+7+i$ $(0 \leq i \leq 2t+1)$, $n_{6t+6+i} = 2t+7+i$ $(0 \leq i \leq 2t-3)$;

$p_{4i} = 8t-2i+8$, $p_{4i+1} = 4t-2i+4$, $p_{4i+2} = 4t-2i+3$, $p_{4i+3} = 8t-2i+7$ $(0 \leq i \leq t)$, $p_{4t+4+i} = 6t+6-i$ $(0 \leq i \leq 2t+1)$, $p_{6t+6+i} = 2t-2-i$ $(0 \leq i \leq 2t-3)$;

$q_{4i} = 4t-2i+4$, $q_{4i+1} = 8t-2i+8$, $q_{4i+2} = 8t-2i+7$, $q_{4i+3} = 4t-2i+3$ $(0 \leq i \leq t)$, $q_{4t+4+i} = 2t+2-i$ $(0 \leq i \leq 2t+1)$, $q_{6t+6+i} = 6t+2-i$ $(0 \leq i \leq 2t-3)$.

**Proof:** Firstly, $T(m+4,m) = \lceil \frac{(m+4)(m+3)}{m} \rceil = \lceil m+7+\frac{12}{m} \rceil = m+8$, when $m \geq 12$. Thus, we only need to prove that the construction is a $(m+4,m)-DCC$. For the Case 1 $(m = 4t-2)$, we have the following verification.

1° The differences $a_{i+1} - a_i$ run over $Z^*_{4t+2} \setminus \{1, 2t+1, 4t+1\}$:

$\{a_{2i+1} - a_{2i}\}_0^{2t-2} = [2t+2, 4t]_2 \cup [2, 2t-2]_2$,

$\{a_{2i+2} - a_{2i+1}\}_0^{2t-3} = [3, 2t-1]_2 \cup [2t+3, 4t-1]_2$,

$a_0 - a_{4t-3} = 2t$.

2° The pairs $(x, x+1)$ are contained in $R$ or $N$, where $x$ fill

$[2t+1, 4t-5]_2$ for $(r_{4i}, r_{4i+1})$, $0 \leq i \leq t-3$;

$[1, 2t-5]_2$ for $(r_{4i+2}, r_{4i+3})$, $0 \leq i \leq t-3$;

$[4t-3, 4t+1]$ for $(r_{4t-8+i}, r_{4t-7+i})$, $0 \leq i \leq 4$;

$[0, 2t-6]_2$ for $(n_{4i}, n_{4i+1})$, $0 \leq i \leq t-3$;

$[2t+2, 4t-4]_2$ for $(n_{4i+2}, n_{4i+3})$, $0 \leq i \leq t-3$;

$[2t-4, 2t]$ for $(n_{4t-8+i}, n_{4t-7+i})$, $0 \leq i \leq 4$.

3° The pairs $(x, x-1)$ are contained in $P$ or $Q$, where $x$ fill

$[7, 2t+1]_2$ for $(p_{4i}, p_{4i+1})$, $0 \leq i \leq t-3$;

$[2t+7, 4t+1]_2$ for $(p_{4i+2}, p_{4i+3})$, $0 \leq i \leq t-3$;

$[1, 5]$ for $(p_{4t-8+i}, p_{4t-7+i})$, $0 \leq i \leq 4$;

$[2t+8, 4t+2]_2$ for $(q_{4i}, q_{4i+1})$, $0 \leq i \leq t-3$;

$[6, 2t]_2$ for $(q_{4i+2}, q_{4i+3})$, $0 \le i \le t-3$;
$[2t+2, 2t+6]$ for $(q_{4t-8+i}, q_{4t-7+i})$, $0 \le i \le 4$.

4° The pairs $(x, x+2t+1)$ are contained in $R$, $N$, $P$ or $Q$, where $x$ fill
$[2t+2, 4t-4]_2 \cup [2, 2t-4]_2$ for $(r_{4i+1}, r_{4i+2})$ and $(r_{4i+3}, r_{4i+4})$, $0 \le i \le t-3$;
$[1, 2t-5]_2 \cup [2t+3, 4t-3]_2$ for $(n_{4i+1}, n_{4i+2})$ and $(n_{4i+3}, n_{4i+4})$, $0 \le i \le t-3$;
$[6, 2t]_2 \cup [2t+6, 4t]_2$ for $(p_{4i+1}, p_{4i+2})$ and $(p_{4i+3}, p_{4i+4})$, $0 \le i \le t-3$;
$[2t+7, 4t+1]_2 \cup [5, 2t-1]_2$ for $(q_{4i+1}, q_{4i+2})$ and $(q_{4i+3}, q_{4i+4})$, $0 \le i \le t-3$;
0 and $2t+1$ for $(r_{4t-3}, r_0)$, $(n_{4t-3}, n_0)$, $(p_{4t-3}, p_0)$ and $(q_{4t-3}, q_0)$.
Obviously, when $t \ge 4$, all pairs $(x, x+2t+1)$ appear in the circuits $R$, $N$, $P$ or $Q$.

5° In such covering there are $m - 12 = 2(2t-7)$ repeated pairs, which form $2t-7$ 2-circuits $(5, 2t+6), (6, 2t+7), \ldots, (2t-4, 4t-3)$ and $(0, 2t+1)$.

For the Case 2 and Case 3, the proofs are similar. We only point out that the $m-12$ repeated pairs form the following circuits:
when $m = 8t$, $t \ge 3$, a $(\frac{m}{2}-8)$-circuit and a $(\frac{m}{2}-4)$-circuit:

$$(6t+9, 6t+10, \ldots, 8t+3, 8t+4, 2t+7, 2t+8, \ldots, 4t+1, 4t+2),$$

$$(2t-2, 2t-3, \ldots, 2, 1, 6t, 6t-1, \ldots, 4t+4, 4t+3);$$

when $m = 8t+4$, $t \ge 2$, two $(\frac{m}{2}-6)$-circuits:

$$(6t+11, 6t+12, \ldots, 8t+7, 8t+8, 2t+7, 2t+8, \ldots, 4t+3, 4t+4),$$

$$(2t-2, 2t-3, \ldots, 2, 1, 6t+2, 6t+1, \ldots, 4t+6, 4t+5).$$

□

**Lemma 4.** *There exists an optimal $(4m-2, 2m) - DCC$ for any integer $m \ge 5$.*

**Construction.**

Let the vertex set of $DK_{4m-2}$ be $X = Z_{4m-5} \cup \{\infty_0, \infty_1, \infty_2\}$. Define two $2m$-circuits on the set $X$ by

$$A = \langle \infty_0, a_0, a_1, \ldots, a_{2m-2} \rangle \text{ and}$$
$$B = \langle \infty_1, b_0, b_1, \ldots, b_{m-2}, \infty_2, c_{m-2}, c_{m-3}, \ldots, c_1, c_0 \rangle,$$

where

$$a_{2i} = \begin{cases} i & (0 \le i \le m-2) \\ i+1 & (i = m-1) \end{cases}, \qquad a_{2i+1} = -(i+1)\ (0 \le i \le m-2),$$

$$b_{2i} = m+i\ (0 \le i \le \lceil \frac{m-3}{2} \rceil), \qquad b_{2i+1} = -(m+i)\ (0 \le i \le \lfloor \frac{m-3}{2} \rfloor),$$

$$c_{2i} = 2m-2+i\ (0 \le i \le \lceil \frac{m-5}{2} \rceil), \quad c_{2i+1} = 2m-3-i\ (0 \le i \le \lfloor \frac{m-5}{2} \rfloor),$$

$$c_{m-3} = \begin{cases} -\frac{m-5}{2} & (m \text{ odd}) \\ \frac{m-2}{2} & (m \text{ even}) \end{cases}, \qquad c_{m-2} = \begin{cases} \frac{m-1}{2} & (m \text{ odd}) \\ -\frac{m-4}{2} & (m \text{ even}) \end{cases}.$$

Developing the $2m$-circuits $A$ and $B$ modulo $4m-5$ we can obtain $8m-10$ $2m$-circuits. Denote $A' = \langle \infty_0, a'_0, a'_1, \ldots, a'_{2m-2} \rangle$, where $a'_i = a_i + \lceil \frac{m+3}{2} \rceil$. To cover the arcs $(\infty_i, \infty_j)$, $0 \le i \ne j \le 2$, we replace the circuits $A$, $B$ and $A'$ obtained above with the following four $2m$-circuits:

$$M = \langle c_{m-5}, c_{m-6}, \ldots, c_1, c_0, \infty_1, \infty_2, \infty_0, a_0, a_1, \ldots, a_{t-1}, g \rangle,$$
$$N = \langle b_3, b_4, \ldots, b_{m-3}, b_{m-2}, \infty_2, \infty_1, \infty_0, a'_0, a'_1, \ldots, a'_{t-1}, h \rangle,$$
$$P = \langle a_{t-1}, a_t, \ldots, a_{2m-3}, a_{2m-2}, \infty_0, \infty_2, c_{m-2}, c_{m-3}, c_{m-4}, c_{m-5}, G \rangle,$$
$$Q = \langle a'_{t-1}, a'_t, \ldots, a'_{2m-3}, a'_{2m-2}, \infty_0, \infty_1, b_0, b_1, b_2, b_3, H \rangle,$$

where $t = 2\lceil \frac{m}{2} \rceil$ and

$$g = \begin{cases} \text{empty} & (m \text{ odd}) \\ \frac{m}{2} & (m \text{ even}) \end{cases}, \qquad h = \begin{cases} \text{empty} & (m \text{ odd}) \\ 1 & (m \text{ even}) \end{cases},$$

$$G = (1, 2, \ldots, \lfloor \frac{m-3}{2} \rfloor, \lfloor \frac{7m-2}{2} \rfloor, \lfloor \frac{7m}{2} \rfloor, \ldots, 4m-6),$$

$$H = (3, 4, \ldots, 2\lfloor \frac{m-3}{2} \rfloor).$$

Then the obtained $8m-9$ $2m$-circuits form a desired $(4m-2, 2m)-DCC$.

**Proof:** Throughout the proof, $m$ is always a fixed integer not less than 6.

1° $T(4m-2, 2m) = \lceil \frac{(4m-2)(4m-3)}{2m} \rceil = \lceil 8m - 10 + \frac{6}{2m} \rceil = 8m-9$.

2° $\{a_{i+1} - a_i\}_i = [2, 4m-6]_2 \cup \{2m-1\}$,
$\{b_{i+1} - b_i\}_i = [2\lfloor \frac{m}{2} \rfloor - 1, 2m-5]_2 \cup [2m+1, 2\lfloor \frac{3m}{2} \rfloor - 3]_2$,
$\{c_i - c_{i+1}\}_i = [1, 2\lfloor \frac{m}{2} \rfloor - 3]_2 \cup [2\lfloor \frac{3m}{2} \rfloor - 1, 4m-7]_2 \cup \{2m-3\}$.
It is easy to see that these differences fill $Z^*_{4m-5} = Z_{4m-5} \setminus \{0\}$.

3° The elements in $A$ are distinct: $\{a_{2i}\}_i = [0, m-2] \cup \{m\}$, $\{a_{2i+1}\}_i = [3m-4, 4m-6]$. The elements in $B$ are distinct: $\{b_{2i}\}_i = [m, \lfloor \frac{3m-2}{2} \rfloor]$, $\{b_{2i+1}\}_i = [\lfloor \frac{5m-6}{2} \rfloor, 3m-5]$, $\{c_{2i}\}_i = [2m-2, \lfloor \frac{5m-9}{2} \rfloor]$, $\{c_{2i+1}\}_i = [\lfloor \frac{3m-1}{2} \rfloor, 2m-3]$, $\{c_{m-3}, c_{m-2}\} = \{\lfloor \frac{5-m}{2} \rfloor, \lfloor \frac{m-1}{2} \rfloor\}$.

4° It is not difficult to see that all ordered pairs in the $2m$-circuits $A$, $B$ and $A'$ and the pairs $(\infty_i, \infty_j)$, $0 \le i \ne j \le 2$, are contained in the $2m$-circuits $M$ or $N$ or $P$ or $Q$.

5° The elements in $M$ (or $N$, or $P$, or $Q$) are distinct.

$$M\colon\ \{c_i\}_i = [\lfloor\frac{3m+1}{2}\rfloor, \lfloor\frac{5m-9}{2}\rfloor], \{a_{2i}\}_i \cup \{g\} = [0, \lfloor\frac{m}{2}\rfloor],$$
$$\{a_{2i+1}\}_i = [\lfloor\frac{7m-10}{2}\rfloor, 4m-6];$$
$$N\colon \{b_{2i}\}_i = [m+2, \lfloor\frac{3m-2}{2}\rfloor], \{b_{2i+1}\}i = [\lfloor\frac{5m-6}{2}\rfloor, 3m-6],$$
$$\{a'_i\}_i \cup \{h\} = [1, m+1];$$
$$P\colon \{a_{2i}\}_i \cup \{c_j\}_j = [\lfloor\frac{m-1}{2}\rfloor, m-2] \cup \{m, \lfloor\frac{3m}{2}\rfloor, \lfloor\frac{5m-8}{2}\rfloor, \lfloor\frac{7m-5}{2}\rfloor\},$$
$$\{a_{2i+1}\}_i = [3m-4, \lfloor\frac{7m-10}{2}\rfloor], G = [1, \lfloor\frac{m-3}{2}\rfloor] \cup [\lfloor\frac{7m-2}{2}\rfloor, 4m-6];$$
$$Q\colon \{a'_{2i}\}_i \cup \{b_j\}_j = [m, \lfloor\frac{3m}{2}\rfloor] \cup \{\lfloor\frac{3m+4}{2}\rfloor, 3m-6, 3m-5\},$$
$$\{a'_{2i+1}\}_i = [\lfloor\frac{7m-4}{2}\rfloor, 4m-5] \cup R,\ H = [3, 2\lfloor\frac{m-3}{2}\rfloor],$$

where $R = \{0, 1\}$ (if $m$ odd) or $\{0, 1, 2\}$ (if $m$ even).

6° In this construction there are $2m-6$ repeated pairs, which form two $(m-3)$-circuits: $(a_{t-1}, g, c_{m-5}, G)$ and $(a'_{t-1}, h, b_3, H)$. □

**Lemma 5.** *There exists an optimal $(4m-3, 2m) - DCC$ for any integer $m \ge 8$.*

**Construction.**

Let the vertex set of $DK_{4m-3}$ be $X = Z_{4m-7} \cup \{\infty_1, \infty_2, \infty_3, \infty_4\}$. We will construct two $2m$-circuits on $X$ by

$$A = \langle \infty_1, a_1, a_2, \ldots, a_{m-1}, \infty_2, b_1, b_2, \ldots, b_{m-1}\rangle \text{ and}$$
$$C = \langle \infty_3, c_1, c_2, \ldots, c_{m-1}, \infty_4, d_1, d_2, \ldots, d_{m-1}\rangle.$$

Developing $A$ and $C$ modulo $4m-7$, we obtain $2(4m-7)$ $2m$-circuits. Denote

$$A' = \langle \infty_1, a'_1, a'_2, \ldots, a'_{m-1}, \infty_2, b'_1, b'_2, \ldots, b'_{m-1}\rangle \text{ and}$$
$$C' = \langle \infty_3, c'_1, c'_2, \ldots, c'_{m-1}, \infty_4, d'_1, d'_2, \ldots, d'_{m-1}\rangle,$$

where $a'_i = a_i + 1$, $b'_i = b_i + 1$, $c'_i = c_i + 1$, $d'_i = d_i + 1$, $1 \le i \le m-1$. To cover the ordered pairs $(\infty - i, \infty - j)$, $1 \le i \ne j \le 4$, we need to break up four $2m$-circuits $A$, $A'$, $C$ and $C'$ and to form five new $2m$-circuits $M$, $N$, $P$, $Q$ and $R$.

**Case 1:** $m$ is odd. Define

$$\begin{aligned} a_{2i} &= -i, & a_{2i-1} &= i-1, \\ b_{2i} &= \frac{m-3}{2}+i, & b_{2i-1} &= \frac{5m-7}{2}-i, \\ c_{2i} &= 2m-3-i, & c_{2i-1} &= i-1, \\ d_{2i} &= -i, & d_{2i-1} &= 3m-7+i, \end{aligned}$$

where $1 \le i \le \frac{m-1}{2}$. Let

$$\begin{aligned} M &= \langle c_{m-1}, \infty_4, \infty_1, \infty_3, a_1, a_2, \ldots, a_{m-1}, \infty_2, b'_1, b'_2, \ldots, b'_{m-4}\rangle, \\ N &= \langle a'_{m-1}, \infty_2, \infty_3, \infty_1, c'_1, c'_2, \ldots, c'_{m-1}, \infty_4, d_1, d_2, \ldots, d_{m-4}\rangle, \\ P &= \langle d_{m-4}, d_{m-3}, d_{m-2}, d_{m-1}, \infty_3, \infty_2, b_1, \ldots, b_{m-1}, \infty_1, \infty_4, d'_1, \ldots, d'_{m-7}\rangle, \\ Q &= \langle d'_{m-6}, d'_{m-5}, \ldots, d'_{m-1}, \infty_3, \infty_4, \infty_2, \infty_1, c_1, c_2, \ldots, c_{m-1}, \overline{Q}\rangle, \\ R &= \langle b'_{m-4}, b'_{m-3}, b'_{m-2}, b'_{m-1}, \infty_1, \infty_2, \infty_3, \infty_4, a'_1, a'_2, \ldots, a'_{m-1}, \overline{R}\rangle, \end{aligned}$$

where $\overline{Q}$ is a $(m-9)$-sequence on the set $[2m, 3m-10]$ and $\overline{R}$ is a $(m-7)$-sequence on the set $[\frac{m+1}{2}, m-3] \cup [m, \frac{3m-9}{2}]$. **Remark:** when $m = 9$, $\overline{Q}$ is empty.

**Case 2:** $m$ is even. Define

$a_{2i} = -i$, $b_{2i} = \frac{5m-6}{2} - i$, $c_{2i} = i-1$, $d_{2i} = 3m-7+i$, $(1 \le i \le \frac{m}{2}-1)$,

$a_{2i-1} = i-1$, $b_{2i-1} = \frac{m-2}{2}+i$, $c_{2i-1} = 2m-3-i$, $d_{2i-1} = -i$, $(1 \le i \le \frac{m}{2})$.

Let

$$\begin{aligned} M &= \langle c'_{m-1}, \infty_4, \infty_1, \infty_3, c'_1, a'_1, a'_2, \ldots, a'_{m-1}, \infty_2, b'_1, b'_2, \ldots, b'_{m-5}\rangle, \\ N &= \langle a_{m-1}, \infty_2, \infty_3, \infty_1, c_2, c_3, \ldots, c_{m-1}, \infty_4, d_1, d_2, \ldots, d_{m-3}\rangle, \\ P &= \langle d_{m-3}, d_{m-2}, d_{m-1}, \infty_3, \infty_2, b_1, b_2, \ldots, b_{m-1}, \infty_1, \infty_4, d'_1, d'_2, \ldots, d'_{m-6}\rangle, \\ Q &= \langle d'_{m-6}, d'_{m-5}, \ldots, d'_{m-1}, \infty_3, \infty_4, \infty_2, \infty_1, c'_2, c'_3, \ldots, c'_{m-1}, \overline{Q}\rangle, \\ R &= \langle b'_{m-5}, b'_{m-4}, \ldots, b'_{m-1}, \infty_1, \infty_2, \infty_4, \infty_3, c_1, a_1, a_2, \ldots, a_{m-1}, \overline{R}\rangle, \end{aligned}$$

where $\overline{Q}$ is a $(m-8)$-sequence on the set $[2m-3, 3m-12]$ and $\overline{R}$ is a $(m-8)$-sequence on the set $[\frac{m}{2}-1, m-6] \cup [3m-2, \frac{7m}{2}-7]$. **Remark:** when $m = 8$ both $\overline{Q}$ and $\overline{R}$ are empty.

**Proof:**

1° $T(4m-3, 2m) = \lceil 8m - 14 + \frac{12}{2m} \rceil = 8m - 13$, when $m \ge 6$.

2° – 6° are only for odd $m \ge 9$ (similarly, for even $m \ge 8$).

2° The elements in $A$ (or $C$) are distinct:

$$\{a_{2i}\}_i = [\frac{7m-13}{2}, 4m-8], \quad \{a_{2i-1}\}_i = [0, \frac{m-3}{2}],$$
$$\{b_{2i}\}_i = [\frac{m-1}{2}, m-2], \quad \{b_{2i-1}\}_i = [2m-3, \frac{5m-9}{2}],$$
$$\{c_{2i}\}_i = [\frac{3m-5}{2}, 2m-4], \quad \{c_{2i-1}\}_i = [0, \frac{m-3}{2}],$$
$$\{d_{2i}\}_i = [\frac{7m-13}{2}, 4m-8], \quad \{d_{2i-1}\}_i = [3m-6, \frac{7m-15}{2}].$$

3° The differences of ordered pairs in $A$ and $C$ are just all elements of $Z^*_{4m-7}$:

$$\{a_{2i}-a_{2i-1}\}_i = [3m-5, 4m-8]_2, \quad \{a_{2i+1}-a_{2i}\}_i = [2, m-3]_2,$$
$$\{b_{2i}-b_{2i-1}\}_i = [2m-3, 3m-6]_2, \quad \{b_{2i+1}-b_{2i}\}_i = [m, 2m-5]_2,$$
$$\{c_{2i}-c_{2i-1}\}_i = [m-1, 2m-4]_2, \quad \{c_{2i+1}-c_{2i}\}_i = [2m-2, 3m-7]_2,$$
$$\{d_{2i}-d_{2i-1}\}_i = [1, m-2]_2, \quad \{d_{2i+1}-d_{2i}\}_i = [3m-4, 4m-9]_2.$$

4° It is not difficult to see that the $2m$-circuits $M$, $N$, $P$, $Q$ and $R$ contain all ordered pairs in $A$, $C$, $A'$ and $C'$, and contain all ordered pairs $(\infty_i, \infty_j)$, $1 \le i \ne j \le 4$. Note that $a_1 = c_1$ and $d_{m-4} = d'_{m-6}$.

5° The elements in $M$ (or $N$, or $P$, or $Q$, or $R$) are distinct:

$$M:\quad c_{m-1} = \frac{3m-5}{2}, \{a_i\}_1^{m-1} = [0, \frac{m-3}{2}] \cup [\frac{7m-13}{2}, 4m-8],$$
$$\{b'_i\}_1^{m-4} = [\frac{m+1}{2}, m-3] \cup [2m-1, \frac{5m-7}{2}];$$
$$N: a'_{m-1} = \frac{7m-11}{2}, \{c'_i\}_1^{m-1} = [1, \frac{m-1}{2}] \cup [\frac{3m-3}{2}, 2m-3],$$
$$\{d_i\}_1^{m-4} = [3m-6, \frac{7m-17}{2}] \cup [\frac{7m-9}{2}, 4m-8];$$
$$P: \{d_i\}_{m-4}^{m-1} = [\frac{7m-17}{2}, \frac{7m-11}{2}], \{b_i\}_1^{m-1} = [\frac{m-1}{2}, m-2] \cup$$
$$[2m-3, \frac{5m-9}{2}],$$
$$\{d'_1\}_1^{m-7} = [\frac{7m-5}{2}, 4m-7] \cup [3m-5, \frac{7m-19}{2}];$$

$Q$: $\{d'_i\}_{m-6}^{m-1} = [\frac{7m-17}{2}, \frac{7m-7}{2}], \{c_i\}_1^{m-1} = [0, \frac{m-3}{2}] \cup$
$[\frac{3m-5}{2}, 2m-4]$,
$\{\overline{Q}\} = [2m, 3m-10]$;
$R$: $\{b'_i\}_{m-4}^{m-1} = \{m-2, m-1, 2m-2, 2m-1\}$,
$\{a'_i\}_1^{m-1} = [1, \frac{m-1}{2}] \cup [\frac{7m-11}{2}, 4m-7], \{\overline{R}\} = [\frac{m+1}{2}, m-3] \cup$
$[m, \frac{3m-9}{2}]$.

6° In this construction there are $2m-12$ repeated pairs, which form a $(2m-12)$-circuit: $\langle b'_{m-4}, c_{m-1}, \overline{Q}, d'_{m-6}, a'_{m-1}, \overline{R}\rangle$. □

Summarizing the results of Lemmas 1 - 5, we obtain the following theorem.

**Theorem 1.** *Let $m \geq 4$ be an even integer. If $C(v, m)$ is determined for any $m+5 \leq v \leq 2m-4$, then $C(v, m)$ is determined for any $v \geq m$.*

**Proof:** In [1] and [2], the values of $C(v, m)$ for $m \leq 10$ were determined completely. Therefore we only need to consider the case $m \geq 12$. By Lemma 1 - 5, the Theorem holds with the possible exception of $(v, m) \in \{(16, 12), (20, 16), (21, 12), (25, 14)\}$. In [3] and [4], J.C. Bermond and V. Fabour proved that, for $m \in \{4, 6, 8, 10, 12, 14, 16\}$ and $v \geq m$, there is a decomposition of $DK_v$ into arc-disjoint $m$-circuits if and only if $v(v-1) \equiv 0 \pmod{m}$ except $(v, m) = (4, 4)$ and $(6, 6)$. Thus, $C(16, 12) = T(16, 12)$ and $C(21, 12) = T(21, 12)$ are obtained. As for the values of $C(20, 16)$ and $C(25, 14)$ we give the constructions of optimal $(20, 16) - DCC$ and $(25, 14) - DCC$ as follows.

An optimal $(20, 16) - DCC$ on the set $Z_{19} \cup \{\infty\}$:

$\langle \infty, 1, 16, 2, 15, 3, 14, 4, 12, 5, 11, 6, 10, 7, 9, 8\rangle$, develop 19,
$\langle 0, 1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 17, 18, 2\rangle$,
$\langle 0, 10, 1, 2, 3, 4, 14, 5, 15, 6, 16, 7, 17, 8, 18, 9\rangle$,
$\langle 0, 17, 15, 13, 11, 9, 7, 5, 3, 1, 18, 16, 14, 12, 10, 8\rangle$,
$\langle 5, 8, 11, 2, 12, 15, 18, 0, 3, 6, 4, 7, 10, 13, 16, 17\rangle$,
$\langle 17, 1, 11, 14, 15, 16, 0, 8, 6, 9, 12, 3, 13, 4, 2, 5\rangle$,
repeated pairs: $(8, 0), (17, 5)$.

An optimal $(25,14)-DCC$ on the set $Z_{25}$:

$\langle 24,1,23,2,22,3,21,4,11,5,10,6,9,7\rangle$, develop 25,
$\langle i,i+1,i+10,i+19,i+5,i+16,i+3,i+15,i+6,i+22,$
$i+21,i+9,i+23,i+12\rangle, 0\le i\le 8,$
$\langle 1,11,21,6,16,2,18,19,8,7,22,12,13,14\rangle$,
$\langle 4,14,24,9,19,3,18,8,22,11,10,23,7,20\rangle$,
$\langle 17,18,4,19,20,9,22,7,6,5,21,11,12,2\rangle$,
$\langle 8,21,20,19,4,15,16,1,13,23,24,0,10,9\rangle$,
$\langle 18,7,17,16,15,2,12,24,14,0,9,10,20,5\rangle$,
$\langle 15,14,13,12,11,1,16,17,7,21,5,20,10,0\rangle$,
$\langle 15,0,16,6,21,22,23,13,3,19,18,17,2,14\rangle$,
$\langle 5,15,1,17,3,14,4,20,21,10,11,24,8,23\rangle$,
$\langle 23,8,18,3,13,0,12,22,6,19,9,24,15,5\rangle$,
repeated pairs: $(5,23)$.

□

## 3 Another directed construction

**Theorem 2.** *There exists an optimal $(6k+11,4k+6)-DCC$ for any integer $k\ge 1$.*

**Construction.**

Let the vertex set of $DK_{6k+11}$ be $X = Z_{6k+9}\cup\{\infty_1,\infty_2\}$. An optimal $(6k+11,4k+6)-DCC$ consists of $9k+19$ $(4k+6)$-circuits. Firstly, define the following $(4k+6)$-circuits on the set $X$:

$$A^j = \langle a_0^j, a_1^j, \ldots, a_{4k+5}^j\rangle,\quad 0\le j\le \lceil\frac{k-2}{4}\rceil \text{ and}$$
$$B^j = \langle b_0^j, b_1^j, \ldots, b_{4k+5}^j\rangle,\quad 0\le j\le \lfloor\frac{k-1}{4}\rceil,$$

where $a_{2i}^j = 3i$, $a_{2i+1}^j = 3i-1-6j$, $b_{2i}^j = 3i$, $b_{2i+1}^j = 3i-2-6j$ $(0\le i\le 2k+2)$. For a circuit $T^j = \langle t_0,t_1,\ldots,t_m\rangle$, denote the circuits

$$T^{-j} = \langle t_m, t_{m-1},\ldots,t_1,t_0\rangle \text{ and } T^j+s = \langle t_0+s, t_1+s,\ldots,t_m+s\rangle.$$

Then $\{A^j+s, A^{-j}+s; 0\le j\le\lceil\frac{k-2}{2}\rceil, s=0,1,2\}$ and $\{B^j+s, B^{-j}+s; 0\le j\le\lfloor\frac{k-1}{2}\rfloor, s=0,1,2\}$ produce

$$6(\lceil\frac{k-2}{4}\rceil+\lfloor\frac{k-1}{4}\rfloor+2) = \begin{cases}3k+9 & (k \text{ odd})\\ 3k+6 & (k \text{ even})\end{cases}$$

$(4k+6)$-circuits.

Furthermore, construct a $(4k+6)$-circuit on $X$ by

$$P = \langle \infty_1, c_0, c_1, \ldots, c_{2k+1}, d_0, d_1, \ldots, d_{2k+1} \rangle.$$

Developing the circuit $P$ modulo $6k+9$ produces $6k+9$ $(4k+6)$-circuits. Breaking one of them (*e.g.*, $P$) into two new circuits:

$$P_1 = \langle \infty_1, \infty_2, d_0, c_0, c_1, \ldots, c_{2k+1}, d_1, d_2, \ldots, d_{2k+1} \rangle \text{ and}$$
$$P_2 = \langle \infty_2, \infty_1, c_0, d_0, d_1, \ldots, d_{2k+1}, c_1, c_2, \ldots, c_{2k+1} \rangle,$$

we obtain $9k+19$ (when $k$ is odd) or $9k+16$ (when $k$ is even) $(4k+6)$-circuits. When $k$ is even, add three $(4k+6)$-circuits $Q+s$, $s=0,1,2$, where

$$Q = \langle q_0, q_1, \ldots, q_{4k+5} \rangle,$$
$$q_{2i} = 3i, q_{2i+1} = 3i - \frac{k+4}{2} - 4\lceil \frac{k}{4} \rceil (0 \le i \le 2k+2).$$

Finally, let us define the parameters $c_i$ and $d_i$ in $P$ as follows.

**Case 1:** $k = 2t-1$.

When $t$ is odd,

$$c_{2i} = 3i (0 \le i \le 2t-1),$$
$$c_{2i+1} = \begin{cases} 9t-1+2i-3\lfloor \frac{i}{3} \rfloor & (0 \le i \le \frac{3t-5}{2}) \\ 12t-6-3i & (\frac{3t-3}{2} \le i \le 2t-2) \\ 12t-1 & (i = 2t-1) \end{cases};$$
$$d_{2i} = 9t-7-3i (0 \le i \le 2t-1),$$
$$d_{2i+1} = \begin{cases} 12t-3-2i+3\lfloor \frac{i}{3} \rfloor & (0 \le i \le \frac{3t-5}{2}) \\ 9t+2+3i & (\frac{3t-3}{2} \le i \le 2t-2) \\ 9t-3 & (i = 2t-1) \end{cases}.$$

When $t$ is even,

$$c_{2i} = 3i (0 \le i \le 2t-1),$$
$$c_{2i+1} = \begin{cases} 9t-1+2i-3\lfloor \frac{i+1}{3} \rfloor & (0 \le i \le \frac{3t-4}{2}) \\ 12t-6-3i & (\frac{3t-2}{2} \le i \le 2t-2) \\ 12t-1 & (i = 2t-1) \end{cases};$$
$$d_{2i} = 9t-4-3i (0 \le i \le 2t-1),$$
$$d_{2i+1} = \begin{cases} 12t-2-2i+3\lfloor \frac{i+1}{3} \rfloor & (0 \le i \le \frac{3t-4}{2}) \\ 9t+5+3i & (\frac{3t-2}{2} \le i \le 2t-2) \\ 9t & (i = 2t-1) \end{cases}.$$

**Case 2:** $k = 2t$.
When $t$ is odd,

$$c_{2i} = \begin{cases} 3i & (0 \le i \le 2t-1) \\ 3t-2 & (i = 2t) \end{cases},$$

$$c_{2i+1} = \begin{cases} 9t+4+2i-3\lfloor\frac{i+1}{3}\rfloor & (0 \le i \le \frac{3t-5}{2}) \\ 12t-6-3i & (\frac{3t-3}{2} \le i \le 2t-2) \\ 12t+3 & (i = 2t-1) \\ 9t & (i = 2t) \end{cases};$$

$$d_{2i} = \begin{cases} 9t-4-3i & (0 \le i \le 2t-1) \\ 6t-5 & (i = 2t) \end{cases},$$

$$d_{2i+1} = \begin{cases} 12t+1-2i+3\lfloor\frac{i+1}{3}\rfloor & (0 \le i \le \frac{3t-5}{2}) \\ 9t+11+3i & (\frac{3t-3}{2} \le i \le 2t-2) \\ 9t+2 & (i = 2t-1) \\ 12t+2 & (i = 2t) \end{cases}.$$

When $t$ is even,

$$c_{2i} = \begin{cases} 3i & (0 \le i \le 2t-1) \\ 6t-2 & (i = 2t) \end{cases},$$

$$c_{2i+1} = \begin{cases} 9t+2+2i-3\lfloor\frac{i}{3}\rfloor & (0 \le i \le \frac{3t-6}{2}) \\ 12t-6-3i & (\frac{3t-4}{2} \le i \le 2t-2) \\ 12t+4 & (i = 2t-1) \\ 9t & (i = 2t) \end{cases};$$

$$d_{2i} = \begin{cases} 9t-4-3i & (0 \le i \le 2t-1) \\ 3t-2 & (i = 2t) \end{cases},$$

$$d_{2i+1} = \begin{cases} 12t+3-2i+3\lfloor\frac{i}{3}\rfloor & (0 \le i \le \frac{3t-6}{2}) \\ 9t+11+3i & (\frac{3t-4}{2} \le i \le 2t-2) \\ 9t+1 & (i = 2t-1) \\ 12t+2 & (i = 2t) \end{cases}.$$

**Proof:**

1° The elements in $A^j$ (for any fixed $j$) are distinct, since $a^j_{2i} \equiv 0 \pmod 3$ and $a^j_{2i+1} \equiv 2 \pmod 3$. Similarly, the elements in each $B^j$ (or $Q$) are distinct.

2° In each $A^j$ there are only two differences of ordered pairs: $-1-6j$ and $4+6j$, where $0 \le j \le \lceil\frac{k-2}{4}\rceil$. It is not difficult to see that $A^j$,

$A^j+1$ and $A^j+2$ cover just all ordered pairs $(x, x-1-6j)$ and $(x, x+4+6j)$, $x \in Z_{6k+9}$. Similarly, for fixed $j$,

$B^j, B^j+1$ and $B^j+2$ cover just all pairs $(x, x-2-6j)$ and $(x, x+5+6j)$;
$A^{-j}, A^{-j}+1$ and $A^{-j}+2$ cover just all pairs $(x, x+1+6j)$ and $(x, x-4-6j)$;
$B^{-j}, B^{-j}+1$ and $B^{-j}+2$ cover just all pairs $(x, x+2+6j)$ and $(x, x-5-6j)$. And, $Q$ cover just all pairs $(x, x-\frac{k+4}{2}-4\lceil\frac{k}{4}\rceil)$ and $(x, x+\frac{k+10}{2}+4\lceil\frac{k}{4}\rceil)$, where $x \in Z_{6k+9}$. Therefore, the $3k+9$ circuits $A^j+s$, $B^j+s$, $A^{-j}+s$, $B^{-j}+s$ and $Q$ (only for even $k$) cover all ordered pairs $(x, x+y)$, where $x \in Z_{6k+9}$ and $y$ run over the set

$$\{\pm d; 1 \le d \le \frac{3k+2}{2}, d \not\equiv 0 \pmod 3\} \cup \{\pm\frac{3k+8}{2}, -\frac{3k+4}{2}, -\frac{3k+10}{2}\}$$
when $k \equiv 0 \pmod 4$,
$$\{\pm d; 1 \le d \le \frac{3k+7}{2}, d \not\equiv 0 \pmod 3\}$$
when $k \equiv 1 \pmod 4$,
$$\{\pm d; 1 \le d \le \frac{3k+4}{2}, d \not\equiv 0 \pmod 3\} \cup \{-\frac{3k+8}{2}, \frac{3k+14}{2}\}$$
when $k \equiv 2 \pmod 4$,
$$\{\pm d; 1 \le d \le \frac{3k+5}{2}, d \not\equiv 0 \pmod 3\} \cup \{\pm\frac{3k+11}{2}\}$$
when $k \equiv 3 \pmod 4$.

The number of these differences $y$ is $2k+6$, no matter what value $k$ is congruent with.

3° The other $4k+2$ differences are occupied in the circuit $P$.

Below, we verify this conclusion only for the case $k \equiv 1 \pmod 4$. The verifications are similar for the other cases. It is easy to see that

$$\{c_{2i+1}-c_{2i}\}_i = \{-3k-4\} \cup [6, \frac{3k+9}{2}]_6 \cup (\cup_{i=1}^{\frac{k-1}{4}} [-\frac{3k+3}{2}-6i, -\frac{3k-1}{2}-6i]),$$
$$\{c_{2i+2}-c_{2i+1}\}_i = [\frac{9k+15}{2}, 6k+6]_6 \cup (\cup_{i=1}^{\frac{k-1}{4}} [\frac{3k+5}{2}+6i, \frac{3k+9}{2}+6i]),$$
$$\{d_{2i+1}-d_{2i}\}_i = \{3k+4\} \cup [\frac{9k+9}{2}, 6k+3]_6 \cup (\cup_{i=1}^{\frac{k-1}{4}} [\frac{3k-1}{2}+6i, \frac{3k+3}{2}+6i]),$$
$$\{d_{2i+2}-d_{2i+1}\}_i = [3, \frac{3k+3}{2}]_6 \cup (\cup_{i=1}^{\frac{k-1}{4}} [-\frac{3k+9}{2}-6i, -\frac{3k+5}{2}-6i])$$

These differences are pairwise disjoint, and

$$\begin{aligned}\{3k+4\}&\cup(\cup_{i=1}^{\frac{k-1}{4}}[\frac{3k-1}{2}+6i,\frac{3k+3}{2}+6i])\\&\cup(\cup_{i=1}^{\frac{k-1}{4}}[\frac{3k+5}{2}+6i,\frac{3k+9}{2}+6i])\\&=\{3k+4\}\cup(\cup_{i=1}^{\frac{k-1}{4}}[\frac{3k-1}{2}+6i,\frac{3k+9}{2}+6i])\\&=[\frac{3k+11}{2},3k+4],\\\{-3k+4\}&\cup(\cup_{i=1}^{\frac{k-1}{4}}[-\frac{3k+5}{2}-6i])\\&\cup(\cup_{i=1}^{\frac{k-1}{4}}[-\frac{3k+3}{2}-6i,-\frac{3k-1}{2}-6i])\\&=[-(3k+4),-\frac{3k+11}{2}],\\ [3,\frac{3k+3}{2}]_6&\cup[6,\frac{3k+9}{2}]_6=[3,\frac{3k+9}{2}]_3,\\ [\frac{9k+9}{2},6k+3]_6&\cup[\frac{9k+15}{2},6k+6]_6=[-\frac{3k+9}{2},-3]_3.\end{aligned}$$

Therefore, these differences contained in the circuit $P$ are just

$$Z_{6k+9}^*\setminus\{\pm d;1\le d\le\frac{3k+7}{2},d\not\equiv0\pmod 3)\},$$

in which those $\pm d$ are contained in the circuits $A^j+s$, $B^j+s$, $A^{-j}+s$ and $B^{-j}+s$ (refer to 2°).

4° The elements in $P$ are distinct.

In fact, we have the following list of $c_i$ and $d_i$, where the elements $x$ (i.e., $c_i$ or $d_i$) are classified into three parts modulo 3.

When $k$ is odd

$$\begin{aligned}x\equiv0\pmod 3:&[0,3k]_3 &&(c_{2i})\\&[3(k+1),3\lfloor\frac{5k+3}{4}\rfloor]_3\cup[\frac{9k+15}{2},3\lfloor\frac{7k+7}{4}\rfloor]_3 &&(c_{2i+1})\\&[3\lfloor\frac{7k+11}{4}\rfloor,-6]_3\cup\{6\lfloor\frac{3k+3}{4}\rfloor\} &&(d_{2i+1}),\end{aligned}$$

$$
\begin{aligned}
x \equiv 1 \pmod 3 :& [\frac{9k+11}{2}, 3\lfloor\frac{7k+3}{4}\rfloor + 1]_3 && (c_{2i+1})\\
& [3\lfloor\frac{7k+7}{4}\rfloor + 1, p]_3 && (d_{2i+1}),\\
x \equiv 2 \pmod 3 :& [6\lfloor\frac{3k+3}{4}\rfloor + 2, 3\lfloor\frac{7k+3}{4}\rfloor - 1]_3 \cup \{-4\} && (c_{2i+1})\\
& [6\lfloor\frac{k+1}{4}\rfloor - 1, 6\lfloor\frac{3k+3}{4}\rfloor - 4]_3 && (d_{2i})\\
& [3\lfloor\frac{7k+7}{4}\rfloor - 1, q]_3 \cup [r, 6\lfloor\frac{k-2}{4}\rfloor + 2]_3 && (d_{2i+1}),
\end{aligned}
$$

where $p = 6k+1$ (when $k \equiv 1 \bmod 4$) or $6k+4$ (when $k \equiv 3 \bmod 4$), $q = 6k-1$ (when $k \equiv 1 \bmod 4$) or $6k+2$ (when $k \equiv 3 \bmod 4$), $r = \frac{3k-19}{4}$ (when $k \equiv 1 \bmod 4$) or $\frac{3k-1}{4}$ (when $k \equiv 3 \bmod 4$).

When $k$ is even

$$
\begin{aligned}
x \equiv 0 \pmod 3 :& [0, 3k-3]_3 && (c_{2i})\\
& [3k, 3\lfloor\frac{5k}{4}\rfloor]_3 \cup [\frac{9k}{2} + 6, 3\lfloor\frac{7k+2}{4}\rfloor]_3 \cup S && (c_{2i+1})\\
& [3\lfloor\frac{7k}{4}\rfloor + 6, r]_3 && (d_{2i+1}),\\
x \equiv 1 \pmod 3 :& \{p\} && (c_{2i})\\
& [\frac{9k}{2} + 4, 3\lfloor\frac{7k-2}{4}\rfloor + 1]_3 && (c_{2i+1})\\
& \{q\} && (d_{2i})\\
& [3\lfloor\frac{7k}{4}\rfloor + 7, -8]_3 \cup T && (d_{2i+1}),\\
x \equiv 2 \pmod 3 :& [6\lfloor\frac{3k+2}{4}\rfloor + 2, 3\lfloor\frac{7k+2}{4}\rfloor - 1]_3 && (c_{2i+1})\\
& [\frac{3k}{2} - 1, \frac{9k}{2} - 4]_3 && (d_{2i})\\
& [3\lfloor\frac{7k}{4}\rfloor + 5, -10]_3 \cup [3\lfloor\frac{k-2}{4}\rfloor - 1, \frac{3}{2}k - 4]_3 &&\\
& \quad \cup \{-7\} && (d_{2i+1}),
\end{aligned}
$$

where $p = 3k-2$ (when $k \equiv 0 \bmod 4$) or $\frac{3k}{2} - 2$ (when $k \equiv 2 \bmod 4$), $q = \frac{3}{2}k - 2$ (when $k \equiv 0 \bmod 4$) or $3k-5$ (when $k \equiv 2 \bmod 4$), $r = 6k+3$ (when $k \equiv 0 \bmod 4$) or $6k$ (when $k \equiv 2 \bmod 4$), $S = \{\frac{9}{2}k\}$ (when $k \equiv 0 \bmod 4$) or $\{\frac{9}{2}k, -6\}$ (when $k \equiv 2 \bmod 4$), $T = \{\frac{9k}{2} + 1\}$ (when $k \equiv 0 \bmod 4$) or $\phi$ (when $k \equiv 2 \bmod 4$). □

## 4 $C(v,m)$ for $m = 12, 14$ and $16$

**Theorem 3.** *For all integers $v \geq 12$ we have $C(v, 12) = T(v, 12)$.*

**Proof:** By Theorem 1, we need only to construct an optimal $(v, 12) - DCC$ for $v = 17, 18, 19$ and $20$. In what follows, the number of circuits in an optimal $DCC$ is denoted by $c$ and the number of repeated arcs (ordered pairs) in an optimal $DCC$ is denoted by $r$.

An optimal $(17, 12) - DCC$ on the set $Z_{16} \cup \{\infty\}$, $c = 23$, $r = 4$.

$\langle \infty, 0, 2, 6, 12, 3, 11, 5, 14, 9, 8, 4 \rangle$, develop 16
$\langle i, i+1, i+15, i+4, i+5, i+3, i+8, i+9, i+7, i+12, i+13, i+11 \rangle$,
$0 \leq i \leq 3$
$\langle 0, 3, 6, 9, 12, 15, 2, 5, 8, 11, 14, 1 \rangle$,
$\langle 3, 0, 13, 10, 7, 4, 1, 14, 11, 8, 5, 2 \rangle$,
$\langle 1, 4, 7, 10, 13, 0, 2, 15, 12, 9, 6, 3 \rangle$,
repeated pairs: $(1, 0, 2, 3)$.

An optimal $(18, 12) - DCC$ on the set $Z_{18}$, $c = 26$, $r = 6$.

$\langle 0, 15, 3, 17, 6, 1, 9, 10, 2, 13, 7, 16 \rangle$, develop 18
$\langle i, i+17, i+3, i+1, i+6, i+5, i+9, i+7, i+12, i+11, i+15, i+13 \rangle$,
$0 \leq i \leq 5$
$\langle 0, 3, 6, 9, 12, 15, 16, 1, 2, 5, 8, 11 \rangle$,
$\langle 1, 4, 7, 10, 13, 16, 15, 0, 11, 14, 17, 2 \rangle$,
repeated pairs: $(15, 16), (1, 2), (0, 11)$.

An optimal $(19, 12) - DCC$ on the set $Z_{18} \cup \{\infty\}$, $c = 29$, $r = 6$.

$\langle \infty, 0, 3, 9, 16, 6, 15, 13, 10, 4, 14, 7 \rangle$, develop 18
$\langle i, i+17, i+3, i+2, i+6, i+5, i+9, i+8, i+12, i+11, i+15, i+14 \rangle$,
$0 \leq i \leq 2$
$\langle i+14, i+15, i+11, i+12, i+8, i+9, i+5, i+6, i+2, i+3, i+17, i \rangle$,
$0 \leq i \leq 2$
$\langle i, i+2, i+15, i+1, i+12, i+14, i+9, i+11, i+6, i+8, i+3, i+5 \rangle$,
$0 \leq i \leq 2$
$\langle 0, 5, 10, 15, 2, 7, 12, 17, 4, 9, 14, 1 \rangle$,
$\langle 1, 6, 11, 16, 3, 8, 13, 0, 2, 4, 5, 7 \rangle$,
repeated pairs: $(1, 0, 2, 4, 5, 7)$.

An optimal $(20, 12) - DCC$ on the set $Z_{20}$, $c = 32$, $r = 4$.

$\langle 19, 2, 18, 3, 17, 4, 16, 5, 15, 6, 14, 0 \rangle$, develop 20
$\langle 2i, 2i + 4, 2i + 5, 2i + 6, 2i + 3, 2i + 18, 2i + 13, 2i + 17, 2i + 19, 2i + 16, 2i + 9, 2i + 2 \rangle, 0 \leq i \leq 9$
$\langle 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 1, 19 \rangle$,
$\langle 19, 17, 15, 13, 11, 9, 7, 5, 3, 1, 18, 0 \rangle$,
repeated pairs: $(0, 19), (1, 18)$.

□

**Theorem 4.** *For all integers $v \geq 14$ we have $C(v, 14) = T(v, 14)$.*

**Proof:** By Theorem 1, we only need to construct an optimal $(v, 14) - DCC$ for $19 \leq v \leq 24$. But $v = 21$ and 22 satisfy the equation $v(v - 1) \equiv 0 \pmod{14}$. Therefore, by [3] (see the proof of Theorem 1), $C(21, 14) = T(21, 14)$ and $C(22, 14) = T(22, 14)$. Using Theorem 2 ($k = 2$), we have $C(23, 14) = T(23, 14)$. As for $v = 19, 20$ and 24 we give the following constructions.

An optimal $(19, 14) - DCC$ on the set $Z_{18} \cup \{\infty\}$, $c = 25$, $r = 8$.

$\langle \infty, 1, 17, 2, 16, 3, 15, 4, 14, 5, 13, 6, 12, 7 \rangle$, develop 18
$\langle 2, 3, 4, 5, 6, 7, 8, 9, 17, 16, 15, 14, 13, 12 \rangle$,
$\langle 11, 8, 5, 2, 6, 10, 14, 0, 17, 4, 1, 16, 13, 12 \rangle$,
$\langle 9, 6, 3, 7, 11, 12, 13, 14, 15, 16, 2, 17, 0, 4 \rangle$,
$\langle 17, 14, 11, 15, 1, 5, 9, 13, 10, 7, 4, 8, 12, 16 \rangle$,
$\langle 13, 17, 3, 5, 7, 9, 10, 11, 12, 14, 16, 0, 1, 2 \rangle$,
$\langle 15, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0 \rangle$,
$\langle 0, 2, 4, 6, 8, 10, 12, 9, 11, 13, 15, 17, 1, 3 \rangle$,
repeated pairs: $(9, 17, 4), (2, 13, 12), (11, 12)$.

An optimal $(20, 14) - DCC$ on the set $Z_{20}$, $c = 28$, $r = 12$.

$\langle 1, 16, 2, 13, 3, 12, 4, 11, 5, 10, 6, 9, 7, 8 \rangle$, develop 20
$\langle i, i + 4, i + 8, i + 12, i + 16, i + 13, i + 10, i + 7, i + 15, i + 3, i + 11, i + 19, i + 1, i + 18 \rangle, 0 \leq i \leq 3$
$\langle 11, 13, 12, 14, 16, 18, 2, 10, 9, 8, 7, 4, 6, 3 \rangle$,
$\langle 13, 15, 17, 19, 7, 6, 5, 4, 3, 2, 1, 0, 8, 10 \rangle$,
$\langle 18, 17, 16, 15, 14, 13, 0, 19, 3, 5, 7, 9, 11, 10 \rangle$,
$\langle 10, 12, 11, 16, 0, 17, 1, 9, 6, 8, 5, 2, 19, 18 \rangle$,
repeated pairs: $(0, 2, 19, 1, 3, 11, 16, 13), (10, 13), (10, 18)$.

An optimal $(24,14)-DCC$ on the set $Z_{24}$, $c=40$, $r=8$.

$\langle 23,1,22,2,21,3,20,4,19,5,18,6,17,7\rangle$, develop 24
$\langle i,i+9,i+18,i+14,i+13,i+16,i+10,i+8,i+4,i+3,i+6,i+7,$
$i+12,i+19\rangle, 0\le i\le 9,$
$\langle 18,19,20,21,22,23,0,1,2,3,4,5,6,11\rangle$,
$\langle 9,7,14,8,6,4,2,0,22,5,3,1,23,21\rangle$,
$\langle 19,4,11,5,23,6,13,7,1,8,2,9,3,21\rangle$,
$\langle 21,6,2,20,5,1,4,0,7,3,10,17,18,23\rangle$,
$\langle 23,2,5,10,4,22,20,18,12,6,0,3,7,8\rangle$,
$\langle 3,2,1,0,23,8,15,9,16,17,22,7,5,12\rangle$,
repeated pairs: $(23,21,9,18,12,3,7,8)$.

□

**Theorem 5.** *For all integers $v\ge 16$ we have $C(v,16)=T(v,16)$.*

**Proof:** By Theorem 1, we only need to construct an optimal $(v,16)-DCC$ for $21\le v\le 28$. These constructions are listed as follows.

An optimal $(21,16)-DCC$ on the set $Z_{21}$, $c=27$, $r=12$.

$\langle 20,1,19,2,18,3,17,4,13,5,12,6,11,7,10,8\rangle$, develop 21
$\langle 0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15\rangle$,
$\langle 0,20,19,18,17,16,15,14,13,12,11,10,9,8,7,6\rangle$,
$\langle 0,10,20,9,19,8,18,7,17,6,16,5,4,3,2,1\rangle$,
$\langle 0,11,1,12,2,13,3,14,4,15,5,16,17,18,19,20\rangle$,
$\langle 15,16,6,17,7,18,8,19,9,20,10,0,1,2,3,4\rangle$,
$\langle 6,5,15,4,14,3,13,2,12,1,11,0,20,19,18,17\rangle$,
repeated pairs: $(0,1,2,3,4,15)$, $(0,20,19,18,17,6)$.

An optimal $(22,16)-DCC$ on the set $Z_{22}$, $c=29$, $r=2$.

$\langle 21,1,20,2,19,3,18,4,15,5,14,6,13,7,12,8\rangle$, develop 22
$\langle i,i+3,i+6,i+9,i+12,i+15,i+18,i+21,i+20,i+8,i+7,i+17,$
$i+16,i+4,i+14,i+2\rangle, i=0,1$
$\langle 1,11,21,2,5,8,9,10,20,19,18,16,15,14,12,13\rangle$,
$\langle 0,1,2,3,4,5,6,7,8,11,14,17,18,19,20,21\rangle$,
$\langle 20,18,6,16,14,13,12,11,10,9,7,5,3,2,1,0\rangle$,
$\langle 20,1,21,19,17,15,13,11,12,0,10,8,6,5,4,2\rangle$,
$\langle 2,12,10,11,9,19,7,6,4,3,13,14,15,16,17,20\rangle$,
repeated pairs: $(2,20)$.

An optimal $(23,16)-DCC$ on the set $Z_{23}$, $c=32$, $r=6$.

$\langle 22,1,21,2,20,3,19,4,15,5,14,6,13,7,12,8\rangle$, develop 23
$\langle i,i+10,i+20,i+7,i+19,i+18,i+21,i+1,i+4,i+5,i+17,i+15,$
$i+14,i+12,i+13,i+2\rangle, i=0,1$
$\langle i+4,i+7,i+10,i+22,i+20,i+9,i+8,i+6,i+16,i+3,i+13,i,$
$i+12,i+15,i+18,i+5\rangle, i=0,1$
$\langle 6,9,12,1,22,11,10,8,18,7,5,3,15,2,14,17\rangle$,
$\langle 22,9,21,19,17,20,0,3,15,4,2,12,10,13,11,14\rangle$,
$\langle 3,6,7,8,9,10,11,12,22,21,20,18,17,16,14,15\rangle$,
$\langle 5,15,16,17,18,19,8,7,6,4,3,2,1,0,22,21\rangle$,
$\langle 14,13,12,11,9,19,20,21,22,0,1,2,3,4,16,5\rangle$,
repeated pairs: $(3,15),(14,22,21,5)$.

An optimal $(24,16)-DCC$ on the set $Z_{24}$, $c=35$, $r=8$.

$\langle 1,23,2,22,3,21,4,20,5,19,6,18,7,15,8,10\rangle$, develop 24
$\langle i,i+1,i+7,i+13,i+17,i+14,i+11,i+8,i+18,i+4,i+23,i+9,$
$i+10,i+5,i+6,i+3\rangle, i=0,1,2$
$\langle i+16,i+20,i,i+4,i+8,i+12,i+13,i+19,i+1,i+22,i+2,i+6,$
$i+5,i+11,i+15,i+10\rangle, i=0,1$
$\langle 6,16,22,17,12,18,19,20,15,21,3,4,14,0,23,5\rangle$,
$\langle 4,5,15,1,0,19,14,9,8,7,2,21,16,17,18,22\rangle$,
$\langle 8,9,4,10,7,17,3,13,23,22,21,20,19,18,0,6\rangle$,
$\langle 5,4,3,2,12,11,21,22,23,20,17,16,15,14,13,8\rangle$,
$\langle 6,10,14,15,16,2,1,20,21,18,17,23,0,3,22,19\rangle$,
$\langle 0,21,7,11,10,9,6,19,23,18,13,12,22,8,3,4\rangle$,
repeated pairs: $(5,6,8),(0,3,4),(6,19)$.

An optimal $(25,16)-DCC$ on the set $Z_{25}$, $c=38$, $r=8$.

$\langle 24,1,23,2,22,3,21,4,15,5,14,6,13,7,12,8\rangle$, develop 25

$\langle i,i+10,i+20,i+19,i+17,i+18,i+7,i+21,i+8,i+9,i+12,i+11,$
$i+24,i+2,i+1,i+13\rangle, 0\leq i\leq 6$

$\langle i+20,i+5,i+18,i+6,i+9,i+7,i+17,i+2,i,i+14,i+3,i+16,$
$i+19,i+22,i+10,i+8\rangle, i=0,1$

$\langle 5,19,18,16,17,15,13,11,10,23,1,24,9,8,6,7\rangle$,
$\langle 24,12,10,9,22,0,1,2,3,4,5,6,20,7,8,21\rangle$,
$\langle 18,21,9,19,4,2,15,16,14,12,24,0,13,11,23,8\rangle$,
$\langle 10,22,7,20,8,11,2,16,5,3,17,6,4,18,1,14\rangle$,
repeated pairs: $(10,8,18,1,13,11,2,14)$.

An optimal $(26,16)-DCC$ on the set $Z_{26}$, $c=41$, $r=6$.

$\langle 25,1,24,2,23,3,22,4,13,5,12,6,11,7,10,8\rangle$, develop 26

$\langle i,i+13,i+1,i+2,i+17,i+18,i+7,i+23,i+22,i+12,i+11,i+24,$
$i+9,i+20,i+4,i+14\rangle, 0\leq i\leq 9$

$\langle 10,24,12,0,14,3,15,25,13,2,16,5,17,6,22,21\rangle$,
$\langle 11,12,13,14,15,16,17,7,19,4,18,8,20,10,9,23\rangle$,
$\langle 21,8,22,7,6,20,5,19,3,17,1,13,25,11,23,10\rangle$,
$\langle 19,9,8,7,21,6,18,2,14,24,11,10,22,25,12,1\rangle$,
$\langle 22,9,21,11,25,1,15,4,16,0,12,24,10,23,8,19\rangle$,
repeated pairs: $(21,10),(22,25,1,19)$.

An optimal $(27,16)-DCC$ on the set $Z_{27}$, $c=44$, $r=2$.

$\langle 26,1,25,2,24,3,23,4,17,5,16,6,15,7,14,8\rangle$, develop 27

$\langle i,i+12,i+24,i+25,i+14,i+3,i+17,i+4,i+9,i+10,i+6,i+16,$
$i+15,i+20,i+18,i+1\rangle, 0\leq i\leq 10$

$\langle 13,11,0,23,24,9,25,10,26,4,14,1,15,2,16,3\rangle$,
$\langle 26,11,23,19,17,0,5,15,14,13,12,1,6,2,7,3\rangle$,
$\langle 18,16,14,12,10,8,6,4,2,25,21,24,20,23,26,22\rangle$,
$\langle 9,12,15,18,21,22,25,1,4,7,10,13,2,5,8,11\rangle$,
$\langle 6,9,7,5,3,8,4,0,26,2,12,11,14,17,20,21\rangle$,
$\langle 21,17,15,13,16,19,22,23,8,9,5,1,24,0,3,6\rangle$,
repeated pairs: $(21,6)$.

An optimal $(28,16)-DCC$ on the set $Z_{28}$, $c=48$, $r=12$.

$\langle 27,1,26,2,25,3,24,4,22,5,21,6,20,7,19,8\rangle$, develop 28

$\langle i,i+5,i+6,i+13,i+12,i+22,i+18,i+19,i+11,i+20,i+27,$
$i+26,i+7,i+10,i+8,i+2\rangle, 0\le i\le 12$

$\langle i+10,i+20,i+23,i+26,i+1,i+4,i+7,i+17,i+22,i+27,i+9,$
$i+5,i+25,i+19,i+15,i+11\rangle, i=0,1$

$\langle 25,7,3,13,18,10,2,12,8,4,24,22,16,21,15,20\rangle$,

$\langle 11,3,4,5,15,7,1,27,25,6,0,26,24,18,14,10\rangle$,

$\langle 7,27,21,26,25,0,3,6,9,19,24,1,23,17,13,5\rangle$,

$\langle 2,24,5,27,4,14,19,26,8,0,22,25,23,21,17,9\rangle$,

$\langle 5,12,4,26,3,25,2,13,9,1,11,7,14,6,16,8\rangle$,

repeated pairs: $(11,10),(14,12,10,8,5,7),(13,11,9,2)$.

□

## References


[1] F.E. Bennett and J.X. Yin, Packings and coverings of the complete directed graph with $k$-circuits, preprint.

[2] J.X. Yin, On coverings of $DK_v$ by even circuits, to appear in *JSPI.*

[3] J.C. Bermond, Cycles dans les graphes et $G$-configurations, Thesis, Univ. of Paris XI (Orsay), April 1975.

[4] J.C. Bermond and V. Faber, Decomposition of the complete directed graph into $k$-circuits, *J. Combin. Theory (B)* **21** (1976), 146–155.