The Homomorphism Factoring Problem

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Abstract

Let H and Y be fixed digraphs, and let h be a fixed homomorphism of H to Y. The *Homomorphism Factoring Problem with respect to* (H, h, Y) is described as follows:

HFP(H, h, Y)
INSTANCE: A digraph G and a homomorphism g of G to Y.
QUESTION: Does there exist a homomorphism f of G to H such that $h \circ f = g$? That is, can the given homomorphism g be factored into the composition of h and some homomorphism f of G to H?

We investigate the complexity of this problem and show that it differs from that of the H-colouring problem, i.e., the decision problem "is there a homomorphism of a given digraph G to the fixed digraph H?", and of restricted versions of this problem. We identify directed graphs H for which any homomorphism factoring problem involving H is solvable in polynomial time. By contrast, prove that for any fixed undirected graph Y which is not a path on at most four vertices, there exists a fixed undirected graph H, which can be chosen to be either a tree or a cycle, and a fixed homomorphism h of H to Y such that HFP(H, h, Y) is NP-complete, and if Y is such a path then HFP(H, h, Y) is polynomial.

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1 Introduction

We use the definitions and notation of [5]. All digraphs in this paper are assumed to be finite and have no multiple arcs. Moreover, they are loopless unless explicitly stated otherwise. Graphs are usually viewed as special digraphs in which each undirected edge is regarded as two oppositely oriented arcs, that is, as their *equivalent digraph*. Thus, definitions made for digraphs implicitly hold for undirected graphs.

Suppose G, H and Y are sets, and $h : H \to Y$ and $g : G \to Y$ are functions. A fundamental question is whether there exists a function $f : G \to H$ such that $h \circ f = g$, that is, whether the function g can be factored into the composition of the functions f and h. In this paper we consider this question when G, H and Y are (the vertex-sets of) directed graphs, and f, g, and h are digraph homomorphisms. This version of the question was suggested by G. Sabbidussi and C. Tardif. Some definitions are required before we can make a formal statement of *the homomorphism factoring problem.*

Let G and H be digraphs. A *homomorphism* of G to H is a function $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. If there is a homomorphism of G to H, we write $G \to H$, or $f : G \to H$ to emphasize the function f. If $f : G \to H$ is a homomorphism, we use $f(G)$ to denote the graph with vertex set $f(V(G))$ and edge set $\{f(x)f(y) : xy \in E(G)\}$. Note that $f(G)$ is a subgraph of H. If $f(G) = H$, that is, both f and the induced function from $E(G)$ to $E(H)$ are onto, then we say H is a *homomorphic image* of G. Note, however, that given a *graph* G, all of its homomorphic images are graphs as well; hence, a given homomorphism problem can be meaningfully restricted to graphs.

A homomorphism of a graph G to K_n is an n-colouring of G. For this reason, a homomorphism of a digraph D to a digraph H is called an H*-colouring* of D [20, 28]. If there is a homomorphism $D \to H$, we say that D is H*-colourable.* The literature contains many papers dealing with homomorphisms of graphs and digraphs. See, for example, [7, 11, 13, 15, 18, 19, 20, 31, 12, 29, 30, 22]. Other references, more specific to the problem we will consider, will be listed momentarily.

We now state the formal definition of *the homomorphism factoring problem.* Let H and Y be fixed digraphs and $h : H \to Y$ a fixed homomorphism.

> HFP(H, h, Y): The homomorphism factoring problem with respect to H, h and Y
> INSTANCE: A digraph G and a homomorphism $g : G \to Y$.
> QUESTION: Does there exist a homomorphism $f : G \to H$ such that $h \circ f = g$?

Each homomorphism factoring problem is clearly in NP.

Homomorphism factoring problems can be viewed in terms of vertex-coloured digraphs. We regard H as a digraph which has been Y-coloured — that is, each vertex of H has been assigned a colour, which is a vertex of Y, so that if u is adjacent to v in H, then the colour of u is adjacent to the colour of v in Y. An instance of the problem consists of a digraph G and a Y-colouring of G. The problem is to decide if there exists a homomorphism of G to H which maps vertices of G to vertices of the same Y-colour in H — that is, if there exists an H-colouring of G with the property that each vertex x of G is coloured by a vertex z of H of the same Y-colour of x, with respect to the given Y-colourings of G and H.

The complexity of deciding the existence of a homomorphism to a fixed digraph H has received considerable attention in the literature [1, 2, 3, 4, 6, 7, 9, 10, 16, 17, 20, 21, 22, 23, 24, 25, 28, 32]. It is stated formally as follows:

H-COL: The H-colouring problem
INSTANCE: A digraph G.
QUESTION: Does there exist a homomorphism $G \to H$?

For *graphs*, Hell and Nešetřil [20] have shown that H-COL is polynomial if H is bipartite, and NP-complete if H is non-bipartite. For digraphs, although many families of digraphs have been completely classified, there is, as yet, no such clear distinction (see [2, 3]).

Furedi, Griggs, and Kleitman [13] asked if knowing G is 3-colourable makes C_5-COL any easier. (K_3-colourability is a necessary condition for C_5-colourability, since $C_5 \to K_3$.) This question is answered in the negative in [6], where it is proved that for any non-bipartite graph H, the H-colouring problem remains NP-complete even if the input is restricted to graphs G such that $G \to K_k$, where k is fixed, unless $K_k \to H$ when the problem is (trivially) polynomial.

A more general version of this problem is stated below. It is closely related to, but different than, the homomorphism factoring problem. Let H and Y be fixed digraphs.

RHP(H, Y): The restricted homomorphism problem with respect to H and Y
INSTANCE: A digraph G and a homomorphism $g : G \to Y$.
QUESTION: Does there exist a homomorphism $f : G \to H$?

It is clear that if H-COL is polynomial, then so is RHP(H, Y) for any Y. In a personal communication, Hell and Nešetřil conjectured that, for graphs, RHP(H, Y) is NP-complete whenever H-COL is NP-complete, unless $Y \to H$ in which case it is polynomial. That is, the information that the input graph admits a homomorphism to a fixed graph Y is no help unless it makes the problem trivial. In [8] we present some tools and verify the conjecture whenever H is a circulant graph and Y is any graph. We also give examples which show the conjecture is not true if extended to digraphs.

If there exists a homomorphism $h : H \to Y$, then the existence of a homomorphism $g : G \to Y$ is necessary for the existence of a homomorphism $f : G \to H$. The restricted homomorphism problem asks if f exists, while the homomorphism factoring problem asks if f exists subject to the condition that $h \circ f = g$.

Let Y be a directed graph which contains a vertex, say y, with a loop. Any digraph G admits a homomorphism to Y: map all vertices of G to y. Furthermore, if $h : H \to Y$ is chosen to be the constant homomorphism $h(x) = y$ for all $x \in V(H)$, then there exists a homomorphism $f : G \to H$ such that $h \circ f = g$ if, and only if, $G \to H$. Hence, in this case, H-COL, RHP(H, Y) and HFP(H, h, Y) are all polynomially equivalent. Given this, and the fact that there is, as yet, no complete classification of H-colouring problems for digraphs, it seems unlikely that we will be able to determine the complexity of HFP(H, h, Y) for all digraphs H and Y, and homomorphisms $h : H \to Y$.

The following is a brief outline of the remainder of the paper. In section two we describe some homomorphism factoring problems that are solvable in polynomial time. The main result of our study is in section three, where we show that for "almost all" graphs Y, there exists a graph H and a homomorphism $h : H \to Y$, such that HFP(H, h, Y) is NP-complete. The complexity of H-COL, RHP(H, Y) and HFP(H, h, Y) is compared in section four. Finally, in section five, we examine a related class of graph homomorphism problems.

2 Polynomial Problems

Let G and H be digraphs. A homomorphism $r : G \to H$ is called a *retraction* if there exists a homomorphism $c : H \to G$ such that $r \circ c$ is the identity map on H. If there exists a retraction of G to H, then H is called a *retract* of G. It follows from the definition that a retract of G is also an induced subdigraph of G. A digraph is called a *core* if it is has no proper subdigraph which is a retract. Every digraph G contains a unique (up to isomorphism) subdigraph H which is both a core and a retract of G [12, 30]; we call H *the core* of G.

Proposition 2.1 *Suppose that $h : H \to Y$ is a retraction. For any homomorphism $g : G \to Y$ there exists a homomorphism $f : G \to H$ such that $h \circ f = g$.*

Proof. Since h is a retraction, there exists $c : Y \to H$ such that $h \circ c$ is the identity map on Y. Let $f = c \circ g$. Then $h \circ f = h \circ c \circ g = g$. □

Corollary 2.2 *If $h : H \to Y$ is a retraction, then HFP(H, h, Y) is polynomial.*

Corollary 2.3 *Let H and Y be digraphs such that Y is the core of H. Then for any homomorphism $h : H \to Y$, HFP(H, h, Y) is polynomial.*

Proof. Since Y is the core of H, any homomorphism $h : H \to Y$ is a retraction. □

In contrast to Corollary 2.3, if h is not a retraction, it is possible for Y to be a retract of H and HFP(H, h, Y) to be NP-complete. We will prove this in Section 4.

We now describe several classes of graphs for which the homomorphism factoring problem is polynomial regardless of the choice of Y and h (cf. Theorem 3.3).

A graph G is called *homomorphically full* if every homomorphic image of G is a subgraph G. Six equivalent characterizations of these graphs are presented in [7], among them the statement that if H is a homomorphically full graph and u and v are nonadjacent vertices of H, then either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$.

Lemma 2.4 *Let H be a homomorphically full graph, let Y be any graph and let $h : H \to Y$ be a homomorphism. For any homomorphism $g : G \to Y$, there exists a homomorphism $f : G \to H$ such that $h \circ f = g$ if, and only if, $g(V(G)) \subseteq h(V(H))$.*

Proof. Clearly if $g(V(G)) \not\subseteq h(V(H))$, then $h \circ f \neq g$ for any $f : G \to H$. On the other hand, suppose $g(V(G)) \subseteq h(V(H))$. Since H is homomorphically full, for any pair of non-adjacent vertices u and v in H, either $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$. Thus, h is a retraction of H to $h(H)$. The result now follows from Proposition 2.1. □

Corollary 2.5 *Let H be a homomorphically full graph, let Y be any graph and let $h : H \to Y$ be a homomorphism. Then HFP(H, h, Y) is polynomial.*

Proof. By Lemma 2.4, an instance (G, g) of HFP(H, h, Y) is a YES instance if, and only if, $g(V(G)) \subseteq h(V(H))$. □

An *edge-coloured digraph*, G, is a $(k+1)$-tuple $(V(G), E_1(G), E_2(G), \ldots, E_k(G))$ where $V(G)$ is the set of vertices and each $E_i(G)$ is a binary relation on $V(G)$ called the *edges of colour i*. Given two edge-coloured digraphs G

and H, a *homomorphism* $f : G \to H$ is a function $f : V(G) \to V(H)$ such that $f(u)f(v) \in E_i(H)$ whenever $uv \in E_i(G)$. Edge-coloured digraph homomorphism problems have been studied in [10]. In particular, we define problem H-colouring where H is a fixed edge-coloured digraph analogously to the case where H is a fixed digraph.

Let G, H, and Y be digraphs and let (G, g) be an instance of HFP(H, h, Y). The following construction transforms the digraphs G and H, using the homomorphisms g and h, into edge-coloured digraphs G_c and H_c, such that (G, g) is a YES instance of HFP(H, h, Y) if, and only if, $G_c \to H_c$.

Suppose Y and H are digraphs and $h : H \to Y$ is a homomorphism. Let $V(Y) = \{y_0, y_1, \ldots, y_k\}$ and let $C = V(Y) \times V(Y)$; the set C is the set of edge colours of our new edge-coloured digraph. We construct the edge-coloured digraph H_c as follows:

- $V(H_c) = V(H)$;
- for each arc uv of H where $h(u) = y_i$ and $h(v) = y_j$, add the arc uv in colour (y_i, y_j) to H_c.

Let G, H, and Y be digraphs and let $h : H \to Y$ and $g : G \to Y$ be homomorphisms. It is easy to check that G_c admits a homomorphism to H_c if and only there exists a homomorphism $f : G \to H$ such that $h \circ f = g$. That is, if and only if (G, g) is a YES instance of HFP(H, h, Y). This gives the following proposition.

Proposition 2.6 *Let H, Y, h, and H_c be defined as above. Then HFP(H, h, Y) polynomially transforms to H_c-colouring.*

We now describe two edge-coloured digraph H-colouring problems that are polynomial, and use them to obtain more polynomial homomorphism factoring problems.

Let H be an edge-coloured digraph with the property that there is an enumeration of the vertices $h_0, h_1, h_2, \ldots, h_n$ such that if (h_i, h_j) is an arc of some colour, then $|i - j| = 1$ and for $i = 0, 1, 2, \ldots n - 1$, there is at least one arc between (h_i, h_{i+1}). If all arcs are the same colour, such a digraph is a superdigraph of an oriented path and a subdigraph of the undirected path of length n. Call such an edge-coloured digraph H a *quasi-path*. The algorithms in [16] or in [9] can easily be modified to prove the following theorem.

Theorem 2.7 [16, 9] *Let H be an edge-coloured quasi-path. Then H-colouring is polynomial.*

The H-colouring problem is also polynomial when the edge-coloured digraph H has the property that each vertex is incident with at most one in-arc of each edge colour and at most one out-arc of each edge colour. This follows from the fact that when a vertex is mapped into such an edge-coloured digraph, the image of each of its neighbours is uniquely determined. Examples of such graphs and digraphs are K_2, directed paths, and directed cycles. All of these are well known polynomial H-colouring problems.

Let $h : H \to Y$ be a homomorphism from a digraph H to a digraph Y. We say h is a *local injection* if, and only if, for each vertex $v \in V(H)$, the function h is an injection on each of the in-neighbourhood and the out-neighbourhood of v.

Two immediate corollaries of Proposition 2.6 and the two comments above are given below.

Corollary 2.8 *Let H and Y be digraphs such that H is a quasi-path (one edge colour), and $h : H \to Y$ a homomorphism. Then $HFP(H, h, Y)$ is polynomial.*

Proof. The edge-coloured digraph H_c, constructed as above, is also a quasi-path. Testing for the existence of a homomorphism to a quasi-path is polynomial. □

Corollary 2.9 *Let H and Y be digraphs and $h : H \to Y$ a homomorphism. If h is a local injection, then $HFP(H, h, Y)$ is polynomial.*

Proof. Since h is one-to-one on the in-neighbourhood of each vertex of H, each vertex of H_c, constructed as above, has at most one in-arc of each edge colour. Similarly, each vertex of H_c has at most out-arc of each edge colour. Therefore, H_c-colouring is polynomial and hence HFP(H, h, Y) is polynomial. □

3 NP-completeness Results

In this section we will show that for "almost all" graphs Y, there exists a graph H and a homomorphism $h : H \to Y$ such that HFP(H, h, Y) is NP-complete, and otherwise the problem is always polynomial (see Theorem 3.3). In fact, H can be chosen to be a tree or a cycle. This result also allows construction of a new class of H-colouring problems which are solvable in polynomial time.

Recall that we denote the path of *length* n by P_n. We assume throughout that $V(P_n) = \{0, 1, \ldots, n\}$ and $E(P_n) = \{i(i+1); i = 0, 1, \ldots, n-1\}$. Similarly, we denote by C_n the cycle of length n, and assume $V(C_n) = \{1, 2, \ldots, n\}$ and $E(P_n) = \{i(i+1); i = 1, \ldots, n-1\} \cup \{n1\}$.

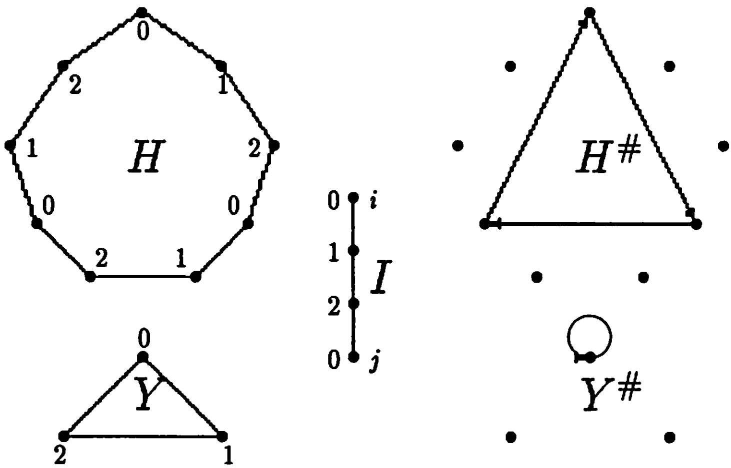

Figure 1: An example of HFP indicator construction

We now introduce the principal tool we use for proving NP-completeness of homomorphism factoring problems. It is a variation of the indicator construction in [20], which has proved to be a powerful tool for constructing polynomial reductions between digraph homomorphism problems [1, 2, 4, 6, 10, 24, 25, 26].

Let H and Y be two digraphs and let $h : H \to Y$ be a homomorphism. Let I be a digraph with distinguished vertices i and j, and let $t : I \to Y$ be a homomorphism. The *indicator construction with respect to* (I, i, j, t) transforms H and Y into two new digraphs $H^{\#}$ and $Y^{\#}$, either of which may contain loops. The vertex-set of $H^{\#}$ is $V(H)$ and $uv \in E(H^{\#})$ if, and only if, there is a homomorphism $r : I \to H$ such that $r(i) = u$, $r(j) = v$, and $h \circ r = t$. The vertex-set of $Y^{\#}$ is $V(Y)$ and the edge-set is the single arc $t(i)t(j)$. In cases of interest to us this arc will be a loop.

Consider the example in Figure 1. The graph H is C_9 and the graph Y is C_3. The numbers beside the vertices in H describe the homomorphism $h : H \to Y$; all vertices labeled with 0 are mapped to the vertex labeled 0 in Y, etc. Similarly, the graph I is P_3 and the homomorphism $t : I \to Y$ is also marked in the figure. The pair uv is an arc of $H^{\#}$ if, and only if, there is a homomorphism $r : I \to H$ with i mapping to u (labeled with 0 – so that $h \circ r(i) = t(i)$), j mapping to v (labeled with 0), and each other vertex x of I mapping to a vertex y of H with $h(y) = t(x)$, i.e., to a vertex having the same label (this is requied in order that $h \circ r(x) = t(x)$). The digraph $Y^{\#}$ contains the single arc from $t(i) = 0$ to $t(j) = 0$. Note that even if one begins with *graphs* the result of the construction will be *digraphs* which are

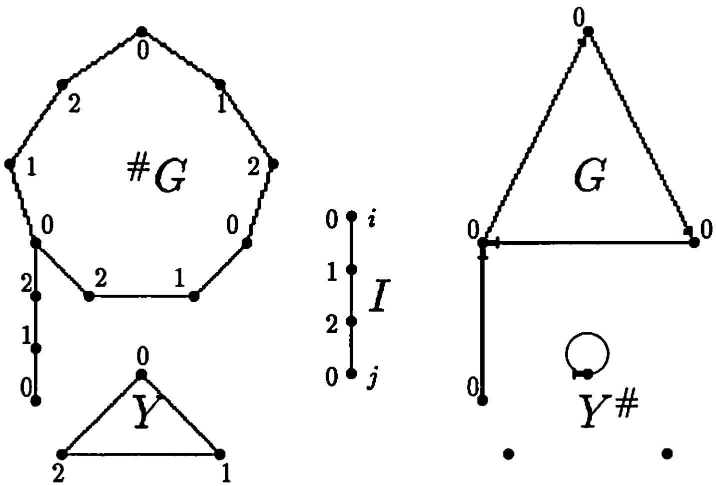

Figure 2: The construction of $^{\#}G$ from G.

not necessarily graphs.

An important point is that the homomorphism $h : H \to Y$ is also a homomorphism from $H^{\#}$ to $Y^{\#}$. To see this, note that given an arc uv of $H^{\#}$ there is homomorphism $r : I \to H$ such that $r(i) = u$, $r(j) = v$, and $h \circ r = t$. Now $t : I \to Y$, $t(i) = (h \circ r)(i) = h(u)$, and $t(j) = (h \circ r)(j) = h(v)$; therefore, $h(u)h(v)$ is an arc in $Y^{\#}$. Hence, HFP$(H^{\#}, h, Y^{\#})$ is a well-defined problem.

Lemma 3.1 *Let H and Y be digraphs and $h : H \to Y$ be a homomorphism. Let I be a digraph with distinguished vertices i and j and $t : I \to Y$ be a homomorphism. Let $H^{\#}$ and $Y^{\#}$ be the result of the indicator construction with respect to (I, i, j, t). Then HFP$(H^{\#}, h, Y^{\#})$ polynomially transforms to HFP(H, h, Y).*

Proof. Let (G, g) be an instance of HFP$(H^{\#}, h, Y^{\#})$, where $g : G \to Y^{\#}$ is a homomorphism. Let $^{\#}G$ be the digraph obtained by taking a copy of $V(G)$ and for each arc $uv \in E(G)$ putting a copy of I in $^{\#}G$ and identifying i with u and j with v. See Figure 2 for an example. The transformed instance of HFP(H, h, Y) is the pair $(^{\#}G, {}^{\#}g)$, where the homomorphism $^{\#}g : {}^{\#}G \to Y$ is defined as follows: (recall $V(Y) = V(Y^{\#})$)

$$^{\#}g(v) = \begin{cases} g(v) & \text{if } v \in V(G) \\ t(v) & \text{otherwise.} \end{cases}$$

The construction can clearly be accomplished in polynomial time.

To see that ${}^{\#}g$ is a homomorphism consider an arc $uv \in E({}^{\#}G)$. (In fact, uv belongs to some copy of I.) If neither u nor v belong to $V(G)$, then ${}^{\#}g(u){}^{\#}g(v) = t(u)t(v) \in E(Y)$ since $t : I \to Y$ is a homomorphism. If $u \in V(G)$, then ${}^{\#}g(u) = g(u) = t(i)$, since $g : G \to Y^{\#}$ and the only arc in $Y^{\#}$ is $t(i)t(j)$. If $v \in V(G)$, then similarily ${}^{\#}g(v) = t(j)$ and hence, ${}^{\#}g(u){}^{\#}g(v) = t(i)t(j) \in E(Y)$. If $v \notin V(G)$, then v belongs to some copy of I and ${}^{\#}g(u){}^{\#}g(v) = t(i)t(v) \in E(Y)$. If $u \notin V(G)$ and $v \in V(G)$, the argument is similar.

Finally, we show that there exists a homomorphism $f : G \to H^{\#}$ such that $h \circ f = g$ if, and only if, there exists a homomorphism ${}^{\#}f : {}^{\#}G \to H$ such that $h \circ {}^{\#}f = {}^{\#}g$.

Suppose there exists $f : G \to H^{\#}$ such that $h \circ f = g$. For each arc xy of $H^{\#}$, fix a homomorphism $r_{xy} : I \to H$ such that $r_{xy}(i) = x$, $r_{xy}(j) = y$, and $h \circ r_{xy} = t$. Define ${}^{\#}f : V({}^{\#}G) \to V(H)$ by

$$ {}^{\#}f(v) = \begin{cases} f(v) & \text{if } v \in V(G) \\ r_{f(x)f(y)}(v) & \text{if } v \text{ belongs to the copy of } I \text{ in } {}^{\#}G \\ & \text{that replaced the edge } xy \in E(G). \end{cases} $$

A similar argument to the above for ${}^{\#}g$ shows that ${}^{\#}f$ is a homomorphism. Further, if $v \in V(G)$, then $h \circ {}^{\#}f(v) = h \circ f(v) = g(v) = {}^{\#}g(v)$, and if v belongs to the copy of I in ${}^{\#}G$ that replaced the edge $xy \in E(G)$, then $h \circ {}^{\#}f(v) = h \circ r_{f(x)f(y)}(v) = t(v) = {}^{\#}g(v)$.

On the other hand, suppose there exists a homomorphism ${}^{\#}f : {}^{\#}G \to H$ such that $h \circ {}^{\#}f = {}^{\#}g$. Define $f : G \to H^{\#}$ by $f(v) = {}^{\#}f(v)$, for all $v \in V(G)$. Then $f(x)f(y) = {}^{\#}f(x)\,{}^{\#}f(y)$, and we claim this is an arc of $H^{\#}$. To see this, recall the each arc $xy \in E(G)$ was replaced, in ${}^{\#}G$, by a copy I_{xy} of I. Hence, ${}^{\#}f\big|_{I_{xy}}$ is a homomorphism of I to H, and so ${}^{\#}f(x)\,{}^{\#}f(y)$ is an arc of $H^{\#}$. □

Let $k, m \geq 1$ be integers. We define $S_{k,m}$ to be the generalized star obtained from m copies of P_k by choosing exactly one distinguished vertex in each copy of P_k, either vertex 0 or vertex k, and then identifying all these distinguished vertices. We refer to the copies of P_k as *branches* of $S_{k,m}$.

Suppose t is a homomorphism of P_k to a graph Y such that $t(0) = t(k)$. Then t induces a *natural homomorphism* s of $S_{k,m}$ to Y: map each copy of P_k in $S_{k,m}$ to Y using t.

A *good indicator* for a graph Y is a quadruple $(P_k, 0, k, t)$, where t is a homomorphism of P_k to Y such that $t(0) = t(k)$, and which also has following property: (*) If s is the natural homomorphism (induced by t) of $S_{k,m}$ to Y, and r is a homomorphism of P_k to $S_{k,m}$ such that $s \circ r = t$, then r maps P_k identically onto some branch of $S_{k,m}$. At first glance, property (*) may seem strange, but it turns out to be exactly what we need to prove our main result.

Lemma 3.2 *There exists a good indicator for each connected graph $Y \notin \{P_0, P_1, P_2, P_3\}$.*

Proof. We construct a good indicator for Y in each of the following three cases

Case 1: Y contains a cycle. Let $C = c_0, c_1, \ldots, c_{n-1}, c_0$ be a cycle in Y. Note that $n \geq 3$. Define $t : P_n \to Y$ by $t(k) = c_{k \pmod n}$, $k = 0, 1, \ldots, n$. The quadruple $(P_n, 0, n, t)$ is a good indicator for Y if condition (*) is satisfied. Let $r : P_n \to S_{n,m}$ be a homomorphism such that $s \circ r = t$, where s is the natural homomorphism, defined as above. Then, since $s \circ r(i) = c_{i \pmod n}$, for all i, $0 \leq i \leq n$, it follows that (*) holds.

Case 2: Y contains a vertex of degree at least three. Let y be a vertex in Y with neighbours u, v, w. Let $t : P_6 \to Y$ be defined by:

$$\begin{aligned} t(0) = t(2) = t(4) = t(6) &= y \\ t(1) &= u \\ t(3) &= v \\ t(5) &= w \end{aligned}$$

Again, $(P_6, 0, 6, t)$ is a good indicator if (*) is satisfied. Consider the natural homomorphism $s : S_{6,m} \to Y$. There are m vertices of $S_{6,m}$ mapped to v by s. Hence any homomorphism $r : P_6 \to S_{6,m}$, such that $s \circ r = t$, must map vertex 3 of P_6 to one of these vertices. Such a mapping has a unique extension to a homomorphism of P_6 to $S_{6,m}$ under the condition $s \circ r = t$. Hence, (*) is satisfied.

Case 3: Y is a path of length at least four. Label the first five vertices of such a path with y_0, y_1, y_2, y_3, y_4. Define a homomorphism $t : P_{12} \to Y$ so that for $i = 0, 1, \ldots, 12$, $t(i)$ equals y_0, y_1, y_2, y_1, y_2, y_3, y_4, y_3, y_2, y_3, y_2, y_1, y_0, respectively.

Similarly to the previous case (by considering images of vertex 6), any homomorphism $r : P_{12} \to S_{12,m}$ such that $s \circ r = t$ has a unique extention to a homomorphism of P_{12} to $S_{12,m}$. It follows that (*) is satisfied, and $(P_{12}, 0, 12, t)$ is a good indicator.

The result now follows, since any connected graph without a cycle, without a vertex of degree at least three, and without a path of length at least four must be one of P_0, P_1, P_2, or P_3. □

Theorem 3.3 *For each connected graph $Y \notin \{P_0, P_1, P_2, P_3\}$, there exists a graph H and a homomorphism $h : H \to Y$ such that HFP(H, h, Y) is NP-complete. For each graph $Y \in \{P_0, P_1, P_2, P_3\}$ and for all graphs H and all homomorphisms $h : H \to Y$, HFP(H, h, Y) is polynomial.*

Proof. First suppose Y is a connected graph and is not one of P_0, P_1, P_2, or P_3. By Lemma 3.2 there exists a good indicator $(P_k, 0, k, t)$ for Y.

Let D be a digraph for which D-colouring is NP-complete. Construct the graph H by beginning with a copy D and replacing each arc $uv \in E(D)$ with a copy of P_k, identifying 0 with u and k with v. There is a natural homomorphism h of H to Y: map each of these copies of P_k to Y using t.

Consider HFP$(H^{\#}, h, Y^{\#})$, the result of the indicator construction with respect to $(P_k, 0, k, t)$ on HFP(H, h, Y). Since $t(0) = t(k)$, the single arc in the digraph $Y^{\#}$ is a loop.

We claim that $H^{\#}$ consists of a digraph isomorphic to D together with several isolated vertices. This follows from the fact that $(P_k, 0, k, t)$ is a good indicator. Clearly each arc of D gives rise to an arc in $H^{\#}$. We must prove that $H^{\#}$ has no other arcs.

Let $r : P_k \rightarrow H$ be a homomorphism such that $h \circ r = t$. The function r must map P_k into one or more copies of P_k that replaced arcs of D. However, in the latter case, these several copies of P_k must all share a common vertex. Since the diameter of $r(P_k)$ is at most k (because r is homomorphism), r can be viewed as a homomorphism of P_k to a copy of $S_{k,m}$, for some positive integer m. Condition (*) asserts that r maps P_k identically onto some branch of $S_{k,m}$, i.e., there are no other arcs in $H^{\#}$.

Since $Y^{\#}$ is a loop, HFP$(H^{\#}, h, Y^{\#})$ is polynomially equivalent to $H^{\#}$-colouring (see the comment in the introduction). Moreover, since we can assume any instance of $H^{\#}$-colouring is connected (and trivally contains at least one arc), $H^{\#}$-colouring is polynomially equivalent to D-colouring. Therefore, HFP(H, h, Y) is NP-complete.

Now suppose that $Y \in \{P_0, P_1, P_2, P_3\}$. Let H be a graph. Since a graph G admits a homomorphism to a connected graph Y if, and only if, each connected components of G admits a homomorphism to Y, we assume in what follows that the input graph G is connected. Given a graph G which is not connected, one can apply the algorithms described below to each connected component of G.

An obvious necessary condition for (G, g) to be a YES instance of HFP(H, h, Y) is that $g(V(G)) \subseteq h(V(H'))$ for some connected component H' of H. Suppose $Y = P_0, P_1$ or P_2, and $h : H \rightarrow Y$ is a homomorphism. A homomorphism of a connected graph F *onto* P_0, P_1 or P_2 must be a retraction. Therefore, for each connected component H' of H, h is a retraction of H' to $h(H')$. Hence, (G, g) is a YES instance if, and only if, $g(V(G)) \subseteq h(V(H'))$ for some connected component H' of H.

Now suppose $Y = P_3$, h is a homomorphism $h : H \rightarrow Y$ and (G, g) is an instance of HFP(H, h, Y). If g is not onto Y, then as above (G, g) is a YES instance if, and only if, $g(G) \subseteq h(H')$ for some connected component H' of H. Therefore, assume $g(G) = Y$. Further suppose $h(H') = Y$ for some connected component H' of H; otherwise, (G, g) is a NO instance.

Let $W : (v = p_0)p_1 \ldots (p_n = u)$ be a shortest path in H from a vertex v, such that $h(v) = 0$, to a vertex u, such that $h(u) = 3$. Since W is a shortest path, no internal vertex of W is mapped to 0 or to 3. The vertices in W have the consecutive images under h: $0, 1, 2, 1,\ 2, 1, 2, \ldots,\ 1, 2, 3$. It is straightforward to check that there is a retraction r of H to W (recall that W is a subgraph of H) such that $h \circ r = h$. Let Q be the shortest path in G between x and y where the x and y are taken over all pairs x and y such that $g(x) = 0$ and $g(y) = 3$. This is the Shortest Pairs Problem, which is polynomial [14]. Again there is a retraction t from G to Q such that $g \circ t = g$. Finally, (G, g) is a YES instance of HFP(H, h, Y) if, and only if, there exists $f : Q \rightarrow P$ such that $h \circ f = g$. This is true if, and only if, the length of Q is greater than or equal to the length of W. □

The above theorem applies only when Y is connected. However, the answer for an instance (G, g) of HFP(H, h, Y) is YES if, and only if, the answer is YES for each instance consisting of a connected component of G and the restriction of g to that component. Since a homomorphic image of a connected graph is connected, each of these instances involve a single connected component of Y. Let $Y_1, Y_2, \ldots, Y_n$ be the connected components of Y, and for $i = 1, 2, \ldots, n$,, let H_i be the subgraph of H induced by $h^{-1}(V(Y_i))$. Further, let h_i be the restriction of h to $V(H_i)$. Note that h_i is a homomorphism of H_i to Y_i. The above discussion implies that HFP(H, h, Y) is polynomial if each problem HFP(H_i, h_i, Y_i) is polynomial, and is NP-complete if some problem HFP(H_i, h_i, Y_i) is NP-complete. It is therefore sufficient to consider homomorphism factoring problems in which that graph Y is connected.

In the proof of Theorem 3.3 the result of the indicator construction, $H^{\#}$, is always a digraph isomorphic to D together with some isolated vertices. Since there exist bipartite digraphs, D, for which D-colouring is NP-complete, we have the following result.

Corollary 3.4 *Let Y be a connected graph and suppose that $Y \notin \{P_0, P_1, P_2, P_3\}$. Then there exists a bipartite graph H and a homomorphism $h : H \rightarrow Y$ such that HFP(H, h, Y) is NP-complete.*

The following two corollaries suggest that even a restriction of H to special classes of graphs does not yield polynomial homomorphism factoring problems. There exist oriented trees, T, such that T-colouring is NP-complete, [16] and [22]. Also, there exists oriented cycles, C, such that C-colouring is NP-complete, [23]. Since a good indicator is a path, if the digraph D is choosen in the proof of Theorem 3.3 to be an oriented tree (resp. an oriented cycle), then H is also a tree (resp. a cycle).

Corollary 3.5 *Let Y be a connected graph and suppose that $Y \notin \{P_0, P_1,$*

P_2, P_3}. Then there exists a tree H and a homomorphism $h : H \to Y$ such that HFP(H, h, Y) is NP-complete.

Corollary 3.6 *Let Y be a connected graph and suppose that $Y \notin \{P_0, P_1, P_2, P_3\}$. Then there exists a cycle H and a homomorphism $h : H \to Y$ such that HFP(H, h, Y) is NP-complete.*

Theorem 3.3 is stated for graphs. However, the polynomial algorithms presented in the final case works when Y is a directed path of length at most 3. This gives the following.

Corollary 3.7 *Let Y be a directed path of length 0, 1, 2, or 3. Then for all digraphs H and all homomorphisms $h : H \to Y$, the problem HFP(H, h, Y) is polynomial.*

Using the above corollary and the following lemma, we have a new class of H-colouring problems that are all polynomial. This result has also been announced by Feder and by Hell, Nešetřil, and Zhu. Let $Hom(G, Y)$ denote the set of all homomorphisms from G to Y.

Lemma 3.8 *Let H and Y be digraphs and $h : H \to Y$ a homomorphism such that HFP(H, h, Y) is polynomial. Suppose that Y has the property that for any digraph G the set $Hom(G, Y)$ can be constructed in polynomial time. Then H-colouring is polynomial.*

Proof. We produce a polynomial time Turing reduction of H-colouring to HFP(H, h, Y). Let G be an instance of H-colouring. Construct $Hom(G, Y)$. Denote the elements of $Hom(G, Y)$ by $g_1, g_2, \ldots, g_m$. For each g_i, test whether (G, g_i) is a YES instance of HFP(H, h, Y). Since $Hom(G, Y)$ can be constructed in polynomial time and HFP(H, h, Y) is polynomial, this process is a Turing reduction of H-colouring to HFP(H, h, Y).

We claim there exists an i ($1 \leq i \leq m$) such that (G, g_i) is a YES instance of HFP(H, h, Y) if and only if G is a YES instance of H-colouring. On the one hand, the existence of such an i implies there exists $f : G \to H$ such that $h \circ f = g_i$. Trivially, G is a YES instance of H-colouring. On the other hand, if G is a YES instance of H-colouring, then there exists $f : G \to H$. The homomorphism $h \circ f : G \to Y$ must be g_i for some i since $Hom(G, Y)$ contains all homomorphisms from G to Y. Hence, (G, g_i) is a YES instance of HFP(H, h, Y). □

Corollary 3.9 *Let H be a digraph such that H admits a homomorphism to a directed path of length at most three. Then H-colouring is polynomial.*

Proof. Let Y be a directed path of length k, where $k \leq 3$, and let $h : H \rightarrow Y$ be a homomorphism. By the Corollary 3.7, HFP(H, h, Y) is polynomial. It is easy to check that given any digraph G, there exist at most k homomorphisms from G to Y. Since Y is a fixed path, k is a constant. Hence, by Lemma 3.8, H-colouring is polynomial. □

We conclude this Section by showing that if h is not a retraction, it is possible for Y to be a retract of H and HFP(H, h, Y) NP-complete. Let $W \notin \{P_0, P_1, P_2, P_3\}$ be a digraph which is a not a core. By Theorem 3.3 there exists a digraph Z and a homomorphism $c : Z \rightarrow W$ such that HFP(Z, c, W) is NP-complete. Let W' be a digraph isomorphic to the core of W, and set $H = Z \cup W \cup W'$ (disjoint union) and $Y = W \cup W'$ (disjoint union). We claim that HFP(H, h, Y) is NP-complete. Since $Z \rightarrow W$, the digraph Y is a retract of H. Let h be the extension of c to H obtained by mapping the copy of $W \cup W'$ in H onto the copy of W' in Y. It remains to show that HFP(H, h, Y) is NP-complete. This follows from Proposition 3.10 below, which gives a version of proof by restriction for homomorphism factoring problems.

Proposition 3.10 *Let H be a digraph and $h : H \rightarrow Y$ a homomorphism. Suppose H' is an induced subgraph of H and $Y' = h(H')$ is an induced subgraph of Y. Then HFP(H', h', Y') polynomially transforms to HFP(H, h, Y), where h' denotes the restriction of h to $V(H')$.*

Proof. Let (G, g) be an instance of HFP(H', h', Y'). The same ordered pair is the transformed instance of HFP(H, h, Y). The result is clear, since a homomorphism $f : G \rightarrow H$ such that $h \circ f = g$ also satisfies $h' \circ f = g$. □

By the Proposition, HFP(Z, c, W) polynomially transforms to HFP(H, h, Y). Thus the latter problem is NP-complete, as claimed.

4 Comparison of Complexity

Consider a sequence $c_{H-COL}, c_{RHP(H,Y)}, c_{HFP(H,h,Y)}$, where c_{Π} denotes the complexity of problem Π. If each element of the sequence is "polynomial" or "NP-complete", then there are eight possible sequences. We investigate here which of these are possible.

In the introduction we remarked that if the directed graph Y contains a vertex y with a loop, and $h : H \rightarrow Y$ is the constant homomorphism $h(x) = y$ for all $x \in V(H)$, then H-COL, RHP(H, Y) and HFP(H, h, Y) are all polynomially equivalent. Since H-COL can be polynomial or NP-complete, this takes care of the two sequences with identical terms. (Note that there are also examples in which Y has no loops.)

Consider the sequence NP-complete, polynomial, NP-complete. To see that this is possible, we proceed as follows: Let H' be the graph consisting of a fifteen-cycle and a twenty-cycle joined at vertex 0, and $Y' = C_5$. Let $h' : H' \to Y'$ be the homomorphism which takes vertex i of each cycle in H to vertex i (mod 5) of Y. Let t be the homomorphism of P_5 to C_5 defined by $t(i) = i$ (mod 5). The result of applying the indicator construction with respect to $(P_5, 0, 5, t)$ to H', h', and Y' is the digraph $H^{\#}$ consisting of a directed four-cycle and a directed three-cycle joined at a vertex, and the digraph $Y^{\#}$ which has a loop. Thus, HFP$(H^{\#}, h', Y^{\#})$ is polynomially equivalent to $H^{\#}$-COL. The latter problem is proved to be NP-complete in [2]. Hence, HFP(H', h', Y') is also NP-complete. Now let H be the graph $H' \cup C_3$, and let $Y = Y' \cup C_3$. Then H-COL is NP-complete [20], and since $Y \to H$, RHP(H, Y) is polynomial. Let h be the extension of h' to a homomorphism of H to Y that maps the copy of C_3 in H to the copy of C_3 in Y. Then HFP(H, h, Y) is NP-complete by Proposition 3.10.

We now show that the sequence NP-complete, NP-complete, polynomial is possible. For any positive integer k, C_{2k+1}-COL is NP-complete [20]. It is proved in [26], and is implicit in [29] that for $k \geq 2$, RHP(C_{2k+1}, C_{2k-1}) is NP-complete. On the other hand, we have the following:

Theorem 4.1 *For any homomorphism $h : C_{2k+1} \to C_{2k-1}$, HFP$(C_{2k+1}, h, C_{2k-1})$ is polynomial.*

Proof. Let the vertices of C_{2k+1} be $\{l_1, l_2, \ldots, l_{2k+1}\}$ and the vertices of C_{2k-1} be $\{s_1, s_2, \ldots, s_{2k-1}\}$. There is essentially one homomorphism of C_{2k+1} to C_{2k-1}, so without loss of generality we can assume h is the following homomorphism:

$$\begin{aligned} h(l_1) &= h(l_{2k}) = s_1 \\ h(l_{2k-1}) &= h(l_{2k+1}) = s_{2k-1} \\ h(l_i) &= s_i \qquad \text{for } 2 \leq i \leq 2k-2 \end{aligned}$$

Let (G, g) be an instance of HFP(C_{2k+1}, h, C_{2k-1}). Let $A \subseteq V(G)$ be the vertices that are mapped by g to s_{2k-1} and are adjacent to a vertex mapped to s_{2k-2}. Similarily, let $B \subseteq V(G)$ be the vertices that are mapped by g to s_1 and are adjacent to a vertex mapped to s_2.

We claim that there exists a homomorphism $f : G \to C_{2k+1}$ such that $f \circ h = g$ if and only if there is no pair of vertices $a \in A$ and $b \in B$ such that $ab \in E(G)$.

Suppose that such a homomorphism f and vertices a and b as above exist. By the definition of A there exists $z \in V(G)$ such that $az \in E(G)$ and $g(z) = s_{2k-2}$. Since l_{2k-2} is the unique vertex of C_{2k+1} that is mapped by h to s_{2k-2}, we have $f(z) = l_{2k-2}$. It follows that $f(a) = l_{2k-1}$, and similarly,

$f(b) = l_1$. However, l_{2k-1} and l_1 are not adjacent in C_{2k+1}; hence, f is not a homomorphism, a contradiction.

On the other hand, assume the vertices a and b do not exist. Then the following function is the desired homomorphism f:

$$f(u) = \begin{cases} l_i & \text{if } g(u) = s_i \text{ for } 2 \leq i \leq 2k-2 \\ l_{2k+1} & \text{if } g(u) = s_{2k-1} \text{ and } u \notin A \\ l_{2k-1} & \text{if } u \in A \\ l_{2k} & \text{if } g(u) = s_1 \text{ and } u \notin B \\ l_1 & \text{if } u \in B \end{cases}$$

□

If H-COL is polynomial, then clearly so is RHP(H, Y) for any digraph Y. Hence the two sequences beginning polynomial, NP-complete are impossible.

Consider the sequence polynomial, polynomial, NP-complete. That such a sequence is possible follows from Theorem 3.3, with $Y = P_5$, say. (Since Y is bipartite and $H \rightarrow Y$, H is also bipartite; thus, both H-COL and RHP(H, Y) are polynomial.)

The final sequence is NP-complete, polynomial, polynomial. This sequence can be realized as follows: Let H be a non-bipartite core. Both RHP(H, H) and HFP(H, h, H), where $h : H \rightarrow H$ is any homomorphism, are trivially polynomial; whereas, H-COL is NP-complete.

5 The Two Homomorphism Problem

In this final section, we examine a slightly different homomorphism factorization problem. In some sense this problem is more restricted than those studied in the rest of this paper, yet again almost all problems are NP-complete.

Let Y be a fixed digraph:

THP(Y): The Two Homomorphism Problem for Y.
INSTANCE: A digraph H and a two homomorphisms $h_1 : H \rightarrow Y$ and $h_2 : H \rightarrow Y$.
QUESTION: Does there exist a homomorphism $f : H \rightarrow H$ such that $h_1 \circ f = h_2$?

We now state our main result for the Two Homomorphism Problem.

Theorem 5.1 *Let Y a connected graph and $Y \notin \{P_0, P_1, P_2, P_3\}$. Then THP$(Y)$ is NP-complete.*

Proof. By Corollary 3.4, there exists a bipartite graph H and a homomorphism $h : H \to Y$ such that HFP(H, h, Y) is NP-complete. Since any instance of HFP(H, h, Y) containing a non-bipartite graph G can easily be determined to be a NO instance (since $G \not\to H$), it suffices to restrict attention to bipartite input graphs. We can also assume G contains at least one edge. We present a polynomial transformation of (this restriction of) HFP(H, h, Y) to THP(Y).

Let (G, g) be an instance of HFP(H, h, Y), where G is bipartite and has at least one edge.

Let y_0y_1 be an edge of Y. We begin by examining two special cases. First, if $g(G)$ is y_0y_1, then (G, g) is a YES instance of HFP(H, h, Y) if and only if H contains an edge uv such that $h(u) = y_0$ and $h(v) = y_1$. Second, if $h(H)$ is y_0y_1, then (G, g) is a YES instance of HFP(H, h, Y) if and only if $g(G)$ is the edge y_0y_1. Therefore, assume neither $g(G)$ nor $h(H)$ is y_0y_1.

Let H' be the union of G and H. Let f_1 be the homomorphism that maps H to y_0y_1 and is equal to g on G. Similarly, let f_2 be the homomorphism that maps G to y_0y_1 and is equal to h on H. The instance (H', f_1, f_2) is a YES instance of $THP(Y)$ if and only if (G, g) is a YES instance of HFP(H, h, Y).

Suppose (G, g) is a YES instance of HFP(H, h, Y). This implies there is $f : G \to H$ such that $h \circ f = g$. Let uv be an edge in G such that $f_2(u) = y_0$ and $f_2(v) = y_1$. Let t be the homomorphism H' to H' defined by:

$$t(x) = \begin{cases} f(x) & \text{if } x \in V(G) \\ u & \text{if } x \in V(H) \text{ and } f_1(x) = y_0 \\ v & \text{if } x \in V(H) \text{ and } f_1(x) = y_1 \end{cases}$$

It is easy to check that $f_2 \circ t = f_1$.

On the other hand, suppose (H', f_1, f_2) is a YES instance of $THP(Y)$. Let $t : H' \to H'$ be a homomorphism such that $f_2 \circ t = f_1$. Consider t restricted to G. This is a homomorphism from G to H'. Either $t(G)$ is a subgraph of G or $t(G)$ is a subgraph of H. Since $f_1(G)$ is not y_0y_1 and $f_2(G)$ is y_0y_1, it must be the case that $t(G)$ is contained in H. By restricting t to G, f_1 to G and f_2 to H, we have $h \circ t = g$. Therefore, (G, g) is a YES instance of HFP(H, h, Y). □

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