Mixed Ramsey Numbers: Harmonious Chromatic Number versus Independence Number *

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Abstract

The *harmonious chromatic number* of a graph G, denoted $h(G)$, is the smallest number of colors needed to color the vertices of G so that adjacent vertices receive different colors and no two edges have the same pair of colors represented at their endvertices. The *mixed harmonious Ramsey number* $H(a, b)$ is defined to be the smallest integer p such that if a graph G has p vertices, then either $h(G) \geq a$ or $\alpha(G) \geq b$. For certain values of a and b, we determine the exact value of $H(a, b)$. In some other cases, we are able to determine upper and lower bounds for $H(a, b)$.

Given a graph G, let $|G|$ denote the number of vertices, $q(G)$ the number of edges, $\alpha(G)$ the (vertex) independence number, $\text{cl}(G)$ the clique number, $\Delta(G)$ the maximum vertex-degree, $\omega(G)$ the number of components, $\chi_1(G)$ the edge chromatic number and $\chi_2(G)$ the total chromatic number of G. Also, let $\overline{G}$ denote the complement of G. The Ramsey number $R(a, b)$ is defined as

$$R(a, b) = \inf\{p : |G| = p \Rightarrow \text{cl}(G) \geq a \text{ or } \alpha(G) \geq b\},$$

that is, $R(a, b)$ is the smallest integer p such that any graph on p vertices either has a complete subgraph with a vertices or an independent set of b vertices. One way to generalize this concept is as follows: let f and g be functions from the set of all graphs to the nonnegative integers. Then the *mixed Ramsey number* $R(f, g; a, b)$ is defined by

$$R(f, g; a, b) = \inf\{p : |G| = p \Rightarrow f(G) \geq a \text{ or } g(G) \geq b\}.$$

When no confusion results, f and g are often suppressed in the notation. Achuthan[1], Fink[4] and Wang[11] studied the mixed Ramsey numbers for the

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pair of functions $\chi_2(G)$ and $\Delta(\overline{G})$ and Lesniak-Foster, Polimeni and Vanderjagt[6] investigated them for the pair $\chi_1(G)$ and $\alpha(G)$ and for the pair $\chi_1(G)$ and $\Delta(\overline{G})$.

In this paper we study the mixed Ramsey numbers for the harmonious chromatic number and the independence number. The *harmonious chromatic number* of a graph G, denoted $h(G)$, is the smallest number of colors needed to color the vertices of G so that adjacent vertices receive different colors and no pair of colors occurs on more than one edge. For background on this parameter, see [7] and [12]. We define the *mixed harmonious Ramsey number* $H(a,b)$ to be $H(a,b) = R(h,\alpha\colon a,b)$, that is,

$$H(a,b) = \inf\{p : |G| = p \Rightarrow h(G) \geq a \text{ or } \alpha(G) \geq b\}.$$

We will derive several upper bounds, lower bounds, and exact formulas for $H(a,b)$ for various values of a and b.

We have not been able to find a general formula for $H(a,b)$. However, formulas have been found for special cases, as provided in the next several theorems. The proof of the first of these is straightforward and will be omitted.

Theorem 1 *$H(1,n) = H(n,1) = 1$ and $H(2,n) = H(n,2) = n$, for $n \geq 1$.*

Theorem 2 *If $a \geq 3$ and $b > \binom{a-1}{2}$, then $H(a,b) = b + \binom{a-1}{2}$.*

Proof Let

$$A := \binom{a-1}{2},$$

and let G consist of A independent edges together with $b - A - 1$ additional independent vertices. Then $|G| = A + b - 1$, $h(G) = a - 1$, and $\alpha(G) = b - 1$, so $H(a,b) \geq A + b$.

Now, let G be a graph with $A + b$ vertices, and suppose $h(G) \leq a - 1$. Then $q(G) \leq A$, and so

$$\alpha(G) \geq \omega(G) \geq p(G) - q(G) \geq b.$$

Therefore, $H(a,b) \leq A + b$. ∎

To prove our next theorem, two background results will be needed. Turán[10] proved the following fundamental result, which gives a lower bound for the number of edges in a graph in terms of its order and its independence number:

Theorem 3 *(Turan's Theorem) Let G be a graph with p vertices and with $\alpha(G) \leq \Lambda$. Then*

$$q \geq p\left\lfloor\frac{p}{\Lambda}\right\rfloor - \frac{\Lambda}{2}\left\lfloor\frac{p}{\Lambda}\right\rfloor - \frac{\Lambda}{2}\left\lfloor\frac{p}{\Lambda}\right\rfloor^2$$

Let $\tau(G)$ denote the maximum number of edge-disjoint triangles in a graph G. Schönheim[9] obtained an explicit formula for $\tau(K_m)$:

Theorem 4

$$\tau(K_m) = \left\lfloor \frac{m}{3} \left\lfloor \frac{m-1}{2} \right\rfloor \right\rfloor$$

except when $m \equiv 5 \pmod 6$, *in which case the formula is less by 1.*

Theorem 5 *Let* a *and* b *be positive integers,* $a \geq 3$, *that satisfy one of the following conditions:*

1. $a \equiv 2, 4 \pmod 6$ *and* $\binom{a-1}{2} - 2\tau(K_{a-1}) + 1 \leq b \leq \binom{a-1}{2}$

2. $a \equiv 0, 1, 3, 5 \pmod 6$ *and* $\binom{a-1}{2} - 2\tau(K_{a-1}) \leq b \leq \binom{a-1}{2}$

Then

$$H(a,b) = \left\lceil \frac{\binom{a-1}{2} + 3b - 2}{2} \right\rceil .$$

Proof Since the formula is easily verified for $3 \leq a \leq 6$, we can assume that $a \geq 7$. Throughout the proof, let

$$A := \binom{a-1}{2}$$

and let

$$F(a,b) := \left\lceil \frac{A + 3b - 2}{2} \right\rceil .$$

We first show that $H(a,b) \geq F(a,b)$. Let $p := F(a,b) - 1$ and let $G := (3b - 3 - p)K_2 \cup (p - 2b + 2)K_3$. Then $\alpha(G) = b - 1$ and $|G| = p$. Also $q(G) \leq A$ and

$$p - 2b + 2 = \left\lceil \frac{A - b}{2} \right\rceil \leq \tau(K_{a-1}).$$

Hence $h(G) \leq a - 1$, and so $H(a,b) \geq F(a,b)$.

We now show that $H(a,b) \leq F(a,b)$. Let $p := F(a,b)$ and let G be any graph with $|G| = p$ and $\alpha(G) \leq b - 1$. If $b < \frac{1}{3}A + 2$, then

$$\left\lfloor \frac{p}{b-1} \right\rfloor = 3.$$

Otherwise, if $b \geq \frac{1}{3}A + 2$, we see that

$$\left\lfloor \frac{p}{b-1} \right\rfloor = 2.$$

In either case, by Theorem 3, we have that

$$\begin{aligned}\binom{h(G)}{2} &\geq |E(G)| \\ &\geq p\left\lfloor\frac{p}{b-1}\right\rfloor - \frac{b-1}{2}\left\lfloor\frac{p}{b-1}\right\rfloor - \frac{b-1}{2}\left\lfloor\frac{p}{b-1}\right\rfloor^2 \\ &\geq A+1,\end{aligned}$$

and therefore, $h(G) \geq a$. Hence, $H(a,b) \leq F(a,b)$, and the result holds. ∎

For some other values of a and b, the next theorem gives a lower bound for $H(a,b)$.

Theorem 6 *If* $a, b \geq 3$ *and* $a \geq \binom{b-1}{2} + 1$, *then*

$$H(a,b) \geq a + \binom{b-1}{2}. \tag{1}$$

Proof Let $A := \left\lfloor\frac{a+B-1}{b-1}\right\rfloor$, $B := \binom{b-1}{2}$, and

$$\begin{aligned} G &:= \{b-1-(a+B-1)+(b-1)A\}\, K_A \\ &\cup \{a+B-1-(b-1)A\}\, K_{A+1}. \end{aligned}$$

Observe that $A \geq b-2$. Let $v_1^i, v_2^i, \ldots, v_{b-2}^i$ be any $b-2$ vertices of the ith component of G. For $1 \leq i \leq j \leq b-2$, color v_j^i and v_i^{j+1} with color

$$(i-1)b - \binom{i+1}{2} + j - i + 2.$$

Finally, color all other vertices with distinct colors. This is easily seen to be a harmonious coloring of G, and so

$$h(G) \leq a + B - 1 - \sum_{k=1}^{b-2} k = a - 1.$$

Also, $\alpha(G) = b-1$ and $|G| = a + B - 1$. Therefore, $H(a,b) \geq a + B$. ∎

In general, inequality (1) cannot be replaced by equality, as will be shown later in this paper. However, for certain values, the inequality is sharp. To prove this, we need two lemmas. The proof of the first is straightforward and will be omitted.

Lemma 1 *For any graph* G, $\alpha(G) \geq p(G) - \binom{h(G)}{2}$. ∎

Lemma 2 *If $|G| = 8$ and $h(G) = 4$, then $\alpha(G) \geq 4$. Moreover, if u is a vertex such that $\alpha(G-u) \leq 3$, then there is another vertex v with the same color as u that is in a triangle.*

Proof Since $h(G) = 4$, G has at most 6 edges. By a complete enumeration of all graphs with at most 6 edges, the lemma is easily verified. ∎

Theorem 7 *We have, for small b, the following formulas for $H(a,b)$:*

1. *For $a \geq 3$, $H(a,3) = a+1$.*
2. *For $a \geq 4$, $H(a,4) = a+3$.*
3. *For $a \geq 7$, $H(a,5) = a+6$.*

Proof We will prove only the last part, since the others follow similarly. Let $a \geq 7$. By Theorem 6, $H(a,5) \geq a+6$. Let G be a graph with $a+6$ vertices, and assume that G has a harmonious coloring with $a-1$ colors. Let $V_1, V_2, \ldots, V_{a-1}$ be the color classes and let $p_i = |V_i|$. Then without loss of generality, $p_1 \geq p_2 \geq \cdots \geq p_{a-1}$.

Case 1: $p_1 \geq 5$. Then clearly $\alpha(G) \geq 5$.

Case 2: $p_1 = 4$. Since $p(G) - 4 > a-2$, it follows that $p_2 \geq 2$. So, by Lemma 1, $\alpha(G) \geq 5$.

Case 3: $p_1 = p_2 = 3$. Again, by Lemma 1, $\alpha \geq 5$.

Case 4: $p_1 = 3$ and $p_2 < 3$. It follows that $p_i = 2$ for $i = 2, 3, \ldots, 6$. Let $V_i = \{u_i, v_i\}$, where, without loss of generality, $v_i \in N_G(V_1)$, for $2 \leq i \leq 6$. Let $H = G[\{u_2, u_3, \ldots, u_6\}]$. Suppose that H is not complete. We may assume that u_2 is not adjacent to u_3. Then $V_1 \cup \{u_2, u_3\}$ is an independent set of 5 elements. Otherwise, if H is complete, then $\{v_2, v_3, \ldots, v_6\}$ is such a set.

Case 5: $p_1 < 3$. It follows that $p_i = 2$ for $i = 1, 2, \ldots, 7$. Let $H = G[V_1 \cup V_2 \cup \cdots \cup V_7]$. Suppose that $\deg_H(u) \leq 2$ for some vertex u. Then, without loss of generality, $u \in V_1$, and $N_H(u) \subset V_2 \cup V_3$. So by Lemma 2, $G[V_4 \cup V_5 \cup V_6 \cup V_7]$ has an independent set of 4 vertices and hence $\alpha \geq 5$.

Otherwise, H is 3-regular. Then $V_i = \{u_i, v_i\}$ for $1 \leq i \leq 7$, where, without loss of generality, u_1 is adjacent to v_2, v_3, and v_4. Note that if $K_4 \subset H$ or $K_4 - e \subset H$, then clearly $\alpha(H) \geq 5$. Now, we may assume that $\alpha(G[\{u_i\} \cup V_5 \cup V_6 \cup V_7]) < 4$, for $i = 2, 3, 4$. For otherwise, an independent set in one of these induced subgraphs together with u_1 would form an independent set of order 5. Then by Lemma 2, u_i is contained in a triangle in $G[\{u_i\} \cup V_5 \cup V_6 \cup V_7]$ for $i = 2, 3, 4$. Now, since H is 3-regular and does not contain $K_4 - e$, these triangles must be vertex-disjoint, and without loss of generality, these triangles must consist of vertices $\{u_2, v_5, u_6\}$, $\{u_3, v_6, u_7\}$, and $\{u_4, v_7, u_5\}$ respectively. Again, since H doesn't contain $K_4 - e$, we may assume that v_1 is adjacent to v_5, v_6, and v_7. Then $\{v_1, u_1, u_5, u_6, u_7\}$ is an independent set and so $\alpha(G) \geq 5$.

The five cases exhaust all possibilities, so the result follows. ∎

The next theorem provides an upper bound for $H(a,b)$.

Theorem 8 *For $a, b \geq 3$ let $\Psi := \frac{(a-1)(a-2)}{b-1}$ and $\Omega := \left\lfloor \frac{1+\sqrt{1+4\Psi}}{2} \right\rfloor$. Then*

$$H(a,b) \leq \left\lfloor (b-1)\left\lceil \frac{\Omega}{2} + \frac{1}{2} + \frac{\Psi}{2\Omega} \right\rceil \right\rfloor + 1. \tag{2}$$

Proof Let G be a graph with r vertices such that $\alpha(G) \leq b-1$ and $h(G) \leq a-1$. Then by Theorem 3,

$$\binom{a-1}{2} \geq |E(G)| \geq r\left\lfloor \frac{r}{b-1} \right\rfloor - \frac{b-1}{2}\left\lfloor \frac{r}{b-1} \right\rfloor - \frac{b-1}{2}\left\lfloor \frac{r}{b-1} \right\rfloor^2.$$

Simplifying, we find that

$$\left\lfloor \frac{r}{b-1} \right\rfloor^2 - \frac{2r}{b-1}\left\lfloor \frac{r}{b-1} \right\rfloor + \left\lfloor \frac{r}{b-1} \right\rfloor + \Psi \geq 0.$$

Let $f(x) := \lfloor x \rfloor^2 - 2x\lfloor x \rfloor + \lfloor x \rfloor + \Psi$, for $x \geq 0$. Then the following facts are evident:

(a) $f(x)$ is continuous.
(b) $f(x) = x - x^2 + \Psi$, for $x = 0, 1, 2, \ldots$.
(c) $f(x) > 0$, for $0 \leq x \leq 1$.
(d) $f(x)$ is strictly decreasing for $x \geq 1$.
(e) If n is a positive integer and $n \leq x \leq n+1$, then $f(x) = n^2 - 2xn + n + \Psi$.

From (a)-(d), we conclude that $f(x)$ has exactly one positive root x^* and $\Omega \leq x^* \leq \Omega + 1$. Then from (e), we deduce that

$$x^* = \frac{\Omega}{2} + \frac{1}{2} + \frac{\Psi}{2\Omega}.$$

Observe that $f(x) < 0$ for $x > x^*$. Now, let $r^* = (b-1)x^*$. Then, since $\lfloor r^* \rfloor + 1 > r^*$, it follows that no graph of order $\lfloor r^* \rfloor + 1$ exists with $\alpha \leq b-1$ and $h \leq a-1$. Therefore, $H(a,b) \leq \lfloor r^* \rfloor + 1$. ∎

Inequality (2) happens to be sharp for certain diagonal values (i.e., where $a = b$). This is proved in the next theorem.

Theorem 9 *If n is a prime power, then*

$$H(n^2+n+2, n^2+n+2) = n^3 + 2n^2 + 2n + 2.$$

Proof Let $G := (n^2+n+1)K_{n+1}$. Since n is a prime power, it is well-known that there exists a projective plane π of order n containing $N = n^2+n+1$ points and lines. (See [8]) Let $p_1, p_2, \ldots, p_N$ and $L_1, L_2, \ldots, L_N$ be the points and lines of π respectively. Then by coloring the vertices of the ith copy of K_{n+1} with the "colors" (points) that are contained in L_i, we obtain a harmonious coloring of G

using N colors. So $h(G) \le N = n^2+n+1$. Then, since $p(G) = n^3+2n^2+2n+1$, and $\alpha(G) = n^2+n+1$, we obtain that $H(n^2+n+2, n^2+n+2) \ge n^3+2n^2+2n+2$.

Now using the notation of Theorem 8, we see that if $a = b = n^2 + n + 2$, then $\Psi = n^2 + n$ and $\Omega = n + 1$. Therefore, by this same theorem, we see that $H(n^2 + n + 2, n^2 + n + 2) = n^3 + 2n^2 + 2n + 2$. Hence, the result follows. ∎

The previous theorem allows us to find the asymptotic value of $H(k, k)$.

Corollary 1 *$H(k, k)$ is asymptotically equal to $k^{1.5}$.*

Proof For $k \ge 2$, let

$$x_k := \frac{-1 + \sqrt{4k-7}}{2}.$$

It is well-known [2]that if x is sufficiently large, then there is a prime r such that

$$x - \frac{1}{10}x^{\frac{2}{3}} \le r \le x.$$

Therefore, if k is sufficiently large, then there exist primes r_1 and r_2 such that

$$x_k - x_k^{.7} \le r_1 \le x_k \le r_2 \le x_k + x_k^{.7}.$$

Therefore,

$$\begin{aligned}
&(x_k - x_k^{.7})^3 + 2(x_k - x_k^{.7})^2 + 2(x_k - x_k^{.7}) + 2 \\
\le\ & H(k, k) \\
\le\ & (x_k + x_k^{.7})^3 + 2(x_k + x_k^{.7})^2 + 2(x_k + x_k^{.7}) + 2
\end{aligned}$$

So, $H(k, k) = x_k^3 + o(x_k^3) = k^{1.5} + o(k^{1.5})$. ∎

The next theorem is proved using a classic result on Ramsey numbers. (See [3]). The proof is due to Andras Gyarfas[5].

Theorem 10 *For $m \ge 1$. $H(2^m, 4m) \ge 2^{m+1} - 1$.*

Proof It is well-known that $R(2m, 2m) \ge 2^m$. Consider a Red-Blue edge-coloring of K_{2^m-1} that contains no monochromatic K_{2m}. Let $\{v_1, v_2, \ldots, v_{2^m-1}\}$ be the vertices of K_{2^m-1}. Let graph G have vertices

$$V(G) = \{x_1, x_2, \ldots, x_{2^m-1}, y_1, y_2, \ldots, y_{2^m-1}\}$$

where $x_i \sim x_j$ if v_i and v_j are joined by a blue edge, and $y_i \sim y_j$ if v_i and v_j are joined by a red edge, and where all other vertices in G are non-adjacent. Then clearly $h(G) = 2^m - 1$. Now assume that G has an independent set of $4m$ vertices. This set contains $2m$ x_i's or $2m$ y_i's. Therefore K_{2^m-1} contains a monochromatic K_{2m} which is a contradiction. Hence, $\alpha(G) \le 4m - 1$. Therefore, $H(2^m, 4m) \ge p(G) + 1 = 2^{m+1} - 1$. ∎

In Table 1 , we present all values of $H(a, b)$ that we have found for small a and b, either by the theorems of this paper or by ad hoc methods. Based on evidence from this table, it is natural to conjecture that $H(a, b) = H(b, a)$ for all $a, b \geq 1$. But this is not the case. For by Theorem 10, $H(1024, 40) \geq 2047$, while by Theorem 8, $H(40, 1024) \leq 1765$, rather a large difference. The first part of this example also shows that the inequality (1) in Theorem 6 cannot be replaced by equality.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
3	1	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
4	1	4	5	7	8	9	10	11	12	13	14	15	16	17	18	19
5	1	5	6	8	10	11	13	14	15	16	17	18	19	20	21	22
6	1	6	7	9	11	13	15	16	18	19	21	22	23	24	25	26
7	1	7	8	10	13	15	17	19	20	22	23	25	26	28	29	31
8	1	8	9	11	14			22	23	25	26	28	29	31	32	34
9	1	9	10	12	15							31	33	34	36	37
10	1	10	11	13	16								37	38	40	41
11	1	11	12	14	17											
12	1	12	13	15	18											
13	1	13	14	16	19											
14	1	14	15	17	20									53		
15	1	15	16	18	21											
16	1	16	17	19	22											

Table 1. Values of $H(a, b)$ for small a and b.

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