A New Type of Freeman-Youden Rectangle

D.A. Preece
Institute of Mathematics and Statistics
Cornwallis Building
University of Kent at Canterbury
Canterbury, Kent
England CT2 7NF

N.C.K. Phillips
Computer Science Department
Southern Illinois University
Carbondale, Illinois
USA 62901-4408

ABSTRACT. A Freeman-Youden rectangle (FYR) is a Graeco-Latin row-column design consisting of a balanced superimposition of two Youden squares. There are well known infinite series of FYRs of sizes $q \times (2q+1)$ and $(q+1) \times (2q+1)$ where $(2q+1)$ is a prime power congruent to 3 (modulo 4). Any member of these series is readily constructed from an initial column whose entries are specified very simply in terms of powers of a primitive root of GF$(2q+1)$. We now show that, for $q \geq 9$, initial columns for FYRs of the above sizes can be specified more generally, which allows us to obtain many more FYRs, which are unlike any that have previously appeared in the literature. We present enumerations for $q = 9$ and $q = 11$, and we tabulate new FYRs for these values of q. We also present some new FYRs for $q = 15$.

1 Introduction with definitions

An $r \times t$ Youden square [7] is a rectangular array of t symbols in r ($< t$) rows and t columns such that each symbol occurs just once in each row and no more than once in each column, the subsets of symbols in the columns being the blocks of a symmetric balanced incomplete block design (SBIBD, often called a symmetric 2-design).

JCMCC **25** (1997), pp. 65-78

Preece [7], Bailey, Preece and Rowley [1] and Preece and Vowden [8] considered what they termed 'balanced superimpositions of one Youden square on another'. The first such superimposition to have been published seems to have been a 4×7 specimen given by Clarke [2] (page 99), who obtained it from G.H. Freeman. We therefore, for ease of reference, rename the 'balanced superimpositions ...' as Freeman-Youden rectangles (FYRs). FYRs of size 5×11 include the following two examples, in each of which the ith letter of each pair ($i = 1, 2$) belongs to the ith of the superimposed Youden squares:

BI	CJ	DK	EA	FB	GC	HD	IE	JF	KG	AH	
EK	FA	GB	HC	ID	JE	KF	AG	BH	CI	DJ	
FH	GI	HJ	IK	JA	KB	AC	BD	CE	DF	EG	(1)
JG	KH	AI	BJ	CK	DA	EB	FC	GD	HE	IF	
DC	ED	FE	GF	HG	IH	JI	KJ	AK	BA	CB	

and

BC	CD	DE	EF	FG	GH	HI	IJ	JK	KA	AB	
EI	FJ	GK	HA	IB	JC	KD	AE	BF	CG	DH	
FK	GA	HB	IC	JD	KE	AF	BG	CH	DI	EJ	(2)
JH	KI	AJ	BK	CA	DB	EC	FD	GE	HF	IG	
DG	EH	FI	GJ	HK	IA	JB	KC	AD	BE	CF	

Each of these two FYRs is cyclically generated from its first column, using the cycle of letters $(AB \dots K)$.

For a formal definition of an FYR, we say that the superimposition has 4 factors, namely 'rows', 'columns', 'symbols S1 of the first Youden square', and 'symbols S2 of the second Youden square'. The rows-factor is orthogonal to each of the others in that each row has exactly one entry from each column, exactly one for each symbol S1, and exactly one for each symbol S2. The pairwise relationships of the other 3 factors are, however, relationships of balance. Numbering these factors as 0 (for columns), 1 (for S1), and 2 (for S2), we define an $r \times t$ FYR as an $r \times t$ rectangular array such that

1. each entry is an ordered pair x, y where x is drawn from a set S1 of t elements, and y is drawn from a set S2 of t elements;

2. if the elements from either S1 or S2 are disregarded, the array becomes an $r \times t$ Youden square;

3. if n_{21} is the $t \times t$ (0, 1)-matrix whose (i, j)th element $(i, j = 1, 2, \dots, t)$ is the number of times that the ith element of S2 is paired with the jth element of S1, then n_{21} is the incidence matrix of an SBIBD;

4. if n_{10} is the $t \times t$ (0,1)-matrix whose (i,j)th element $(i,j = 1,2,\ldots,t)$ is the number of times that the ith element of S1 occurs in the jth column, and n_{20} is defined similarly for S2, and we write n_{01} for the transpose of n_{10}, etc., then

$$\begin{aligned} n_{01}n_{12}n_{20} + n_{02}n_{21}n_{10} &= n_{12}n_{20}n_{01} + n_{10}n_{02}n_{21} \\ &= n_{20}n_{01}n_{12} + n_{21}n_{10}n_{02} = fI + gJ \end{aligned} \tag{3}$$

where f and g are integers, I is the $t \times t$ identity matrix, and J is the $t \times t$ matrix whose elements are all 1.

If the elements of S1 and S2 in each of the FYRs (1) and (2) are taken in their natural orders, then (1) satisfies

$$n = n_{10} = n_{02} = n_{12} \tag{4}$$

and (2) satisfies

$$n = n_{10} = n_{02} = n_{21} \tag{5}$$

where $n + n' = J - I$ in both instances. Thus the matrix in (3) becomes

$$-3I + 23J \text{ for } (1)$$

and

$$8I + 22J \text{ for } (2).$$

Each of the 5×11 FYRs (1) and (2) can be converted into a cyclically generated 6×11 FYR by adding a row $AA\ BB\ \ldots\ KK$. These larger FYRs still satisfy (4) and (5) respectively, but now with $n + n' = J + I$.

Suppose an FYR is of size $q \times (2q+1)$ or $(q+1) \times (2q+1)$ where q is odd, $q > 1$, and where $(2q+1)$ is a prime power congruent to 3 (modulo 4). Then it is of 'type 1' if it can be written in such a form that it satisfies (4); it is of 'type 2' if it can be written so as to satisfy (5). The existence of the two types is implicit in [3] and explicit in [1, 6]. In order that a given FYR of type 1 may be written so as to satisfy (4), a permutation of the factors 0, 1, 2 may be needed.

2 The well known series of FYRs of sizes $q \times (2q+1)$ and $(q+1) \times (2q+1)$

By relabelling $A, B, \ldots, J, K$ as $0, 1, \ldots, 9, t$ (where t denotes 10), the initial columns

$$(BI\ \ EK\ \ FH\ \ JG\ \ DC)$$

of the cyclically generated 5×11 FYR (1), and

$$(BC\ \ EI\ \ FK\ \ JH\ \ DG)$$

of the cyclically generated 5×11 FYR (2), can be rewritten as, respectively,

$$(1,8 \quad 4,t \quad 5,7 \quad 9,6 \quad 3,2) \tag{6}$$

and

$$(1,2 \quad 4,8 \quad 5,t \quad 9,7 \quad 3,6). \tag{7}$$

Each of these is of the form

$$(x^0, x^j \quad x^2, x^{2+j} \quad x^4, x^{4+j} \quad x^6, x^{6+j} \quad x^8, x^{8+j}) \tag{8}$$

where $x = 2$ is a primitive root of GF(11) and where each integer equal to a power of x is reduced modulo 11. Similarly, the initial columns of the 6×11 FYRs already discussed are of the form

$$(0,0 \quad x^0, x^j \quad x^2, x^{2+j} \quad x^4, x^{4+j} \quad x^6, x^{6+j} \quad x^8, x^{8+j}). \tag{9}$$

In (8) and (9), the entries $x^0, x^2, \ldots, x^8$ are the quadratic residues of GF(11), but we choose j so that the entries $x^j, x^{2+j}, \ldots, x^{8+j}$ are the non-quadratic residues. The quadratic residues form a well known difference set, modulo 11, as do the non-quadratic residues; if these difference sets are augmented by including the element 0 additionally, we still have well known difference sets, modulo 11. For the initial column (6), which is for an FYR of type 1, the differences

$$x^{i+j} - x^i, \quad i = 0,2,4,6,8$$

are the non-quadratic residues, but for the initial column (7), which is for an FYR of type 2, the differences are the quadratic residues.

If we disregard the ordering of each pair y, z in the initial column (6) or (7), each set of pairs is a 'starter' in the sense of, for example, Kocay, Stinson and Vanstone [5], Wallis [9 (p. 212), 10 (pp. 599 and 624)] and Dinitz [4]. But we have imposed the following extra requirements on a starter:

(i) Each of its pairs must consist of one quadratic residue and one non-quadratic residue.

(ii) Each pair must be ordered, with the quadratic residue coming first.

(iii) The within-pairs ordered differences, second element minus first element, must all be non-quadratic residues (for FYRs of type 1) or all quadratic residues (type 2).

These requirements have the following further implication:

(iv) If a type 2 FYR has an initial column comprising the set $\{(y, z)\}$ of ordered pairs formed from a starter satisfying (i), (ii) and (iii), then the sets $\{(z - y, -y)\}$ and $\{(-z, y - z)\}$ similarly constitute initial columns of FYRs of type 2, arithmetic being within the relevant field.

A starter is 'strong' [9 (p. 267)] if its pairs y, z are such that all its sums $y + z$ are distinct and non-zero, addition again being within the relevant field. Section 4 below shows that not all starters used for FYRs are strong, even though (6) and (7), when regarded as starters, are strong.

The generalisation of the above results from the sizes 5×11 and 6×11 to the sizes $q \times (2q + 1)$ and $(q + 1) \times (2q + 1)$ where q is odd, $q > 1$, and $(2q + 1)$ is a prime power congruent to 3 (modulo 4), is so obvious that we omit detail here. We merely note that the initial columns of the FYRs are of the respective forms

$$(x^0, x^j \quad x^2, x^{2+j} \quad \ldots \quad x^{2q-2}, x^{2q-2+j}) \tag{10}$$

and

$$(0, 0 \quad x^0, x^j \quad x^2, x^{2+j} \quad \ldots \quad x^{2q-2}, x^{2q-2+j}) \tag{11}$$

where x is a primitive root of the appropriate Galois field. If $(2q + 1)$ is prime, the quadratic residues, the non-quadratic residues, the quadratic residues augmented by 0, and the non-quadratic residues augmented by 0, all form well known difference sets, modulo $(2q + 1)$, and the FYRs are cyclically generated, with cycles of length $(2q + 1)$.

3 Adjugacy in FYRs of types 1 and 2

If we multiply each of (6) and (7) throughout by the primitive root $x = 2$, reverse the order of each pair, and reorder the sequence of pairs so that the quadratic residues are again in natural order, we obtain, respectively,

$$(1, 7 \quad 4, 6 \quad 5, 2 \quad 9, 8 \quad 3, t) \tag{6a}$$

and

$$(1, 6 \quad 4, 2 \quad 5, 8 \quad 9, t \quad 3, 7) \tag{7a}$$

These too are initial columns of 5×11 FYRs of types 1 and 2 respectively. In vocabulary taken from the theory of Latin squares, the FYRs obtainable from (6a) and (7a) are adjugates of, respectively, the FYRs obtainable from (6) and (7). In this aspect of adjugacy, the 5×11 FYR of type 2 that is obtainable from the initial column

$$(1, t \quad 4, 7 \quad 5, 6 \quad 9, 2 \quad 3, 8) \tag{7b}$$

is self-adjugate.

However, as is suggested by property (iv) above, another type of adjugacy is available for our FYRs of type 2. Consider again the initial column (7) for an FYR of type 2. If we label each entry of the FYR as f, g, h where f is the column number (counting from 0), g is the member of S1, and h is the member of S2, then the entries of the initial column are, in row order,

$$0,1,2 \quad 0,4,8 \quad 0,5,t \quad 0,9,7 \quad 0,3,6$$

So, because of the cyclic development (modulo 11) of the FYR, the successive rows contain the following entries where the second component of the triple is 0:

$$t,0,1 \quad 7,0,4 \quad 6,0,5 \quad 2,0,9 \quad 8,0,3$$

Now, cyclically permuting each triple so that the zero comes first (which is equivalent to a cyclic permutation of the factors 0, 1, 2), we have

$$0,1,t \quad 0,4,7 \quad 0,5,6 \quad 0,9,2 \quad 0,3,8$$

which gives us the initial column (7b) above; we have moved from the pairs y, z in (7) to the pairs $z-y, -y$ in 7(b). Thus the FYR obtainable from the initial column (7b) is an adjugate of the FYR obtainable from the initial column (7). Similarly the FYR obtainable from (7a), comprising the pairs $-z, y-z$, is an adjugate of the FYR obtainable from (7b).

These concepts of adjugacy, appropriately generalised, are needed for classifying our new FYRs. For convenience, we henceforth refer to two initial columns as being adjugates of one another if the corresponding FYRs are adjugates of one another.

4 The new Freeman-Youden rectangles of sizes 9×19 and 10×19

To obtain FYRs of sizes $q \times (2q+1)$ and $(q+1) \times (2q+1)$, with q restricted as already specified, initial columns of the form (10) or (11) are not needed. More generally for the sizes $q \times (2q+1)$ we can use any starter whose q pairs can be arranged so as to satisfy the conditions (i), (ii) and (iii) above. For $q \geq 9$, this enables us to produce new FYRs.

We consider first $q = 9$, for which we need the quadratic and non-quadratic residues of GF(19). Using the primitive root $x = 2$ of GF(19), we have, in systematic order,

quadratic residues (Q)	1	4	16	7	9	17	11	6	5
non-quadratic residues (N)	2	8	13	14	18	15	3	12	10

Accordingly, the 9×19 FYRs obtained by the well known construction are the 4 type 1 examples and 5 type 2 examples whose initial columns are as follows, where the SS numbers refer to 'strong starters' from the table of Kocay, Stinson and Vanstone [5 (pp. 52-53)] and Dinitz [4 (p. 470)]:

type 1:

SS46:

$$(1,15\ \ 4,3\ \ 16,12\ \ 7,10\ \ 9,2\ \ 17,8\ \ 11,13\ \ 6,14\ \ 5,18)$$

adjugate of SS46:

$$(1,14\ \ 4,18\ \ 16,15\ \ 7,3\ \ 9,12\ \ 17,10\ \ 11,2\ \ 6,8\ \ 5,13)$$

SS52:

$$(1,3\ \ 4,12\ \ 16,10\ \ 7,2\ \ 9,8\ \ 17,13\ \ 11,14\ \ 6,18\ \ 5,15)$$

adjugate of SS52:

$$(1,13\ \ 4,14\ \ 16,18\ \ 7,15\ \ 9,3\ \ 17,12\ \ 11,10\ \ 6,2\ \ 5,8)$$

type 2:

SS29:

$$(1,2\ \ 4,8\ \ 16,13\ \ 7,14\ \ 9,18\ \ 17,15\ \ 11,3\ \ 6,12\ \ 5,10)$$

adjugates of SS29:

$$(1,10\ \ 4,2\ \ 16,8\ \ 7,13\ \ 9,14\ \ 17,18\ \ 11,15\ \ 6,3\ \ 5,12)$$
$$(1,18\ \ 4,15\ \ 16,3\ \ 7,12\ \ 9,10\ \ 17,2\ \ 11,8\ \ 6,13\ \ 5,14)$$

SS51:

$$(1,12\ \ 4,10\ \ 16,2\ \ 7,8\ \ 9,13\ \ 17,14\ \ 11,18\ \ 6,15\ \ 5,3)$$

adjugate of SS51:

$$(1,8\ \ 4,13\ \ 16,14\ \ 7,18\ \ 9,15\ \ 17,3\ \ 11,12\ \ 6,10\ \ 5,2)$$

But many 9×19 type 2 FYRs of different construction are available, including one whose initial column is this:

$$(1,12\ \ 4,10\ \ 16,14\ \ 7,8\ \ 9,13\ \ 17,3\ \ 11,18\ \ 6,15\ \ 5,2). \qquad (12)$$

To study this novel 9×19 example, we write down its "associated partial Latin square" (APLS), of size 9×9, obtained as follows. Write the non-quadratic residues, in order, in a border column

$$(r_1\ \ r_2\ \ \ldots\ \ r_9)'$$

to the left of the APLS, and the quadratic residues, in order, in a border row

$$(c_1\ \ c_2\ \ \ldots\ \ c_9)$$

above the APLS. Then the (i,j)th entry of the APLS $(i,j = 1,2,\ldots,9)$ is entered as $r_i - c_j$ (modulo 19) if this is a quadratic residue, but is left blank otherwise. So the APLS is

	1	4	16	7	9	17	11	6	5
2	1	17	5			4			16*
8	7	4	11	1*			16		
13		9	16	6	4*			7	
14			17*	7	5	16			9
18	17			11	9	1	7*		
15		11			6	17	4	9*	
3			6			5*	11	16	17
12	11*			5			1	6	7
10	9	6*			1			4	5

(13)

where entries with asterisks are those that correspond to the pairings in the initial column (12). These entries constitute a transversal of the APLS, i.e. they are all different and there is one in each row and one in each column of the APLS. Indeed, any transversal of the APLS will give us the initial column of a 9×19 FYR of type 2.

The pattern of asterisks in (13) shows that initial column (12) has some symmetry. If we multiply (12) throughout by x^2, with x still taken as the primitive root 2 of GF(19), and then again multiply by x^2, we obtain successively (after re-ordering the 9 entries of each outcome, to put the quadratic residues in natural order) two further initial columns

$$(1,8\;\; 4,10\;\; 16,2\;\; 7,18\;\; 9,13\;\; 17,14\;\; 11,12\;\; 6,15\;\; 5,3)$$

and

SS50:

$$(1,12\;\; 4,13\;\; 16,2\;\; 7,8\;\; 9,15\;\; 17,14\;\; 11,18\;\; 6,10\;\; 5,3).$$

A further multiplication by x^2, followed by re-ordering of entries, returns us to (12). But we can also move to adjugates by multiplying (12) by $x = 2$, and reversing the ordering of each pair, to obtain (again after re-ordering of entries) this further initial column:

$$(1,8\;\; 4,10\;\; 16,14\;\; 7,18\;\; 9,13\;\; 17,3\;\; 11,12\;\; 6,15\;\; 5,2).$$

Multiplying this by x^2 and x^4 we also obtain, as before,

$$(1,8\;\; 4,13\;\; 16,2\;\; 7,18\;\; 9,15\;\; 17,14\;\; 11,12\;\; 6,10\;\; 5,3)$$

and

$$(1,12\;\; 4,13\;\; 16,14\;\; 7,8\;\; 9,15\;\; 17,3\;\; 11,18\;\; 6,10\;\; 5,2).$$

Thus (12) belongs to a set of $3 + 3 = 6$ initial columns that meet our requirements for an FYR of type 2; these 6 initial columns correspond to 6 transversals in the APLS.

More usually for $q = 9$, an initial column that meets our requirements for type 2 belongs to a set of $9 + 9 = 18$ distinct possible initial columns, not $3+3 = 6$; these 18 are obtained from any one of themselves by multiplication by x^2 and x as just described. This is true of each of the following initial columns, each of which is from a different set of 18 possibilities; none of the 18 is based on a strong starter:

$$(1,2\ \ 4,10\ \ 16,14\ \ 7,18\ \ 9,13\ \ 17,3\ \ 11,8\ \ 6,15\ \ 5,12), \tag{14}$$

$$(1,2\ \ 4,15\ \ 16,14\ \ 7,13\ \ 9,18\ \ 17,3\ \ 11,8\ \ 6,10\ \ 5,12) \tag{15}$$

and

$$(1,2\ \ 4,13\ \ 16,14\ \ 7,18\ \ 9,15\ \ 17,3\ \ 11,8\ \ 6,10\ \ 5,12). \tag{16}$$

Thus we have accounted for $6 + 18 + 18 + 18 = 60$ transversals of the APLS, apart from the 5 transversals on or parallel to the main diagonal. Complete enumeration by computer confirms that $5+60 = 65$ is indeed the total number of distinct possible initial columns that meet our requirements for an FYR of type 2; of these 65 initial columns, the first $5 + 6 = 11$ are based on strong starters, and the remaining $18+18+18 = 54$ are not. The situation is summarised by our tabulation of the new initial columns (12), (14), (15) and (16) in Table 1.

Table 1

Initial columns for new 9×19 Freeman-Youden rectangles of type 2

Set no.	Set size	Text ref.	Initial column
1	6	(12)	(1,12 4,10 16,14 7,8 9,13 17,3 11,18 6,15 5,2)
2	18	(14)	(1,2 4,10 16,14 7,18 9,13 17,3 11,8 6,15 5,12)
3	18	(15)	(1,2 4,15 16,14 7,13 9,18 17,3 11,8 6,10 5,12)
4	18	(16)	(1,2 4,13 16,14 7,18 9,15 17,3 11,8 6,10 5,12)

Of the 52 inequivalent strong starters in the table of Kocay, Stinson and Vanstone [5 (pp. 52-53)] and Dinitz [4 (p. 470)], only 6 satisfy condition (i) above. These are strong starters 29, 46, 50, 51 and 52, which have already been referred to, plus strong starter 30, which cannot be ordered so as to satisfy condition (iii); this anomalous strong starter, rearranged so that condition (ii) is satisfied and so that the quadratic residues are in ascending order, is as follows:

$$[1,2\ \ 4,13\ \ 16,18\ \ 7,14\ \ 9,15\ \ 17,12\ \ 11,3\ \ 6,10\ \ 5,8].$$

Exhaustive enumeration of transversals of the APLS, and exhaustive enumeration of eligible starters, are not the only ways of finding initial columns that meet our requirements for an FYR of type 2. For example, the APLS (13) for the intial column (12) contains the following two 2×2 subsquares:

$$\begin{array}{cc} 11 & 1* \\ 17* & 7 \end{array} \quad \text{(rows 2 and 4)} \quad \text{and} \quad \begin{array}{cc} 17 & 7* \\ 11* & 1 \end{array} \quad \text{(rows 5 and 8).}$$

Clearly, the asterisks may all be shifted to the opposite positions, to create a new admissible pattern of asterisks; the corresponding initial column belongs to the same set of 18 initial columns as does (14). A similar shift of asterisks would have been possible even if the pattern had been

$$\begin{array}{cc} 11 & 1* \\ 17* & 7 \end{array} \quad \text{and} \quad \begin{array}{cc} 17 & 11* \\ 7* & 1 \end{array}$$

Other simple shifts of asterisks are possible too; for example, in the asterisked APLS for the initial column (16) we may change

$$\begin{array}{ccc} 1* & 5 & . \\ 7 & 11* & 1 \\ 11 & . & 5* \end{array} \quad \text{to} \quad \begin{array}{ccc} 1 & 5* & . \\ 7 & 11 & 1* \\ 11* & . & 5 \end{array}$$

Having found the 60 new possible initial columns for 9×19 FYRs of type 2, we now turn to the possibility of type 1 FYRs of the same size. For this we need the "complementary partial Latin square" (CPLS), in which the differences $r_i - c_j$ are entered if they are non-quadratic residues:

	1	4	16	7	9	17	11	6	5
2				14	12		10	15	
8					18	10		2	3
13	12					15	2		8
14	13	10					3	8	
18		14	2					12	13
15	14		18	8					10
3	2	18		15	13				
12		8	15		3	14			
10			13	3		12	18		

In this sparser array, there are no transversals except parallel to the main diagonal, so no new type 1 FYRs of size 9×19 are found.

If we add the pair $0, 0$ to each of the 60 new initial columns for 9×19 FYRs of type 2, we obtain 60 new initial columns for 10×19 FYRs of type 2. But we have no new type 1 FYRs of size 10×19.

5 The new Freeman-Youden rectangles of sizes 11×23 and 12×23

As q increases, subject only to $(2q+1)$ being a prime power congruent to 3 (modulo 4), we can expect the numbers of non-isomorphic FYRs of sizes $q\times(2q+1)$ and $(q+1)\times(2q+1)$ to increase, reflecting the ever greater room to manoeuvre when searching for transversals in the $q\times q$ APLS and the $q\times q$ CPLS. Indeed, for $q=11$, unlike $q=9$, the CPLS has transversals that are not parallel to the main diagonal; so new 11×23 and 12×23 FYRs of both of types 1 and 2 are found.

Computer enumeration shows that there are 336 possible initial columns for an 11×23 FYR of type 2 that is analogous to our 9×19 FYRs of type 2. This total of 336 is (as expected) much larger than the corresponding total of 65 for size 9×19. Continuing the approach already used, these two totals can be partitioned as follows:

$$9\times 19 \text{ (type 2)}: \quad 65 = 5 + 1.(3+3) + 3.(9+9)$$
$$11\times 23 \text{ (type 2)}: \quad 336 = 6 + 0 + 15.(11+11)$$

Specifically there are, for size 11×23, the 6 possible initial columns of form (10), plus 15 sets of possible initial columns, each such set containing $11+11=22$ members; these 15 sets are analogous (except as described below) to the corresponding 3 sets, each of size $9+9=18$, for size 9×19. Representative members of the sets are in Table 2. The other 21 members of each set are obtained by multiplying the representative member by successive even powers of the primitive root $x=5$ of GF(23), as described above for 9×19 FYRs, and by multiplying by successive odd powers and reversing pairs, again as above.

Unlike for the 4 new sets of 9×19 FYRs of type 2, the 15 new sets of 11×23 FYRs of type 2 can be grouped, in threes, into 5 'adjugacy classes' which, in the notation of Table 2, are a, b, c, d and e. Thus, for example, the initial columns that are labelled a1, a2 and a3 in Table 2 are adjugates of one another, and so the corresponding FYRs are in the same adjugacy class a; if the entries in a1 are denoted y, z, then those in a2 and a3 are, respectively, $z-y, -y$ and $-z, y-z$. The same relationships hold for adjugacy classes b, c, d and e.

Turning now to 11×23 FYRs of type 1 that are analogous to the FYRs so far considered, we find that there are 49 possible initial columns, namely 5 that are of the form (10), and 44 others that fall into 2 sets each comprising $11+11=22$ members. Representative members of the 2 sets are in Table 2.

An oddity of Table 2, when regarded as a table of starters, is that it contains just one strong starter, namely that for b2. Thus Table 2 gives us just a single set of 22 FYRs based on a strong starter.

If we add the pair 0, 0 to each of the new initial columns for 11 × 23 FYRs, we obtain corresponding new initial columns for 12 × 23 FYRs.

Table 2
Initial columns for new 11 × 23 Freeman-Youden rectangles of types 1 and 2

Set code	Initial column
Type 1:	
A	(1,15 2,22 4,11 8,7 16,21 9,19 18,14 13,5 3,20 6,17 12,10)
B	(1,15 2,7 4,21 8,5 16,14 9,19 18,10 13,20 3,22 6,17 12,11)
Type 2:	
a1	(1,5 2,10 4,20 8,17 16,22 9,11 18,21 13,14 3,15 6,19 12,7)
a2	(1,10 2,14 4,22 8,21 16,19 9,15 18,11 13,17 3,5 6,7 12,20)
a3	(1,17 2,20 4,10 8,11 16,5 9,22 18,19 13,15 3,7 6,4 12,21)
b1	(1,5 2,10 4,20 8,17 16,22 9,11 18,7 13,14 3,21 6,19 12,15)
b2	(1,10 2,14 4,22 8,21 16,19 9,15 18,20 13,17 3,11 6,7 12,5)
b3	(1,17 2,5 4,10 8,20 16,11 9,22 18,19 13,15 3,7 6,14 12,21)
c1	(1,5 2,10 4,20 8,14 16,17 9,11 18,21 13,22 3,15 6,19 12,7)
c2	(1,7 2,14 4,22 8,21 16,19 9,10 18,11 13,17 3,5 6,15 12,20)
c3	(1,14 2,20 4,10 8,11 16,5 9,17 18,19 13,15 3,7 6,22 12,21)
d1	(1,5 2,10 4,17 8,14 16,11 9,21 18,20 13,22 3,19 6,7 12,15)
d2	(1,17 2,5 4,22 8,21 16,20 9,10 18,7 13,19 3,11 6,15 12,14)
d3	(1,14 2,11 4,7 8,20 16,22 9,17 18,19 13,15 3,21 6,10 12,5)
e1	(1,5 2,20 4,10 8,11 16,17 9,22 18,7 13,15 3,19 6,14 12,21)
e2	(1,7 2,10 4,22 8,17 16,20 9,11 18,21 13,14 3,15 6,19 12,5)
e3	(1,10 2,14 4,7 8,21 16,11 9,15 18,19 13,17 3,5 6,22 12,20)

6 Some new 15 × 31 Freeman-Youden rectangles

This paper focusses on the series of sizes $q \times (2q + 1)$ for FYRs, where $(2q + 1)$ is a prime power congruent to 3 (modulo 4) and where $q \geq 9$. Taking $q = 9$ has indicated that special interest may attach to such sizes for which q is not prime, as possible initial columns for the FYRs can then fall into a set containing $2p$ members, where p is a factor of q, as well as into sets containing $2q$ members. For $q = 9$ there is just one of the smaller sets ($p = 3$), namely set 1 of Table 1. However, for $q = 15$, which gives us FYRs of size 15 × 31, there are more such smaller sets, and these arise for both $p = 3$ and $p = 5$. To avoid serious notational complications, we give no theory for these. We merely outline constructions in a way that shows clearly what is involved.

If we use the primitive root $x = 3$ of GF(31), we can write the successive quadratic (Q) and non-quadratic (N) residues as follows:

Q	1	9	19	16	20	25	8	10	28	4	5	14	2	18	7
N	3	27	26	17	29	13	24	30	22	12	15	11	6	23	21

Accordingly, the well known construction for a 15×31 FYR of type 1 gives us, for example, the following initial column that starts with x^0, x^3:

$$(1,27\ \ 9,26\ \ 19,17\ \ 16,29\ \ 20,13\ \ 25,24\ \ 8,30\ \ 10,22\ \ \ldots)$$

However, the second elements of the pairs in positions $1, 4, 7, \ldots$ can be moved to positions $4, 7, 10, \ldots$ respectively to give the initial column of another FYR of type 1:

$$(1,23\ \ 9,26\ \ 19,17\ \ 16,27\ \ 20,13\ \ 25,24\ \ 8,29\ \ 10,22\ \ \ldots)$$

This initial column belongs to a set with $p = 3$.

Again using the above Q/N table, we see that an initial column arising from the well known construction for a 15×31 FYR of type 2 is the following, which starts with x^0, x^1:

$$(1,3\ \ 9,27\ \ 19,26\ \ 16,17\ \ 20,29\ \ 25,13\ \ 8,24\ \ 10,30\ \ \ldots)$$

This column too can be modified as before, except that movement is now from positions $4, 7, 10, \ldots$ to positions $1, 4, 7, \ldots$ respectively:

$$(1,17\ \ 9,27\ \ 19,26\ \ 16,24\ \ 20,29\ \ 25,13\ \ 8,12\ \ 10,30\ \ \ldots)$$

This initial column for a 15×31 FYR of type 2 belongs to a set with $p = 3$.

Another initial column arising from the well known construction for a 15×31 FYR of type 2 is the one that starts with x^0, x^5:

$$(1,26\ \ 9,17\ \ 19,29\ \ 16,13\ \ 20,24\ \ 25,30\ \ 8,22\ \ 10,12\ \ \ldots)$$

We now modify this as before, except that the movement is from positions $1, 6, 11$ to positions $6, 11, 1$ respectively:

$$(1,6\ \ 9,17\ \ 19,29\ \ 16,13\ \ 20,24\ \ 25,26\ \ 8,22\ \ 10,12\ \ \ldots)$$

This initial column for a 15×31 FYR of type 2 belongs to a set with $p = 5$. Another such initial column can be created by making, additionally, coresponding movements from positions $2, 7, 12$ to positions $7, 12, 2$ respectively, or from positions $3, 8, 13$ to positions $8, 13, 3$ respectively.

Acknowledgments: The authors are very grateful to Dr K. Arasu (Wright State University, Dayton, Ohio, USA) for a conversation that led to the ideas developed in this paper. The second author's contribution to this paper was made whilst he held a Visiting Research Fellowship in the Institute of Mathematics and Statistics, University of Kent at Canterbury, England.

References

[1] R.A. Bailey, D.A. Preece and C.A. Rowley, Randomization for a balanced superimposition of one Youden square on another, *J. R. Statist. Soc. B* **57** (1995), 459–469.

[2] G.M. Clarke, A second set of treatments in a Youden square design, *Biometrics* **19** (1963), 98–104.

[3] G. Dall'Aglio, Blocs incomplets équilibrés orthogonaux, *Colloq. Int. CNRS* **110** (1963), 195–214.

[4] J.H. Dinitz, Starters, in "The CRC Handbook of Combinatorial Designs" (ed. C.J. Colbourn and J.H. Dinitz), pp. 467–473; Boca Raton: CRC Press (1996).

[5] W.L.Kocay, D.R.Stinson and S.A. Vanstone, On strong starters in cyclic groups, *Discrete Mathematics* **56** (1985), 45–60.

[6] D.A. Preece, Some row and column designs for two sets of treatments, *Biometrics* **22** (1966), 1–25.

[7] D.A. Preece, Fifty years of Youden squares: a review, *Bull. Inst. Maths Appplics* **26** (1990), 65–75.

[8] D.A. Preece and B.J. Vowden, Graeco-Latin squares with embedded superimpositions of Youden squares, *Discrete Mathematics* **138** (1995), 353–363.

[9] W.D. Wallis, *Combinatorial Designs*; New York: Marcel Dekker (1988).

[10] W.D. Wallis, One-factorizations of complete graphs, in "Contemporary Design Theory: A Collection of Surveys" (ed. J.H. Dinitz and D.R. Stinson), pp. 593-631; New York: Wiley (1992).