Stability number of a subclass of chair-free, net-free graphs

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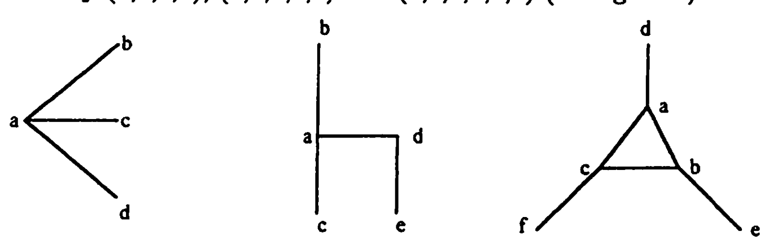
ABSTRACT. We describe a class of graphs Γ for which the stability number can be obtained in polynomial time. A graph in class Γ is chair-free, net-free and has the property that the claw-centers form an independent set.

1 Introduction

We consider only finite, undirected graphs $G = (V, E)$ of order n. For a subset S of G, $G[S]$ denotes the subgraph induced by S. Three graphs, a claw $(a; b, c, d)$, a chair $(a; b, c, d, e)$ and a net $(a, b, c, ; d, e, f)$ play an important role in our paper. They are the unique graphs whose degree sequences are respectively (3,1,1,1), (3,1,1,2,1) and (3,3,3,1,1,1) (see figure 1).



the claw (a;b,c,d) the chair (a;b,c,d,e) the net (a,b,c;d,e,f)

Figure 1

The vertex of degree 3 of the first graph is called the center of the claw (or the claw-center). We denote by Y the set of all claw-centers in G.

Finding the stability number $\alpha(G)$ (also named independence number, vertex packing number) of a graph G is in general a difficult problem. The vertex packing problem (VPP) remains NP-complete even for the class of triangle-free and cubic planar graphs [4]. It is well known however that the VPP is polynomial for bipartite graphs and perfect graphs. This general result on perfect graphs is obtained by Grötschel et al. [5] as a consequence of the fact that Linear Programming is polynomial. For several subclasses of perfect graphs, combinatorial algorithms exist (see [4]). As an independent set in a graph G corresponds to a matching in its edge graph $L(G)$ and vice versa, efficient algorithms exist for solving VPP in line graphs. It is remarkable that the VPP is also polynomially solvable for the class of claw-free graphs [11] and [12], a class containing strictly the class of line graphs. Iterative procedures which, at each step, construct from a graph G another smaller graph G' such that $\alpha(G) = \alpha(G') - k$ where k is a positive known integer have been given by several authors. In [10], this procedure was used to provide another polynomial algorithm for solving the VPP if G is claw-free, two different reduction operations were introduced with $k \in \{1, 2\}$. Via the study of pseudo-boolean functions, Ebenegger et al. [14] proposed a general method with $k = 1$. This method was named *struction* (STability number RedUCTION). Although the order of G' does not decrease in general, for some classes of graphs this drawback disappears and a polynomial algorithm follows whenever the transformation $G \rightarrow G'$ is closed for these classes. This technique has been applied for CAN-free graphs (claw-, antenna-,net-free graphs) [9] and CN-free graphs (claw-free, net-free graphs) [8], subclasses of claw-free graphs. Other reduction procedures which are in fact special versions of the struction have been used to prove that the VPP is polynomial for bull- and chair-free in [13] and for $AH-$free graphs [7]. Using the particular struction given in [8], we prove in this paper that the VPP is polynomial for a new subclass of graphs (which we call Γ), defined as follows

Definition 1 *A graph G belongs to class Γ if*

(i) *G is chair-free*

(ii) *G is net-free*

(iii) *Y is an independent set.*

Graphs in this class are not necessarily perfect or claw-free. Moreover, this class contains strictly the classes of CAN-free, CN-free graphs, but is different from the other classes listed above. We shall say that a vertex is *special* if it does not center a claw.

In the next section, we recall the specialized struction $G \to G'$ developed in [8] for CN-free graphs. As we shall see this struction is particularly suitable for the class Γ. For our purposes, we define the struction as the transformation $G \to G'$, centered at a special vertex 0 (if it exists). At each stage of the transformation, provided that a special vertex exists, we have $\alpha(G) = \alpha(G') + 1$.

In section 3, we shall show that the class of chair-free, net-free graphs is closed under the struction, that is, G' is chair-free, net-free graph.

In section 4, we prove that if $G \in \Gamma$, then at each stage, a special vertex 0 exists. As $| V(G') | \leq | V(G) - 2$, a polynomial algorithm for finding the stability number in this class of graphs can be obtained.

2 Struction of chair-free, net-free graphs

Following [8], we shall use the following notation. We write $[a, b]$ to mean that a, b are joined by an edge while $\overline{[a, b]}$ denotes the absence of an edge (or the presence of a *"nonedge"*). The *open neighborhood*, and the *closed neighborhood* of u are respectively denoted $N(u) = \{x \in V \mid xu \in E\}$ and $N[u] = \{u\} \cup N(u)$. Given a special vertex 0, it is convenient to define $N_0(a) = N[a] \cap N(0)$ for all $a \in V \setminus \{0\}$.

Throughout, we consider a struction centered at a vertex 0 which is assumed to be special (for $CN-$free graphs, this vertex is any vertex of G).

Let $\preceq$ be a partial preorder defined on $N(0)$ by $a \preceq b$ if $N_0[a] \subseteq N_0[b]$. We write $a \equiv b$ if $a \preceq b$ and $a \succeq b$. The vertices of $N(0)$ will be numbered from 1 to $| N(0) |$ and it will be convenient to refer to these vertices with their associated numbers. So, a total order $<$ on the vertices of $N(0)$ is created. The struction for the class of chair-free, net-free graphs, inspired from that given in [8], is described as follows

(a) Preliminaries

1) choose a special vertex 0 (if it exists)
2) number the vertices of $N(0)$ from 1 to $| N(0) |$ in such way that
   (i) if $a \preceq b$ and $a \not\succeq b$ then $a < b$
   (ii) if $a \equiv b$, $a \not\preceq x$, $x \not\preceq a$ and $a < b$ then either $x < a$ or $x > b$
3) let $J = \{i \in N(0) \mid$ there exists $j \in N(0)$ with $j > i$ and $\overline{[i, j]}\}$.

(b) Construction of G'

1) for each $i \in J$ introduce a new vertex i^*
2) remove the vertices of $N[0]$ and set $R := V(G) \setminus N[0]$
3) let N be the set of new vertices in G'

4) link every pair of N (new vertices), so N is a clique
5) link a new vertex i^* to vertex r in R if in G we have either $[i, r]$ or $[j, r]$ for every $j > i$ such that $\overline{[i, j]}$.

A vertex b in G is a follower of another vertex a if $\overline{[a, b]}$, $a, b \in N(0)$ and $b > a$. Unless otherwise specified, we shall assume throughout that

(1) G is a chair-free, net-free graph containing a special vertex 0

(2) G' is the resulting graph form a struction centered at a special vertex 0

(3) if $a, b, c, \ldots$ are new vertices (elements of N), then $a', b', c', \ldots$ are vertices in G corresponding to $a, b, c, \ldots$ respectively and $a'', b'', c'', \ldots$ are followers of $a', b', c', \ldots$ respectively.

To ensure the polynomial character of the determination of the stability number of our class of graphs, the following two results [8] are essential.

Proposition 1 *Let G be a chair-free, net-free graph with $\alpha(G) > 1$, containing a special vertex 0 and let G' be the graph obtained by struction centered at 0. Then $\alpha(G') = \alpha(G) - 1$.*

Proof. Only minor changes are needed to adapt the proof given in [8].

a) We first show that if $S \neq \emptyset$ is an independent set in G, there is an independent set S' in G' with $|S'| = |S| - 1$. As 0 is special, we have $|S \cap N(0)| \leq 2$. If $S \cap N(0) = \emptyset$ then take $S' = S \setminus \{x\}$, where x is any vertex in $S \cap R$. If $S \cap N(0) = \{i\}$ then $S' = S \setminus \{i\}$. Finally if $S \cap N(0) = \{i, j\}$ then assume $0 < i < j$, $\overline{[i, j]}$ and take $S' = (S \setminus \{i, j\}) \cup \{i^*\}$, where i^* is the new vertex corresponding to i. To see that S' is stable, suppose $[i^*, r]$ for some r in $S \cap R$. By construction, $\overline{[i^*, r]}$ in G' since $\overline{[i, r]}$ in G and $[j, r]$ in G must hold for any follower of i, in particular j. This is a contradiction since $\overline{[i, r]}$, $\overline{[j, r]}$ in G. Therefore (a) holds and thus $\alpha(G') \geq \alpha(G) - 1$.

(b) Next we prove that for any stable set S' of G' there exists a stable set S in G such that $|S| = |S'| + 1$. Clearly $|S' \cap N| \leq 1$ since N is a clique. If $S' \cap N = \emptyset$ then $S' \subset R$ and we can take $S = S' \cup \{0\}$. Suppose now $S' \cap N = \{a\}$. Let a' correspond to a in G. We shall prove that a follower a'' of a' exists for which $S = (S' \setminus \{a\}) \cup \{a', a''\}$ is an independent set in G. By construction, a', a'' are nonadjacent, $\overline{[a', r]}$ in G for any vertex r of $S' \setminus \{a\}$ otherwise we have $[a, r]$ in G'. So $N(a') \cap (S' \setminus \{a\}) = \emptyset$. Furthermore for any follower a'' of a' we can assume that $|N(a'') \cap (S' \setminus \{a\})| = 1$. To justify this, suppose first $N(a'') \cap (S' \setminus \{a\}) = \emptyset$ for some a''. Then $S = (S' \setminus \{a\}) \cup \{a', a''\}$ is an independent set in G and we are done. Next, if there exist $r, s \in (S' \setminus \{a\}) \cap N(a'')$ then $(a''; r, s, 0, a')$ is a chair. Another contradiction is easily obtained if two followers (a'', a''' say) exist since then $[a'', a''']$ because $(0; a', a'', a''')$ is not a claw and hence $(0, a'', a'''; a', r, s)$ is a net (we assume $[a'', r]$, $[a''', s]$). It remains to consider the case where a' has only one follower a''. In one hand, a'' must be adjacent to one and only one vertex of R, r say. On the other hand if $[a'', r]$ in G then $[a, r]$ in G' since a'' is the unique follower of a'. This last contradiction proves (b) and thus $\alpha(G) \leq \alpha(G') + 1$. Combining (a) and (b) we get $\alpha(G') = \alpha(G) - 1$. □

As for 0 and the last vertex $\mid N(0) \mid$ no new vertex is created in G', the following simple proposition guarantees a rapid convergence of the transformation, provided that a special vertex exists at each stage.

Proposition 2 *[8] Let G be a nontrivial chair-free, net-free graph containing a special vertex 0 and let G' be the graph obtained by struction centered at 0. Then* $\mid V(G') \mid \leq \mid V(G) \mid -2$.

3 Closedness of the class of chair-free, net-free graphs

In this section, we prove that the struction recalled in section 2, centered at a special vertex 0, transforms a chair-free, net-free graph into a graph which is also chair-free, net-free.

Lemma 1 *Let G be a net-free graph with a special vertex 0 and let G' be obtained by a struction (centered at 0) from G. Assume $a \in N$ and a' correspond to a in G. Then*

(i) *$[a, x]$ in G' implies either $[a', x]$ or $[a'', x]$ for every follower a'' of a'*

(ii) *$[\overline{a, x}]$ in G' implies $[\overline{a', x}]$ and $[\overline{a'', x}]$ in G for at least one follower a'' of a'*

(iii) *$[\overline{a, x}]$, $[\overline{x, y}]$, $[\overline{y, a}]$ in G' imply $[\overline{a', u}]$ and $[\overline{a'', u}]$ for $u = x, y$ and some follower a'' of a'.*

Proof. (i) and (ii) are consequences of the definition of the struction. Part (iii) is proved in ([8], Lemma 3.2). □

Lemma 2 *Let G be a chair-free, net-free graph with a special vertex 0. If $B = (r, a, b, 0, x, s)$ (see figure 2) is an induced subgraph in G, then $a \preceq b$.*

Proof. To prove $a \preceq b$, assume by contradiction the existence of a vertex $v \in N_0(a) \setminus N_0(b)$ in G (recall that $a \preceq b$ if $N_0(a) \subseteq N(b)$). Set $H := B + v$. Since $(0; x, b, v)$ cannot be a claw, we have $[x, v]$. Also $[v, r]$ otherwise $H - \{0, s\}$ is a chair and $[\overline{v, s}]$ otherwise $H - \{a, x\}$ is a chair. But now $H - \{a\}$ is a net. □

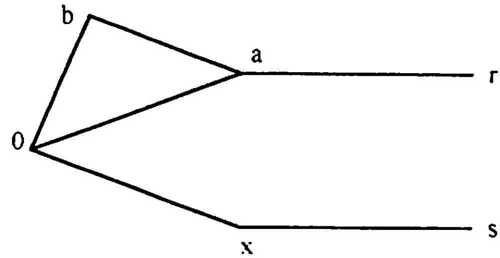


Figure 2

Lemma 3 *Let G be a chair-free, net-free graph with a special vertex* 0 *and let G' be obtained by a struction (centered at* 0*) from G. Assume G' contains an induced claw $(a; x, y, z)$ and assume $N \cap \{a, x, y, z\} \neq \emptyset$. Then the following hold*

(i) $a \in N$

(ii) *a', a'', x', x'' are all distinct if $x \in N$, where a'' is any follower of a' and x'' is a follower of x' chosen so that $\boxed{x'', y}$ and $\boxed{x'', z}$.*

(iii) *under the conditions of (ii), $[a', v]$ and $[a'', v]$ for $v = x', x''$*

(iv) *a' and its followers center claws in G.*

Proof.

(i) By contradiction of (i), assume without loss of generality $N \cap \{x, y, z\} = \{x\}$. By Lemma 1(iii), there exists a follower x'' of x' in G such that $\boxed{x'', y}$ and $\boxed{x'', z}$. Thus $\{x', x'', y, z\}$ is an independent set in G. Because $[a, x]$ in G', we have either $[a, x']$ or $[a, x'']$. But then $(a; y, z, u, 0)$ with $u \in \{x', x''\}$ is a chair, a contradiction (note that $\boxed{0, y}$ and $\boxed{0, z}$ since $y, z \notin N$). Because, $N \cap \{a, x, y, z\} \neq \emptyset$, it follows obviously $a \in N$.

(ii) Since $a, x \in N$ and N is a clique by construction, we have $y, z \notin N$. By Lemma 1(iii), we can choose x'' so that $\boxed{x'', u}$ for $u = y, z$. By construction, $\boxed{x'', u}$. Suppose (ii) false, that is, at least two vertices among $\{a', a'', x', x''\}$ are not distinct. Because $a' \neq a''$ and $x' \neq x''$, we only have to consider the following two cases.

(1) $a' \in \{x', x''\}$
Because $[a, y]$ and $[a, z]$ in G' and $\boxed{a', y}$ and $\boxed{a', z}$ since $a' \in \{x', x''\}$, it follows that $[a'', y]$ and $[a'', z]$. But then $(a''; y, z, 0, a')$ is a chair.

(2) $a'' \in \{x', x''\}$
Exchanging a' and a'', we are back to the previous case.

(iii) By (ii), we can choose a'', x'' so that a', a'', x', x'' are all distinct and $\{x', x'', y, z\}$ is an independent set in G. Let F be the subgraph induced by $\{0, a', a'', x', x'', y, z\}$ in G, where 0 is the center of the struction. We first prove that $[a', x']$. Suppose on the contrary $\overline{[a', x']}$. Then $[a'', x']$ and $[a', x'']$ otherwise either $(0; a', x', a'')$ or $(0; a', x', x'')$ is a claw. We cannot have $[a', y]$ and $[a', z]$ otherwise $F - \{a'', x''\}$ is a chair. Similarly we cannot have $[a'', y]$ and $[a'', z]$ for otherwise either $[a'', x'']$ and $(a''; x', u, x'', a')$ with $u \in \{y, z\}$ is a chair (recall that $|N(a') \cap \{y, z\}| \leq 1$) or $\overline{[a'', x'']}$ and $(a''; y, z, 0, x'')$ is a chair. Because $[a, y]$ and $[a, z]$ in G', we may assume, without loss of generality, $[a', y]$, $\overline{[a', z]}$, $[a'', z]$ and $\overline{[a'', y]}$. By Lemma 2 applied to configuration $B = (y, a', x', 0, a'', z)$, we have $a'' \preceq x'$ and hence $a' \preceq a'' \preceq x'$. Therefore x' is a follower of a', which implies $[x'', z]$ since $[a, z]$ in G' and $\overline{[a', z]}$ in G. This is a contradiction to the assumption $\overline{[x, z]}$. Therefore $[a', x']$. As x' and x'' play a symmetrical role for the above arguments, we may assume $[a', x'']$ as well. To finish the proof of (iii) we need only to show $[a'', x'']$ as we can deduce $[a'', x']$ by symmetry. Suppose, by contradiction, $\overline{[a'', x'']}$. Because $(0; a'', x', x'')$ is not a claw, we have $[a'', x']$. Clearly we cannot have $[a'', y]$ and $[a'', z]$ otherwise $F - \{a', x'\}$ is a chair. If $[a'', z]$ then $\overline{[a'', y]}$ and hence $[a', y]$. But then $F - \{0, z\}$ is a chair. If $\overline{[a'', z]}$ then $[a'', y]$ and hence $[a', z]$. But then $F - \{0, y\}$ is a chair. The proof of (iii) is now complete.

(iv) By (i), $a \in N$ and by construction, N is a clique. So, we only have to consider the following two cases

(1) $N \cap \{a, x, y, z\} = \{a\}$
Let a' correspond to a in G and choose any follower, a'' say, of a'. Suppose, by contradiction, at least one of a', a''(a'' say), does not center a claw. Since $[a, u]$ for $u = x, y, z$ in G, we have either $[a', u]$ or $[a'', u]$. By assumption, $|N(a'') \cap \{x, y, z\}| \leq 1$ otherwise $(a''; 0, x, y$ for instance) is a claw. So, assume, without loss of generality, $[a', x]$, $[a', y]$, $\overline{[a'', x]}$, $\overline{[a'', y]}$. But then $(a'; x, y, 0, a'')$ is a chair. Thus (1) must be rejected.

(2) $N \cap \{a, x, y, z\} = \{a, x\}$
Let x' correspond to x in G and choose x'' such that $\overline{[x'', y]}$ and $\overline{[x'', z]}$. By (ii), a', a'', x', x'' are all distinct. As in (1), suppose that a'' does not center a claw. By (iii), $[a'', x']$ and $[a'', x'']$. Therefore $\overline{[a'', y]}$ and $\overline{[a'', z]}$, which imply $[a', y]$ and $[a', z]$ since $[a, y]$ and $[a, z]$ in G'. But then $(a'; y, z, 0, a'')$ is a chair.

The proof of Lemma 3 is now complete. $\square$

Proposition 3 *Let G be a chair-free, net-free graph with a special vertex* 0 *and let G' be obtained by a struction (centered at* 0*) from G. Then G' is chair-free.*

Proof. Assume that G' contains a chair $(a; b, c, d, e)$. At least one of a, b, c, d, e is a new vertex for otherwise this chair exists in G. By Lemma 3(i) applied to claw $(a; b, c, d)$, no one of b, c, d is the only one new vertex, in particular we cannot have $d, e \in N$. So, we have only three cases to consider

(1) $N \cap \{a, b, c, d\} = \{a\}$
Let a' correspond to a and choose a'' so that $\boxed{[a'', e]}$. Since either $[a', d]$ or $[a'', d]$, assume, without loss of generality, $[a', d]$. For $u = b, c$, $\boxed{[a', u]}$ for otherwise $(a'; 0, u, d, e)$ is a chair in G. As $[a, u]$ in G', we must have $[a'', b]$ and $[a'', c]$. But then $(a''; b, c, 0, a')$ is a chair in G. So (1) is impossible.

(2) $N \cap \{a, b, c, d\} = \{a, b\}$
Let a' correspond to a and choose a'' such that $\boxed{[a'', e]}$. Let b' correspond to b and choose b'' such that $\boxed{[b'', c]}$ and $\boxed{[b'', d]}$. Note also that $\boxed{[b', e]}$. This is possible by Lemma 1(iii). By Lemma 3(ii), a', a'', b', b'' are all distinct. Assume first $[a', d]$. By Lemma 3(iii) applied to the claw $(a; b, c, d)$, we have $[a', u]$, $[a'', u]$ for $u = b', b''$. Observe that $[b'', e]$ otherwise $(a'; b', b'', d, e)$ is a chair. Next, we see that $\boxed{[a', c]}$ otherwise $(a'; b', c, d, e)$ is a chair. Therefore $[a'', c]$ since $[a, c]$ in G'. But now $(a''; b', c, b'', e)$ is a chair. By similar arguments, a contradiction arises if $\boxed{[a', d]}$ and hence $[a'', d]$. Thus (2) cannot occur.

(3) $N \cap \{a, b, c, d\} = \{a, d\}$
Consider a' and a'' as in (2) and let d' correspond to d and choose d'' such that $\boxed{[d'', b]}$ and $\boxed{[d'', c]}$. By Lemma 3(ii), a', a'', d', d'' are all distinct. Assume first $[a', c]$. By Lemma 3(iii), $[a', u]$, $[a'', u]$ for $u = d', d''$. We claim that $\boxed{[a', b]}$ otherwise $(a'; b, c, u, e)$ is a chair, with $u \in \{d', d''\}$ chosen so that $[u, e]$ (since $[d, e]$ in G', we have either $[d', e]$ or $[d'', e]$). Therefore $[a'', b]$ since $[a, b]$ in G'. But now $(u; e, a', a'', b)$ is a chair. The same arguments lead to a contradiction if $\boxed{[a', c]}$ and hence $[a'', c]$.

The proof of Proposition 3 is now complete. $\square$

Proposition 4 *Let G be a chair-free, net-free graph with a special vertex* 0 *and let G' be obtained by a struction (centered at* 0*) from G. Then G' is net-free.*

Proof. Assume that $(a, b, c; d, e, f)$ is an induced net in G' and at least one of its vertices is new. Four cases will be considered.

(1) $N \cap \{a, b, c, d, e, f\} = \{d\}$
Let d' correspond to d. Then $\boxed{d', u}$ for $u = b, c, e, f$. By Lemma 1(iii), we can choose d'' so that $\boxed{d'', c}$ and $\boxed{d'', e}$. Now, $\boxed{d', a}$ otherwise $(a, b, c; d', e, f)$ is a net. Since $[d, a]$ in G', we have $[d'', a]$. We next observe that $\boxed{d'', f}$ otherwise $(d''; a, f, 0, d')$ is a chair. But then $(a, b, c; d'', e, f)$ is a net. Thus (1) is impossible.

(2) $N \cap \{a, b, c\} = \{a\}$
Let a' correspond to a. Then $\boxed{a', u}$ for $u = e, f$. By Lemma 1(iii), we can choose a'' so that $\boxed{a'', e}$ and $\boxed{a'', f}$. Suppose, without loss of generality, $[a', c]$. Then $[a', b]$ otherwise $(c; f, a', b, e)$ is a chair. But then $(a', b, c; 0, e, f)$ is a net. Thus (2) must be rejected. Note that the arguments for the proof of (2) do not use the vertex d. So, no further consideration is needed if $N \cap \{a, b, c\} = \{a, d\}$. Obviously this case covers the ones where b or c is a new vertex.

(3) $N \cap \{a, b, c\} = \{a, b\}$
Let a', b' correspond to a and b. Then $\boxed{a', u}$ for $u = e, f$ and $\boxed{b', v}$ for $v = d, f$. By Lemma 1(iii), there exist a'' and b'' so that $\boxed{a'', u}$ and $\boxed{b'', v}$.

(3.1) a', a'', b', b'' are all distinct.
We cannot have $[u, c]$ and $[u, d]$ for $u \in \{a', a''\}$ otherwise $(u; 0, d, c, f)$ is a chair. Similarly we cannot have $[v, c]$ and $[v, e]$ for $v \in \{b', b''\}$ otherwise $(v; 0, e, c, f)$ is a chair. Since $[a, c]$, $[a, d]$, $[b, c]$ and $[b, e]$ in G', we may assume, without loss of generality, $[a', c]$, $[a'', d]$, $\boxed{a', d}$, $\boxed{a'', c}$, $[b', e]$, $[b'', c]$, $\boxed{b', c}$, $\boxed{b'', e}$.
Suppose now $\boxed{a', b''}$. Then $[a', b']$, $[a'', b'']$ since 0 does not center a claw. We note that $\boxed{a'', b'}$ otherwise $(b'; e, a'', a', c)$ is a chair. By Lemma 2, applied to the subgraph induced by $\{d, a'', 0, a', c, b'\}$ we have $a' \preceq b'$. Therefore $a' < b' < b''$ and hence b'' is a follower of a'. Because $[a, d]$ in G' and $\boxed{a', d}$, we must have $[b'', d]$, a contradiction which implies $[a', b'']$. For symmetrical reasons, we have $[a'', b']$. Now $\boxed{a', b'}$ otherwise $(b'; e, a'', a', c)$ is a chair. But now $(0, b', a''; a', e, d)$ is a net.

(3.2) $a'' = b''$.
As 0 cannot center a claw, we have $[a', b']$ with $a' \neq b'$. Since $b'' = a''$ and $\boxed{a'', e}$ we must have $[b', e]$ since $[b, e]$ in G'. Similarly $[a', d]$ because $\boxed{a'', d} = \boxed{b'', d}$. But now $(0, a', b'; a'', d, e)$ is a net.

(3.3) $a' = b''$ (or $a'' = b'$).
As 0 cannot center a claw, we have $[a'', b']$. Because $[b, e]$ in G' and $b' = a''$, we have $[b', e]$. Similarly $[a'', d]$ since $a' = b''$ and $\overline{[b'', d]}$. But now $(0, a'', b'; a', d, e)$ is a net.

(4) $N \cap \{a, b, c\} = \{a, b, c\}$
Let a', b', c' correspond respectively to a, b and c. Then $\overline{[a', u]}$ for $u = e, f$, $\overline{[b', v]}$ for $v = d, f$ and $\overline{[c'', w]}$ for $w = e, d$. By Lemma 1(iii), there exist a'', b'' and c'' so that $\overline{[a'', u]}$, $\overline{[b'', v]}$ and $\overline{[c'', w]}$. Let $\alpha \in \{a', a''\}$, $\beta \in \{b', b''\}$ and $\gamma \in \{c', c''\}$. By assumption and the definition of G', we may choose α, β, γ so that $N(\alpha) \cap \{d, e, f\} = \{d\}$, $N(\beta) \cap \{d, e, f\} = \{e\}$, $N(\gamma) \cap \{d, e, f\} = \{f\}$. We recall that $[0, \alpha]$, $[0, \beta]$ and $[0, \gamma]$.

(4.1) α, β, γ are all distinct.
If $[\alpha, \gamma]$, $[\beta, \gamma]$ and $[\gamma, \alpha]$ then $(\alpha, \beta, \gamma; d, e, f)$ is a net. If $[\alpha, \gamma]$, $[\beta, \gamma]$ and $\overline{[\gamma, \alpha]}$ then $(\beta; \gamma, e, \alpha, d)$ is a chair. If $[\alpha, \beta]$ $\overline{[\beta, \gamma]}$ and $\overline{[\gamma, \alpha]}$ then $(0, \alpha, \beta; \gamma, d, e)$ is a net. If $\overline{[\alpha, \gamma]}$, $\overline{[\beta, \gamma]}$ and $\overline{[\gamma, \alpha]}$ then $(0; \alpha, \beta, \gamma)$ is a claw.

(4.2) $\alpha = \beta \neq \gamma$.
If $\overline{[\alpha, \gamma]}$ then $(\alpha; d, e, 0, \gamma)$ is a chair. If $[\alpha, \gamma]$ then $(\alpha; d, e, \gamma, f)$ is a chair.

(4.3) $\alpha = \beta = \gamma$.
Assume, without loss of generality, that $\alpha = a'$. Consider now a''. We know that $\overline{[a'', a']}$, $[a'', e]$, $[a'', f]$. A final contradiction arises since $(\alpha; e, f, 0, a'')$ is a chair.

So, Proposition 4 is now proved. □

4 Stability number of the class Γ of graphs

Theorem 1 *The stability number of a graph G in class Γ can be obtained in polynomial time.*

Proof. We first prove that Γ is closed under the struction. By Propositions 3 and 4, G' is chair-free, net-free whenever a special vertex exists in G. If $G \in \Gamma$, we certainly have such a vertex. So it remains to prove that (iii) of Definition 1 holds for G'. By contradiction, suppose (iii) false and let a, b two adjacent vertices, centering claws, in G'. By Lemma 3(i), it suffices to consider the following two cases.

(1) $a, b \in N$.
For $u = a, b$, let u' correspond to u in G and u'' be a follower of u'. By Lemma 3, a', a'', b', b'' are all distinct and they all center claws. If we

choose any three vertices among the four, two of them are adjacent because 0 does not center a claw. But then two adjacent vertices in G center claws, a contradiction.

(2) $N \cap \{a,b\} = \{b\}$
By Lemma 3(iv), both b', b'' center claws in G. Let x, y, z be chosen so that $(a; x, y, z)$ is a claw (it is possible that b is one of these three vertices). If x, y, z are all in G' then $(a; x, y, z)$ is a claw in G and a contradiction arises since either $[a', b]$ or $[a'', b]$ in G. So, we may assume that x is new vertex. We can choose, by Lemma 1(iii), x'' such that $\overline{[x'', y]}$, $\overline{[x'', z]}$. Moreover $\overline{[x', y]}$, $\overline{[x', z]}$ by definition of the struction. Since $[a, x]$ in G' then either $(a; x', y, z)$ or $(a; x'', y, z)$ is a claw in G. As above, either $[a', b]$ or $[a'', b]$ in G, we have a contradiction to the assumption that no two adjacent vertices in G center claws.

The first part of the proof is now complete. By Propositions 1 and 2, at each stage of the struction, the order and the stability number of G' decrease. By repeatedly applying the struction, thereby obtaining a sequence $G^{(1)}, G^{(2)}, \dots$ of graphs, we stop as soon as we get $G^{(k)} (1 \leq k \leq n-2)$ for which the stability number is easily obtained. Then $\alpha(G) = k+p$, $(p \geq 1)$ if $\alpha(G^{(k)}) = p$. Obviously a special vertex exists at each stage since (iii) holds. So, a polynomial algorithm giving $\alpha(G)$ is easily gotten for any graph $G \in \Gamma$. Since the transformation $G \rightarrow G'$ is applied at most n times and the whole construction of G' has complexity $O(n^2)$ [8], it is clear that the whole complexity of the algorithm is $O(n^3)$. □

Remark 1 *The class of graphs whose claw-centers form an independent set is investigated in [1] for its domination and hamiltonian properties.*

References

[1] A. Ainouche, i-Partly claw-free graphs. *Preprint 1994.*

[2] Z. Benmeziane, Etude de classes de graphes partiellement sans $K_{1,3}$. *Thèse de Magister USTHB-IM.* 1995.

[3] V. Chvàtal and P. Erdös, A note on hamiltonian circuits. *Discrete Math.* **2** (1973), 111–113.

[4] M.C. Golumbic, *Algorithmic graph Theory and Perfect Graphs.* Academic Press, New York, 1980.

[5] M. Grötschel, L. Lovàsz and A. Schrijver, Polynomial algorithms for perfect graphs. *Annals of Discrete Mathematics* **21** (North Holland, Amsterdam), 1984, 325–356.

[6] A. Hertz, Quelques utilisations de la struction. *Discrete Math.* **59** (1986), 79–89.

[7] A. Hertz, D. de Werra, On the Stability Number of AH-Free Graphs. *J. of Graph Theory,* **17**, No. 1, (1993), 53–63.

[8] P.L. Hammer, N.V.R. Mahadev and D. de Werra, The struction of a graph:application to CN-free graphs. *Combinatorica* **5** (1985), 141–147.

[9] P.L. Hammer, N.V.R. Mahadev and D. de Werra, Stability in CAN-free graphs. *J. Combin. Theory B* **38** (1985), 23–30.

[10] L. Lovàsz and M.D. Plummer, Matching Theory. *Annals of Discrete Mathematics* **29** (North Holland, Amsterdam), 1986.

[11] G.J. Minty, On maximal independent set of vertices in claw-free graphs. *J. Combin. Theory Series B* **28** (1980), 284–304.

[12] N. Sbihi, Algorithme de recherche d'un stable de cardinalité maximum dans un graphe sans étoile. *Discrete Math.* **29** (1980), 53–76.

[13] C. De Simone, A. Sassano, Stability number of bull- and chair-free graphs. *Discrete Applied Math.* **41** (1993), 121–129.

[14] Ch. Ebenegger, P.L. Hammer and D. de Werra, Pseudo-Boolean functions and stability of graphs. *Annals of Discrete Math.* **19** (1984), 83–98.

[15] D. de Werra, Some properties of the struction of a graph, *SIAM J. Algebraic Discrete Methods* **5** (1984), 239–243.