Symmetric functions and the theorem of the arithmetic and geometric means

Vesselin Gasharov
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1003, USA
e-mail: gasharov@math.lsa.umich.edu

ABSTRACT. We show that results analogous to the theorem of the arithmetic and geometric means hold for the three multiplicative fundamental bases of the vector space of symmetric functions - the elementary symmetric functions, the homogeneous symmetric functions, and the power sum symmetric functions. We give examples to show that no such results hold for the two non-multiplicative fundamental bases - the Schur functions and the monomial symmetric functions.

1 Introduction

Let $x_1, x_2, \ldots, x_n$ be commuting indeterminates. A symmetric function (or a symmetric polynomial) in $x_1, x_2, \ldots, x_n$ is a polynomial in $\mathbf{Z}[x_1, x_2, \ldots, x_n]$ which is invariant under any permutation of the variables. The vector space of all such polynomials is denoted by Λ_n. It has five "fundamental" bases whose elements are indexed by partitions λ: the elementary symmetric functions e_λ, the homogeneous symmetric functions h_λ, the power sum symmetric functions p_λ, the Schur functions s_λ, and the monomial symmetric functions m_λ. We adhere to the terminology and notation of [2], which is the standard reference for the theory of symmetric functions. (Usually, the symmetric functions are considered as formal power series in infinitely many variables, but here we consider them as actual functions in finitely many real variables.)

The well-known theorem of the arithmetic and geometric means (see for example [1, Theorem 9]) is the following:

Theorem 1.1 *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers and $b = \frac{a_1+a_2+\cdots+a_n}{n}$ be their mean. Then*

$$a_1 a_2 \cdots a_n \leq b^n. \tag{1}$$

Note that the left-hand side of (1) is $e_n(a_1, a_2, \ldots, a_n)$, while the right-hand side is $e_n(b, b, \ldots, b)$. In §2 we will show that the theorem of the arithmetic and geometric means can be generalized to arbitrary elementary symmetric functions, i.e., Theorem 1.1 remains valid if we replace the left and right-hand sides of (1) by $e_\lambda(a_1, a_2, \ldots, a_n)$ and $e_\lambda(b, b, \ldots, b)$ respectively, where λ is any partition. In §3 and §4 we show that the opposite inequality holds for the homogeneous and power sum symmetric functions. It turns out that the two non-multiplicative bases, the monomial symmetric functions and the Schur functions, do not behave well when the arguments are replaced by their mean; Examples are given in §5.

2 The elementary symmetric functions

We denote by $\mathbf{R}$ the set of real numbers, and by $\mathbf{R}_+$ the set of nonnegative real numbers. For $a \in \mathbf{R}$ we denote by $|a|$ the absolute value of a. Let $\| \;\|$ be the norm on $\mathbf{R}^n$ given by: $\|(a_1, a_2, \ldots, a_n)\| = \max\{|a_i| : i = 1, 2, \ldots, n\}$.

Lemma 2.1 *Let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbf{R}_+^n$ and $\bar{\mathbf{a}} = (b, b, \ldots, b) \in \mathbf{R}_+^n$, where $b = \frac{a_1+a_2+\cdots+a_n}{n}$. Then there exists a sequence $\mathbf{a}^{(0)} = \mathbf{a}, \mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots$ of elements in $\mathbf{R}_+^n$, such that the following 3 conditions are satisfied:*

1. $\frac{a_1^{(i)}+a_2^{(i)}+\cdots+a_n^{(i)}}{n} = b$ *for any $i \geq 0$, where $a_j^{(i)}$ is the jth element of $\mathbf{a}^{(i)}$;*
2. *For any $i \geq 0$ there exist $j \neq k$, $1 \leq j, k \leq n$, such that $a_l^{(i)} = a_l^{(i+1)}$ for any $l \neq j, k$ and $a_j^{(i+1)} = a_k^{(i+1)} = \frac{a_j^{(i)}+a_k^{(i)}}{2}$;*
3. *$\|\mathbf{a}^{(i)} - \bar{\mathbf{a}}\|$ approaches 0 as i gets larger and larger.*

Proof: If $\|\mathbf{a}^{(0)} - \bar{\mathbf{a}}\| = 0$, then we are done. Otherwise, let j be such that $|a_j^{(0)} - b| = \|\mathbf{a}^{(0)} - \bar{\mathbf{a}}\|$ and let k be such that $a_j^{(0)} - b$ and $a_k^{(0)} - b$ have different signs. (Clearly, such k exists.) Then let $\mathbf{a}^{(1)}$ be the sequence with $a_l^{(1)} = a_l^{(0)}$ for any $l \neq j, k$ and $a_j^{(1)} = a_k^{(1)} = \frac{a_j^{(0)}+a_k^{(0)}}{2}$. It is clear that $\|\mathbf{a}^{(1)} - \bar{\mathbf{a}}\| \leq \|\mathbf{a}^{(0)} - \bar{\mathbf{a}}\|$. Continuing in this way we construct a sequence $\mathbf{a}^{(0)}, \mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots$ which satisfies (i) and (ii). To show that it satisfies (iii), note that $\|\mathbf{a}^{(j)} - \bar{\mathbf{a}}\| < \frac{1}{2}\|\mathbf{a}^{(i)} - \bar{\mathbf{a}}\|$ whenever $j - i \geq n$. □

Theorem 2.2 *Let* a *and* ā *be as in Lemma 2.1. Then for any partition* λ *we have the inequality:*

$$e_\lambda(\mathbf{a}) \leq e_\lambda(\bar{\mathbf{a}}).$$

Proof: If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, then $e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots e_{\lambda_l}$ and $e_{\lambda_i}(c_1, c_2, \ldots, c_n) \geq 0$ for any nonnegative real numbers $c_1, c_2, \ldots, c_n$, so it will be enough to prove Theorem 2.2 for partitions λ with only one part. If $\mathbf{a}^{(0)}, \mathbf{a}^{(1)}, \mathbf{a}^{(2)}, \ldots$ is a sequence of elements in $\mathbf{R}^n_+$, such that $\|\mathbf{a}^{(i)} - \bar{\mathbf{a}}\|$ approaches 0 as i gets larger and larger, then clearly $e_r(a^{(i)})$ approaches $e_r(\bar{\mathbf{a}})$ for any r. Since the e_r are symmetric functions, this shows that Theorem 2.2 follows from Lemma 2.1 and Proposition 2.3 below. □

Proposition 2.3 *Let* $a_1, a_2, \ldots, a_n$ *be nonnegative real numbers and* $c = \frac{a_1 + a_2}{2}$. *Then for any* $r \geq 0$ *we have that* $e_r(a_1, a_2, \ldots, a_n) \leq e_r(c, c, a_3, a_4, \ldots, a_n)$.

Proof: Let t be an indeterminate. Then

$$\sum_{r \geq 0} e_r(a_1, a_2, \ldots, a_n) t^r = \prod_{j=1}^{n} (1 + a_j t)$$

and

$$\sum_{r \geq 0} e_r(c, c, a_3, a_4, \ldots, a_n) t^r = (1 + ct)^2 \prod_{j=3}^{n} (1 + a_j t).$$

So the proof will be complete if we show that the polynomial

$$P(t) = (1 + ct)^2 \prod_{j=3}^{n} (1 + a_j t) - \prod_{j=1}^{n} (1 + a_j t)$$

has nonnegative coefficients. But $P(t)$ can be written in the form

$$P(t) = (c^2 - a_1 a_2) t^2 \prod_{j=3}^{n} (1 + a_j t) = \frac{1}{4}(a_1 - a_2)^2 t^2 \prod_{j=3}^{n} (1 + a_j t),$$

so we are done. □

3 The homogeneous symmetric functions

Theorem 3.1 *Let* a *and* ā *be as in Lemma 2.1. Then for any partition* λ *the following inequality holds:*

$$h_\lambda(\mathbf{a}) \geq h_\lambda(\bar{\mathbf{a}}).$$

As for the elementary symmetric functions, Theorem 3.1 follows from the following:

Proposition 3.2 *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers and $c = \frac{a_1+a_2}{2}$. Then for any $r \geq 0$ we have that $h_r(a_1, a_2, \ldots, a_n) \geq h_r(c, c, a_3, a_4, \ldots, a_n)$.*

Proof: The proof is analogous to that of Proposition 2.3. Let t be an indeterminate. We have the identities:

$$\sum_{r \geq 0} h_r(a_1, a_2, \ldots, a_n)t^r = \prod_{j=1}^{n}(1 - a_j t)^{-1}$$

and

$$\sum_{r \geq 0} h_r(c, c, a_3, a_4, \ldots, a_n)t^r = (1 - ct)^{-2} \prod_{j=3}^{n}(1 - a_j t)^{-1}.$$

So, to finish the proof it suffices to show that the formal power series

$$Q(t) = \prod_{j=1}^{n}(1 - a_j t)^{-1} - (1 - ct)^{-2} \prod_{j=3}^{n}(1 - a_j t)^{-1}$$

has nonnegative coefficients. This is clear from the following expression for $Q(t)$:

$$\begin{aligned} Q(t) &= (c^2 - a_1 a_2)t^2(1 - ct)^{-2} \prod_{j=1}^{n}(1 - a_j t)^{-1} \\ &= \frac{1}{4}(a_1 - a_2)^2 t^2 (1 - ct)^{-2} \prod_{j=1}^{n}(1 - a_j t)^{-1}. \end{aligned}$$

□

4 The power sum symmetric functions

Theorem 4.1 *Let* $\mathbf{a}$ *and* $\bar{\mathbf{a}}$ *be as in Lemma 2.1. Then for any partition λ the inequality*

$$p_\lambda(\mathbf{a}) \geq p_\lambda(\bar{\mathbf{a}})$$

holds.

As for the elementary and homogeneous symmetric functions, Theorem 4.1 follows from the following:

Proposition 4.2 *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers and $c = \frac{a_1+a_2}{2}$. Then for any $r \geq 0$ we have that $p_r(a_1, a_2, \ldots, a_n) \geq p_r(c, c, a_3, a_4, \ldots, a_n)$.*

Proof: Without loss of generality we can assume that $a_1 = c + d$ and $a_2 = c - d$ for some d with $0 \leq d \leq c$. Then

$$\begin{aligned} p_r(a_1, a_2, \ldots, a_n) - p_r(c, c, a_3, a_4, \ldots, a_n) &= a_1^r + a_2^r - 2c^r \\ &= (c+d)^r + (c-d)^r - 2c^r \\ &= \sum_{1 \leq i \leq \frac{r}{2}} 2\binom{r}{2i} c^{r-2i} d^{2i}. \end{aligned}$$

□

5 The Schur functions and the monomial symmetric functions

As we mentioned in the Introduction, Theorems 2.2, 3.1, and 4.1 do not have analogs for the two non-multiplicative bases s_λ and m_λ. For example, if all parts of a partition λ have length 1, then $s_\lambda = e_\lambda$, and if λ has only one part, then $s_\lambda = h_\lambda$, but the elementary and homogeneous symmetric functions behave differently when their arguments are replaced by their mean. In fact, it turns out that we cannot even say whether the values of s_λ or m_λ increase or decrease for a fixed partition λ when passing to the mean of the arguments, as the following examples show.

Example 5.1 *Let $a_1 \geq a_2$ be nonnegative real numbers and let $b = \frac{a_1+a_2}{2}$ and $c = \frac{a_1-a_2}{2}$. Then $a_1 = b + c$, $a_2 = b - c$ and we have that*

$$s_{(5,1)}(b,b) - s_{(5,1)}(a_1, a_2) = s_{(5,1)}(b,b) - s_{(5,1)}(b-c, b+c) = c^6 + 9b^2c^4 - 5b^4c^2$$

which is positive when $c > \sqrt{\frac{\sqrt{101}-9}{2}}b$ and negative when $c < \sqrt{\frac{\sqrt{101}-9}{2}}b$.

Example 5.2 *Let a_1, a_2, b, and c be as in Example 5.1. Then*

$$m_{(4,1)}(b,b) - m_{(4,1)}(a_1, a_2) = m_{(4,1)}(b,b) - m_{(4,1)}(b+c, b-c) = 6bc^4 - 4b^3c^2$$

which is positive when $c > \sqrt{\frac{2}{3}}b$ and negative when $c < \sqrt{\frac{2}{3}}b$.

References

[1] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities*, Cambridge University Press, London (1934)

[2] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, second edition, Oxford University Press, Oxford (1995)