Hamiltonian Paths in Connected Claw-Free Graphs

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ABSTRACT. Let G be a connected claw-free graph of order n. If $G \notin M$ and the minimum degree of G is at least $n/4$, then G is traceable. Where M is a set of graphs such that each element in M can be decomposed into three disjoint subgraphs G_1, G_2, G_3 and $E_G(G_i, G_j) = u_i u_j$, here $1 \leq i$, $j \leq 3$ and $u_i \in G_i$, $1 \leq i \leq 3$.

1 Introduction

We will consider only finite, undirected graphs without loops or multiple edges. We use the notation and terminology in [2]. Let G be a graph of order n, G is called hamiltonian (traceable) if G has a cycle (path) containing n vertices. A graph G is called hamilton-connected if every pair of distinct vertices in G can be connected by a path containing n vertices. A graph G is called claw-free if no induced subgraph of G is isomorphic to $K_{1,3}$. For $v \in V(G)$ and a subgraph H of G, we define $N_H(v) = \{u \in V(H): uv \in E(G)\}$, $d_H(v) = |N_H(v)|$. Let A, B be two disjoint subsets of $V(G)$, we define $E(A, B) = \{ab: a \in A, b \in B; ab \in E(G)\}$.

The following results are due to M. Matthews and D. Sumner.

Theorem 1. [5] *If G is a connected claw-free graph of order n with $\delta \geq (n-2)/3$, then G is traceable.*

Theorem 2. [5] *If G is a 2-connected claw-free graph of order n with $\delta \geq (n-2)/3$, then G is hamiltonian.*

Let H_1, H_2 and H_3 be three disjoint copies of a complete graph of order at least three, choose two distinct vertices u_i, v_i in H_i, $1 \leq i \leq 3$. We define graphs A, B as follows: the vertex set of the graph A is the union of vertex sets of H_1, H_2 and H_3, the edge set of the graph A is the union of edge sets of H_1, H_2, H_3 and $\{u_1u_2, u_2u_3, u_3u_1\}$. The graph B has the same vertex set as the graph A and the edge set of the graph B is the union of the edge set of A and $\{v_1v_2, v_2v_3, v_3v_1\}$. Then the graphs A, B show that the bounds of δ in the Theorems 1, 2 are sharp, respectively.

Let F be the set of graphs defined as follows: If G is in F, then G can be decomposed into three disjoint subgraphs G_1, G_2, G_3 such that $E_G(G_i, G_j) = \{u_iu_j, v_iv_j\}$, $1 \le i, j \le 3$. Where $u_i, v_i \in G_i$, $1 \le i \le 3$. Note that the graph B is an element of F.

H. Li proved the following theorem.

Theorem 3. [4] *If G is a 2-connected claw-free graph of order n such that $G \notin F$ and $\delta \ge n/4$, then G is hamiltonian. The bound $n/4$ is sharp.*

A corresponding theorem on the traceability of connected claw-free graphs is obtained in this paper.

Let M be the set of graphs defined as follows: If G is in M, then G can be decomposed into three disjoint subgraphs G_1, G_2, G_3 such that $E_G(G_i, G_j) = u_iu_j$, where $1 \le i, j \le 3$ and $u_i \in G_i$, $1 \le i \le 3$. Note that the graph A is an element of M.

Theorem 4. *If G is a connected claw-free graph of order n such that $G \notin M$ and $\delta \ge n/4$, then G is traceable. The bound $n/4$ is sharp.*

The sharpness of Theorem 4 can be shown by the set J of graphs defined as follows: Let G_i $(1 \le i \le 4)$ be the complete graphs of order $\delta + 1 \ge 4$ such that $V(G_i) \cap V(G_j) = \emptyset$, where $1 \le i, j \le 3$; $V(G_i) \cap V(G_4) = \{u_i\}$, $1 \le i \le 3$; $E_G(G_i, G_j) = \{u_iu_j\}$, $1 \le i, j \le 3$; $E_G(G_i - \{u_i\}, G_4 - \{u_i\}) = \emptyset$, $1 \le i \le 3$. Thus the graphs in J are connected claw-free non-traceable of order $4\delta + 1$ and not in M.

2 Proof of Theorem 4

We use the following results as our lemmas to prove Theorem 4.

Lemma 1. [3] *Let G be a connected graph such that for every longest path P the sum of the degrees of the two end-vertices of P is at least $|V(P)|+1$. Then G is hamilton-connected.*

Lemma 2. [5] *If G is a connected claw-free graph, then G is either traceable or has a path with $|V(P)| \ge 2\delta + 3$.*

By the proof of the main Theorem in [1], we have the following lemma.

Lemma 3. [1] *Let $P = P[v, w]$ be a longest path with end-vertices v and w in a connected claw-free graph G and H a component of $G - P$. If $u \in H$, $ux, uy \in E$ and y is in the segment of P between v and x. Then there exist vertices a, b which are in the segment of P between y and x, the segment of P between x and w, respectively, such that $\{u, v, a, b\}$ is independent and their neighbors are pairwise disjoint.*

Proof of Theorem 4: By Lemma 2, we can verify that Theorem 4 is true if $n \leq 8$. So we assume that $n \geq 9$. Let G be a graph satisfying the conditions in Theorem 4 and G is not traceable. Let $P = v_1v_2\dots v_s$ be a path of maximum length in G and let H be the component of $G - P$ with the smallest order. By Lemma 2, $s \geq 2\delta+3$. We define the orientation of P to be from v_1 to v_s. If u, v are in $V(P)$, then $P[u,v]$ denotes the consecutive vertices on P from u, v. We will consider $P[u,v]$ both as a path and as a vertex set. We use u^+ to denote the successor of u ($\neq v_s$) on P and u^- the predecessor of u ($\neq v_1$) on P. Let A be a subset of $V(P) - \{v_s\}$, B be a subset of $V(P) - \{v_1\}$, we define $A^+ = \{a^+ : a \in A\}$, $B^- = \{b^- : b \in B\}$.

By the maximality of P and since G is claw-free, we have $x^-x^+ \in E$ for any $x \in N_P(H)$.

Claim 1. *$d_H(u) \geq (5\delta - n + 1)/4$, for any $u \in H$.*

Proof of Claim 1: Let $N_P(u) = \{x_1, x_2, \dots, x_t\}$. By the maximality of P, we have $N_P(u) \subseteq V(P) - \{v_1, v_2, v_{s-1}, v_s\}$ and $|P[x_i^+, x_{i+1}^-]| \geq 3$, $1 \leq i \leq t$.

If $|P[x_i^+, x_{i+1}^+]| \geq 4$, $1 \leq i \leq t$, then $n \geq |\{v_1, v_2, v_{s-1}, v_s\}| + |H| + 4(d_P(u) - 1) + d_P(u) = 5d_P(u) + |H| \geq 5d_P(u) + d_H(u) + 1 \geq 4d_P(u) + d(u) + 1 \geq 4d_P(u) + \delta + 1$,

$$d_P(u) \leq (n - \delta - 1)/4$$

$d_H(u) = d(u) - d_P(u) \geq \delta - (n - \delta - 1)/4 = (5\delta - n + 1)/4$.

If $|P[x_i^+, x_{i+1}^-]| = 3$, for some i, then $N(x_i^{++}) \cap [H \cup N_P(u) \cup N_P^+(u) \cup N_P^-(u) \cup N_P^{++}(u) \cup \{v_1\} - \{x_i^+, x_{i+1}^-\}] = \emptyset$.

Otherwise we can find paths in G which are longer than P. Thus $n \geq d(x_i^{++}) + |H| + 4d_P(u) + 1 - 2 \geq \delta + d_H(u) + 4d_P(u) = \delta + d(u) + 3d_P(u) \geq 2\delta + 3d_P(u)$,

$$d_P(u) \leq (n - 2\delta)/3,$$

$d_H(u) = d(u) - d_P(u) \geq \delta - (n - 2\delta)/3 \geq (5\delta - n + 1)/4$.

Claim 2. *H is hamilton-connected.*

Proof of Claim 2: Suppose that H is not hamilton-connected. By Lemma 1, there exists a longest path $u_1u_2\dots u_m$ in H such that

$$d_H(u_1) + d_H(u_m) \leq m.$$

Note that $N_P(u_1) \neq \emptyset$ or $N_P(u_m) \neq \emptyset$. Otherwise $|V(P)| \leq n - |H| \leq n - (d_H(u_1) + d_H(u_m)) = n - (d(u_1) + d(u_m)) \leq n - 2\delta \leq 2\delta$, contradicting Lemma 2.

Without loss of generality, we assume that $N_P(u_1) \neq \emptyset$. Then we claim that either $N_P(u_m) = \emptyset$ or $N_P(u_1) = N_P(u_m) = \{v_i\}$, for some i, $3 \leq i \leq s - 3$. Suppose to the contrary, then there exist two distinct vertices x, y

in P such that $u_1x, u_my \in E$. Let $N_P(u_1) = \{x_1, x_2, \ldots, x_t\}$ and the order of x_i's appearing on P agrees with the orientation of P.

If $t = 1$, then $x_1 = x \neq y$. Without loss of generality, we assume that $y \in P[x_1^+, v_s]$. By the maximality of P, we have

$$|P[v_1, x_1^{--}]| \geq m, |P[y^{++}, v_s]| \geq m, \text{ and } |P[x_1^{++}, y^{--}]| \geq m.$$

So $n \geq |H| + |V(P)| \geq 3m + 6 \geq 3m + 4d_P(u_1) - 1$.

If $t \geq 2$, By the maximality of P, we have

$$|P[v_1, x_1^{--}]| \geq m, |P[x_t^{++}, v_s]| \geq m.$$

So $n \geq |H| + |V(P)| = |H| + |P[v_1, x_1^{--}]| + |P[x_1^-, x_t^+]| + |P[x_t^{++}, v_s]| \geq 3m + 4d_P(u_1) - 1$.

Hence $n \geq 3m + 4d_P(u_1) - 1$.

Similarly, $n \geq 3m + 4d_P(u_m) - 1$.

Therefore $n \geq 3m + 2(d_P(u_1) + d_P(u_m)) - 1 \geq 3(d_H(u_1) + d_H(u_m)) + 2(d_P(u_1) + d_P(u_m)) - 1 = (d_H(u_1) + d_H(u_m)) + 2(d(u_1) + d(u_m)) - 1 \geq (5\delta - n + 1)/2 + 4\delta - 1 \geq 4\delta + 1$, a contradiction.

If $N_P(u_m) = \emptyset$, let $N_P(u_1) = \{x_1, x_2, \ldots, x_k\}$ and the order of x_i's appearing on P agrees with the orientation of P. By the maxiamlity of P, we have

$$|P[v_1, x_1^{--}]| \geq m, |P[x_k^{++}, v_s]| \geq m, \text{ and}$$

$n \geq |H| + 2m + 4d_P(u_1) - 1 \geq 3d_H(u_1) + 4d_P(u_1) + 3d_H(u_m) - 1 = 3d(u_1) + 3d(u_m) + d_P(u_1) - 1 \geq 6\delta + d_P(u_1) - 1 \geq 4\delta + 1$, a contradiction.

If $N_P(u_m) \neq \emptyset$, then there exists a vertex v_i in P such that $N_P(u_1) = N_P(u_m) = \{v_i\}$. By the maximality of P, we have

$$|P[v_1, v_i^{--}]| \geq m, |P[v_i^{++}, v_s]| \geq m, \text{ and}$$

$n \geq |H| + 2m + 3 \geq 3(d_H(u_1) + d_H(u_m) + 1) \geq 3(d(u_1) + d(u_m) - 1) \geq 6\delta - 3 \geq 4\delta + 1$, a contradiction.

Thus H is hamilton-connected.

Let $H = \{u_1, u_2, \ldots, u_h\}$. Since G is connected, there exist vertices $u_i \in H, x \in V(P)$ such that $u_ix \in E$. Without loss of generality, we assume that $i = 1$. We claim that $N(u_1) \cap [V(P) - \{x\}] = \emptyset$. Otherwise there exists a vertex y in $V(P)$ such that $u_1y \in E$. Without loss of generality, we assume that $y \in P[v_1, x^-]$. By Lemma 3, there exist vertices $a \in P[y^+, x^-]$, $b \in P[x^+, v_s]$ such that $\{u_1, v_l, a, b\}$ is independent and their neighbors are pairwise disjoint, thus

$$n \geq d(u_1) + d(v_1) + d(a) + d(b) + 4 \geq 4\delta + 1, \text{ a contradiction.}$$

Since $d_P(u_1) = 1$, $h = |H| \geq d_H(u_1)+1 = d(u_1) \geq \delta$. By the maximality of P, we have

$$|P[v_1, x^{--}]| \geq h, |P[x^{++}, v_s]| \geq h.$$

So H is a unique component of $G - P$. Otherwise by the choice of H, we have

$$n \geq |V(P)| + 2|H| \geq 2h + 3 + 2h \geq 4\delta + 1. \text{ a contradiction.}$$

We also note that $N(u_i) \cap [V(P) - \{x\}] = \emptyset$, $2 \leq i \leq h$. Otherwise there exists some i, $2 \leq i \leq h$, such that $u_i y \in E$, where $y \in V(P) - \{x\}$. Without loss of generality, we assume that $y \in P[x^+, v_s]$. By the hamilton-connectedness of H and the maximality of P, we have

$$n = |V(P)| + |H| = |P[v_1, x^{--}]| + |P[x^{++}, y^{--}]| + |P[y^{++}, v_s]| + |\{x^-, x, x^+\}| + |\{y^-, y, y^+\}| + h \geq 4h + 6 \geq 4\delta + 1, \text{ a contradiction.}$$

Claim 3. (1) $N(x^-) \cap P[x^{++}, v_s] = \emptyset$, (2) $N(x^+) \cap P[v_1, x^{--}] = \emptyset$.

Proof of Claim 3: (1). Suppose to the contrary, then there exists a vertex $y \in P[x^{++}, v_s]$ such that $x^- y \in E$. By the maximality of P, we have

$$N(y^-) \cap P[v_{s-h}, v_s] = \emptyset, N(y^-) \cap P[v_1, v_h] = \emptyset.$$

Thus $n \geq |H| + |P[v_1, v_h]| + |P[v_{s-h}, v_s]| + d(y^-) + 1 \geq 3h + \delta + 1 \geq 4\delta + 1$, a contradiction.

By a similar argument, we can show that (2) is true.

So we complete the proof of Claim 3.

Moreover, we have $N(x) \cap [V(P) - \{x^-, x^+\}] = \emptyset$. Otherwise suppose there exists a vertex $y \in N(x) \cap [V(P) - \{x^-, x^+\}]$, then $G[x, u_1, x^+, y] = K_{1,3}$ when $y \in P[v_1, x^{--}]$ and $G[x, u_1, x^-, y] = K_{1,3}$ when $y \in P[x^{++}, v_s]$, a contradiction.

Set

$$G_1 = G[H \cup \{x\}].$$
$$G_2 = G[P[v_1, x^-]].$$
$$G_3 = G[P[x^+, v_s]].$$

To complete the proof, we will show that $G \in M$. It suffices to show there exist no edges between the vertex sets $P[v_1, x^{--}]$ and $P[x^{++}, v_s]$.

Suppose to the contrary, then there exist vertices $y \in P[v_1, x^{--}]$, $z \in P[x^{++}, v_s]$ such that $yz \in E$. We first assume that $z = v_s$.

If $N(x^-) \cap P[v_1, y^-] \neq \emptyset$, let $i = \min\{j \colon v_j \in N(x^-) \cap P[v_1, y^-]\}$, the maximality of P and the choice of i imply that

$$|P[v_1, v_i^-]| \geq h, |P[x^{++}, v_s]| \geq h, \text{ and}$$

$n = |V(P)| + |H| = |P[v_1, v_i^-]| + |P[v_i, x^{--}]| + |\{x^-, x, x^+\}| + |P[x^{++}, v_s] + h \geq h + d(x^-) - 2 + 3 + h + h = 3h + d(x^-) + 1 \geq 4\delta + 1$, contradiction.

If $N(x^-) \cap P[v_, y^-] = \emptyset$, the maximality of P also implies that

$$|P[v_1, y^-]| \geq h, |P[x^{++}, v_s]| \geq h, \text{ and}$$

$n = |V(P)| + |H| = |P[v_1, y^-]| + |P[y, x^{--}]| + |\{x^-, x, x^+\}| + |P[x^{++}, v_s]| + h \geq h + d(x^-) - 2 + 3 + h + h = 3h + d(x^-) + 1 \geq 4\delta + 1$, a contradiction.

A symmetric argument shows that we can derive a contradiction when $y = v_1$.

Thus we can assume that $y \neq v_1$, $z \neq v_s$. Note that the above arguments also imply that we can assume that $N(v_s) \subseteq P[x^+, v_s]$, $N(v_1) \subseteq P[v_1, x^-]$.

Case 1. $N(x^-) \cap P[v_1, y^-] \neq \emptyset$, $N(v_s) \cap P[x^+, z^-] \neq \emptyset$.

Let

$$i = \max\{k \colon v_k \in N(x^-) \cap P[v_1, y^-]\}.$$
$$j = \max\{k \colon v_k \in N(v_s) \cap P[x^+, z^-]\}.$$

Then by the maximality of P and the choice of i and j, we have

$$|P[v_i^+, y^-]| + |P[v_j^+, z^-]| \geq h,$$

$$N(x^-) \cap N(v_s) \subseteq \{x^+\},$$

$$P[v_i^+, y^-] \cup P[v_j^+, z^-] \subseteq V(G) - (H \cup N(v^-) \cup N(v_s) \cup \{x^-, v_s\}, \text{ and}$$

$n \geq |H| + |P[v_i^+, y^-]| + |P[v_j^+, z^-]| + |\{x^-, v_s\}| + |N(x^-) \cup N(v_s)| \geq h + h + 2 + d(x^-) + d(v_s) - |N(x^-) \cap N(v_s)| \geq 2\delta + 2 + 2\delta - |\{x^+\}| = 4\delta + 1$, a contradiction.

Case 2. $N(x^-) \cap P[v_1, y^-] = \emptyset$, $N(v_s) \cap P[x^+, z^-] \neq \emptyset$.

Let $j = \min\{k \colon v_k \in N(v_s) \cap P[x^+, z^-]\}$.

If $v_j = x^+$, then the maximality of P and the choice of j imply that

$$|P[v_1, y^-]| \geq h, \text{ and}$$

$n = |V(P)| + |H| = |P[v_1, y^-]| + |P[y, x^{--}]| + |\{x^-, x\}| + |P[x^+, v_s]| + h \geq h + d(x^-) - 2 + 2 + d(v_s) + 1 + h \geq 4\delta + 1$, a contradiction.

If $v_j \neq x^+$, the maximality of P and the choice of j also imply that

$$|P[v_1, y^-]| + |P[x^{++}, v_j^-]| \geq h, \text{ and}$$

$n = |V(P)| + |H| = |P[v_1, y^-]| + |P[x^{++}, v_j^-] + |P[y, x^{--}]| + |\{x^-, x, x^+\}| + |P[v_j, v_s]| + h \geq h + d(x^-) - 2 + 3 + d(v_s) + 1 + h \geq 4\delta + 1$, contradiction.

We therefore have $N(v_s) \cap P[x^+, z^-] = \emptyset$. Thus we can assume that $N(v_s) \subseteq P[z, v_s]$. Symmetrically, we can also assume that $N(v_1) \subseteq P[v_1, y]$.

So $4\delta \geq n = |V(P)| + |H| = |P[v_1, x^-]| + |P[x, z]| + |P[z^+, v_s]| + h \geq |P[v_1, x^-]| + 2 + \delta + h \geq |P[v_1, x^-]| + 2\delta + 2$, and $|P[v_1, x^-]| \leq 2\delta - 2$.

Let $Q = P[v_1, x^-]$. Then $|N_Q(v_1) \cup N_Q^+(x^-)| + |N_Q(v_1) \cap N_Q^+(x^-)| = |N_Q(v_1)| + |N_Q^+(x^-)| = d_Q(v_1) + d_Q(x^-) \geq \delta + \delta - 2 = 2\delta - 2 \geq |Q|$.

So there exists a vertex $w \in N_Q(v_1) \cap N_Q^+(x^-) \subseteq P[v_1, y]$. By the maximality of P, we have

$$|P[y^+, x^{--}]| + |P[x^{++}, z^-]| \geq h, \text{ and}$$

$n = |V(P)| + |H| = |P[v_1, y]| + |P[y^+, x^{--}]| + |P[x^{++}, z^-]| + |\{x^-, x, x^+\}| + |P[z, v_s]| + h \geq d(v_1) + 1 + h + 3 + d(v_s) + 1 + h \geq 4\delta + 1$, a contradiction.

Thus there are no edges between $P[v_1, x^-]$ and $P[x^+, v_s]$. Hence $G \in M$. This completes the proof of Theorem 4.

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