On Packing Designs with Block Size 5 and index $7 \le \lambda \le 21$

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ABSTRACT. Let V be a finite set of order ν. A $(\nu, \kappa\lambda)$ packing design of index λ and block size κ is a collection of κ-element subsets, called blocks, such that every 2-subset of V occurs in at most λ blocks. The packing problem is to determine the maximum number of blocks, $\sigma(\nu\kappa\lambda)$, in a packing design. It is well known that $\sigma(\nu, \kappa\lambda) \le \left[\frac{\nu}{\kappa}\left[\frac{\nu-1}{\kappa-1}\lambda\right]\right] = \Psi(\nu, \kappa, \lambda)$, where $[x]$ is the largest integer satisfying $x \ge [x]$. It is shown here that $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda) - e$ for all positive integers $\nu \ge 5$ and $7 \le \lambda \le 21$ where $e = 1$ if $\lambda(\nu - 1) \equiv 0 \pmod{\kappa - 1}$ and $\lambda\nu\frac{(\nu-1)}{(\kappa-1)} \equiv 1 \pmod{\kappa}$ and $e = 0$ otherwise with the following possible exceptions of (ν, λ) =(28,7) (32,7) (44,7) (32,9) (28,11) (39,11) (28,13) (28,15) (28,19) (39,21).

1 Introduction

A (ν, κ, λ) packing design (or respectively covering design) of order ν, block size κ and index λ is a collection β of κ-element subsets, called blocks, of a ν-set V such that every 2-subset of V occurs in at most (at least) λ blocks.

Let $\sigma(\nu, \kappa, \lambda)$ denote the maximum number of blocks in a (ν, κ, λ) packing design; and $\alpha(\nu, \kappa, \lambda)$ denote the minimum number of blocks in a (ν, κ, λ) covering design. A (ν, κ, λ) packing design with $|\beta| = \sigma(\nu, \kappa, \lambda)$ will be called a maximum packing design. Similarly, a (ν, κ, λ) covering design with $|\beta| = \alpha(\nu, \kappa, \lambda)$ is called a minimum covering design. It is well known [35] that

$$\sigma(\nu, \kappa, \lambda) \le \left[\frac{\nu}{\kappa}\left[\frac{\nu - 1}{\kappa - 1}\lambda\right]\right] = \Psi(\nu, \kappa, \lambda)$$

and

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$$\alpha(\nu,\kappa,\lambda) \geq \left\lceil \frac{\nu}{\kappa} \left\lceil \frac{\nu-1}{\kappa-1}\lambda \right\rceil \right\rceil = \Phi(\nu,\kappa,\lambda)$$

where $\lfloor x \rfloor$ is the largest integer satisfying $\lfloor x \rfloor \leq x$ and $\lceil x \rceil$ is the smallest integer satisfying $x \leq \lceil x \rceil$. Hanani [28] has sharpened this bound in certain cases by proving the following result.

Theorem 1.1. *If* $\lambda(\nu-1) \equiv 0 \pmod{\kappa-1}$ *and* $\lambda\nu(\nu-1)/(\kappa-1) \equiv 1 \pmod{\kappa}$ *then* $\sigma(\nu,\kappa,\lambda) \leq \psi(\nu,\kappa,\lambda) - 1$.

When $\sigma(\nu,\kappa,\lambda) = \psi(\nu,\kappa,\lambda)$ the (ν,κ,λ) packing design is called optimal packing design. Similarly when $\alpha(\nu,\kappa,\lambda) = \phi(\nu,\kappa,\lambda)$ the (ν,κ,λ) covering design is called minimal covering design.

We adopt convention that $\sigma(\nu,\kappa,\lambda) = 0$ for $0 \leq \nu < \kappa$. If in a (ν,κ,λ) packing design a pair, say, (a,b) appears in λ' blocks where $\lambda' \leq \lambda$ then we say that (a,b) is missing $(\lambda - \lambda')$ times. Similarly if a pair (a,b) in a (ν,κ,λ) covering design appears in λ'' blocks where $\lambda'' \geq \lambda$ then we say that (a,b) is repeated $(\lambda'' - \lambda)$ times. Many researchers have been involved in determining the packing number $\sigma(\nu,\kappa,\lambda)$ known up to date (see bibliography). In the case $\kappa = 5$ and $\lambda \geq 6$ we have the following:

Theorem 1.2. *Let* $\nu \geq 5$ *be a positive integer. Then*

(1) $\sigma(\nu,5,1) = \psi(\nu,5,1)$ *for all* $\nu \equiv 3 \pmod{20}$ *and* $\nu \equiv 0 \pmod 4$ $\nu \neq 12, 16$ *with the possible exception of* $\nu = 32$ 48 52 72 132 152 172 232 243 252 272 332 352 432 *[24], [32], [38].*
And $\sigma(12,5,1) = \psi(12,5,1) - 1$, $\sigma(16,5,1) = \psi(16,5,1) - 3$, *[24].*

(2) $\sigma(\nu,5,2) = \psi(\nu,5,2)$ *for all positive even integers* ν *and for* $\nu \equiv 1$ *or* 5 (mod 10), $\nu \neq 15$, *[4], [28], and* $\sigma(\nu,5,2) = \psi(\nu,5,2) - e$ *where* $e = 1$ *if* $\nu \equiv 7$ *or* 9 (mod 10) *or* $\nu = 13$ *with the possible exception of* $\nu = 19$, 27, 147 *and* $e = 0$ *if* $\nu \equiv 3 \pmod{10}$, $\nu \neq 13$, *[3], [5], [6], see also [40].*

(3) (a) $\sigma(\nu,5,3) = \psi(\nu,5,3)$ *for all positive integers* ν, $\nu \not\equiv 0 \pmod 4$ *with the possible exception of* $\nu = 17, 29, 33, 38, 49$, *[7], [8], [11].*

(b) $\sigma(\nu,5,3) = \psi(\nu,5,3)$ *for all positive integer* $\nu \equiv 0 \pmod 4$, $\nu \leq 96$ *with the possible exception of* $\nu = 20, 28, 32, 36, 56, 296$, *[9].*

(4) $\sigma(\nu,5,4) = \psi(\nu,5,4)$ *for all positive integer* ν, $\nu \neq 7$ *and* $\sigma(7,5,4) = \psi(7,5,4) - 1$, *[10].*

(5) $\sigma(\nu,5,5) = \psi(\nu,5,5)$ *for all positive integers* ν, *with the possible exception of* $\nu = 28, 32, 34$, *[8].*

(6) $\sigma(\nu,5,6) = \psi(\nu,5,6)$ *for all positive integers* ν *with the possible exception of* $\nu = 43$. *[11].*

(7) $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda) - e$ *for all positive integers* ν *and* $\lambda = 8, 12, 16$, *[12], [40] with few possible exceptions where* $e = 1$ *if* $\lambda(\nu - 1) \equiv 0 \pmod{\kappa - 1}$ *and* $\lambda\nu(\nu - 1)/(\kappa - 1) \equiv 1 \pmod{k}$ *and* $e = 0$ *otherwise.*

Furthermore, these few possible exceptions were removed later, in an unpublished paper, by Shalaby, [36].

Our interest here is in the case $\kappa = 5$ and $\lambda \geq 7$. Our goal is to prove the following.

Theorem 1.3. *Let* $7 \leq \lambda \leq 21$ *and* ν *be a positive integers. Then* $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda) - e$ *where* $e = 1$ *if* $\lambda(\nu - 1) \equiv 0 \pmod{\kappa - 1}$ *and* $\lambda\nu(\nu - 1)/(\kappa - 1) \equiv 1 \pmod{\kappa}$, *and* $e = 0$ *otherwise with the possible exceptions of* $(\nu, \lambda) = (28, 7)(32, 7)(44, 7)(32, 9)(28, 11)(39, 11)(28, 13)(28, 15)(28, 19)(39, 21)$.

2 Recursive Constructions

In order to describe our recursive constructions we require several other types of combinatorial designs. A balanced incomplete block design, $B[\nu, \kappa, \lambda]$, is a (ν, κ, λ) packing design where every 2-subset of points is contained in precisely λ blocks. If a $B[\nu, \kappa, \lambda]$ exists then it is clear that $\sigma(\nu, \kappa, \lambda) = \lambda\nu(\nu - 1)/\kappa(\kappa - 1) = \psi(\nu, \kappa, \lambda)$ and Hanani [28] has proved the following existence theorem for $B[\nu, 5, \lambda]$.

Theorem 2.2. *Necessary and sufficient conditions for the existence of a* $B[\nu, 5, \lambda]$ *are that* $\lambda(\nu - 1) \equiv 0 \pmod{4}$ *and* $\lambda\nu(\nu - 1) \equiv 0 \pmod{20}$ *and* $(\nu, \lambda) \neq (15, 2)$.

A (ν, κ, λ) packing design (or respectively covering design) with a hole of size h is a triple (V, H, β) where V is a ν-set, H is a subset of V of cardinality h, and β is a collection of κ-element subsets, called blocks, of V such that

1) no 2-subset of H appears in any block;

2) every other 2-subset of V appears in at most (at least) λ blocks;

3) $|\beta| = \psi(\nu, \kappa, \lambda) - \psi(h, \nu, \lambda)$, $(|\beta| = \phi(\nu, \kappa, \lambda) - \phi(h, \kappa, \lambda))$.

Lemma 2.1. *If there exists a* (ν, κ, λ) *packing design with a hole of size* $h \geq \kappa$ *and* $\sigma(h, \kappa, \lambda) = \psi(h, \kappa, \lambda)$ *then* $\sigma(\nu, \kappa, \lambda) = \psi(\nu, \kappa, \lambda)$.

Proof: Form the blocks of an (h, κ, λ) optimal packing design on the points of the hole. Adding the blocks of the packing design with the hole gives a (ν, κ, λ) optimal packing design.

Let κ, λ and ν be positive integers and M be a set of positive integers. A group divisible design GD$[\kappa, \lambda, M, \nu]$ is a triple (V, β, γ) where V is a set of points with $|V| = \nu$, and $\gamma = \{G_1, \dots, G_n\}$ is a partition of V into n sets called groups. The collection β consists of κ-subsets of V, called blocks, with the following properties

1) $|B \cap G_i| \leq 1$ for all $B \in \beta$ and $G_i \in \gamma$;
2) $|G_i| \in M$ for all $G_i \in \gamma$;
3) every 2-subset $\{x, y\}$ of V such that x and y belong to distinct groups is contained in exactly λ blocks.

If $M = \{m\}$ then the group divisible design is denoted by GD$[\kappa, \lambda, m, \nu]$. A GD$[\kappa, \lambda, m, \kappa m]$ is called a transversal design and denoted by T$[\kappa, \lambda, m]$. It is well known that a T$[\kappa, 1, m]$ is equivalent to $\kappa - 2$ mutually orthogonal Latin squares of side m.

In the sequel we shall use the following existence theorem for transversal designs. The proof of this result may be found in [1], [2], [20], [23], [25], [26], [28], [34], [37].

Theorem 2.2. *There exists a T$[6, 1, m]$ for all positive integers m with the exception of $m \in \{2, 3, 4, 6\}$ and the possible exception of $m \in \{10, 14, 18, 22\}$.*

Theorem 2.3. [28] *There exists a T$[7, \lambda, m]$ for all positive integers m and all integers $\lambda \geq 2$.*

The following theorem is à generalization of theorem 2.6 of [7].

Theorem 2.4. *If there exists a GD$[6, \lambda, 5, 5n]$ and a $(20 + h, 5, \lambda)$ packing design with a hole of size h then there exists a $(20(n-1) + 4u + h, 5, \lambda)$ packing design with a hole of size $4u + h$.*

It is clear that to apply the above theorem we require the existence of a GD$[6, \lambda, 5, 5n]$. Our authority for that is the following lemma of Hanani [28].

Lemma 2.2. *There exists a GD$[6, \lambda, 5, 35]$, for $\lambda = 2m + 3n$ where m and n are non negative integers and there exists a GD$[6, \lambda, 5, \nu]$ where $\nu =$ 50, 60, 65 and λ is a positive even number and there exists a GD$[6, \lambda, 5, 40]$ for $\lambda = 3n$, n is a positive integer.*

A truncated transversal design TT(κ, λ, m, u) is a GD$[\{\kappa, \kappa-1\}, \lambda, \{m, u^*\}, (\kappa - 1)m + u]$ where * means there is exactly one group of size u.

Clearly a TT$(k, \lambda, m, 0)$ is equivalent to a T$[\kappa - 1, \lambda, m]$. Furthermore, if $0 \leq u \leq m$ then one may construct a TT(κ, λ, m, u) from a transversal design T$[\kappa, \lambda, m]$ by removing points from the last group, and from all blocks containing them. Thus we have the following existence result.

Theorem 2.5. *There exists a TT$(6, \lambda, m, u)$ for all positive integers m and u where $0 \leq u \leq m$ and $\lambda > 1$.*

Theorem 2.6. *Let m, u and $\lambda > 1$ be positive integers such that $u, m \equiv 0 \pmod 4$, $0 \le u \le m$, then there exists a $GD[5, \lambda, \{m, u^*\}, 5m + u]$ where $*$ means there is exactly one group of size u.*

Proof: Take a $TT(6, \lambda, \frac{m}{4}, \frac{u}{4})$ and inflate this design by a factor of 4. On the blocks of size 5 and 6 construct a $GD[5, 1, 4, 20]$ and $GD[5, 1, 4, 24]$ respectively. These two designs are assured by theorem 2.1 since they are equivalent to a $B[21, 5, 1]$ and $B[25, 5, 1]$ respectively.

Let us add h points to the groups of a $GD[5, \lambda, \{m, u^*\}, 5m + u]$. On the groups of size m construct a $(m + h, 5, \lambda)$ packing design with a hole of size h; and on the last group construct a $(u + h, 5, \lambda)$ optimal packing design. Then the resultant design is a $(5m + u + h, 5, \lambda)$ optimal packing design. We may write this observation as the following.

Theorem 2.7. *If*

(1) *There exists a $GD[5, \lambda, \{m, u^*\}, 5m + u]$ where $u, m \equiv 0 \pmod 4$.*

(2) *There exists a $(u + h, 5, \lambda)$ optimal packing design with $\frac{\lambda(u+h)^2 - \lambda(u+h) + c(u+h) + d}{20}$ blocks where c and d are integers determined by λ, u and h*

(3) *There exists a $(m + h, 5, \lambda)$ packing design with a hole of size h with $\frac{\lambda m^2 + 2\lambda mh + cm - \lambda m}{20}$ blocks where c is as before (we assume this number is an integer).*

Then $\sigma(5m + u + h, 5, \lambda) = \psi(5m + u + h, 5, \lambda)$.

Proof: We need to show that the total number of blocks obtained by this construction is precisely $\psi(5m+u+h, 5, \lambda)$. But a $GD[5, \lambda, \{m, u^*\}, 5m+u]$ has the following number of blocks

$$\lambda(m(m - u) + \frac{3}{2}mu) \qquad \text{(I)}$$

A $(u + h, 5, \lambda)$ optimal packing design has the following number of blocks

$$\frac{\lambda(u + h)^2 - \lambda(u + h) + c(u + h) + d}{20} \qquad \text{(II)}$$

where c and d are integers completely determined by the values of u and h. A $(m + h, 5, \lambda)$ packing design with a hole of size h has the following number of blocks

$$\frac{\lambda m^2 + 2\lambda mh + cm - \lambda m}{20} \qquad \text{(III)}$$

where c is an integer completely determined by m and h. Furthermore, the value of c is determined by the congruence classes of m and u modulo 4, but the value of d depends on the congruence classes modulo 20 of m, u and h. On the other hand

$$\Psi(5m+u+h,5,\lambda) = \frac{\lambda(5m+u+h)^2 - \lambda(5m+u+h) + c(5m+u+h) + d}{20} \qquad \text{(IV)}$$

where c and d are the same integers as in (II) since $5m+u+h$ and $u+h$ are the same congruency mod 4.

Now it is easily checked that the total number of blocks in (I), (II) and 5 times the number of blocks in (III) is equal to the total number of blocks in (IV).

Let κ, λ, ν and m be positive integers. A modified group divisible design, MGD$[\kappa,\lambda,m,\nu]$, is a quadruple (V,β,γ,δ) where V is a set of points with $|V|=\nu$, $\gamma=\{G_1,\ldots,G_m\}$ is a partition of V into m sets, called groups, $\delta=\{R_1,\ldots,R_s\}$ is a partition of V into s sets, called rows, and β is a family of κ-subsets of V, called blocks, with the following properties

1) $|B\cap G_i|\le 1$ for all $B\in\beta$ and $G_i\in\gamma$.

2) $|B\cap R_j|\le 1$ for all $B\in\beta$ and $R_j\in\delta$.

3) $|R_j|=m$ for all $R_j\in\delta$.

4) Every 2-subset $\{x,y\}$ of V such that x and y are neither in the same group nor in the same row is contained in exactly λ blocks

5) $|G_i\cap R_j|=1$ for all $G_i\in\gamma$ and $R_j\in\delta$.

A resolvable modified group divisible design, RMGD$[\kappa,\lambda,m,\nu]$, is a modified group divisible design the blocks of which can be partitioned into parallel classes. It is clear that a RMGD$[5,1,5,5m]$ is the same as RT$[5,1,m]$ with one parallel class of blocks singled out, and since a $RT[5,1,m]$ is equivalent to a T$[6,1,m]$ we have the following.

Theorem 2.8. *There exists a RMGD$[5,1,5,5m]$ for all positive integers m with the exception of $m\in\{2,3,4,6\}$ and the possible exception of $m\in\{10,14,18,22\}$.*

The next theorem is in the form most useful to us.

Theorem 2.9. [3] *If there exists a RMGD$[5,1,5,5m]$ and a GD$[5,\lambda,\{4,s^*\},4m+s]$, where * means there is exactly one group of size s, and there exists a $(20+h,5,\lambda)$ packing design with a hole of size h then there exists a $(20m+4u+h+s,5,\lambda)$ packing design with a hole of size $4u+h+s$ where $0\le u\le m-1$.*

It is clear that the application of the above theorem requires the existence of a GD$[5, 1, \{4, s^*\}, 4m+s]$. We observe that we may choose $s = 0$ if $m \equiv 1 \pmod 5$; $s = 4$ if $m \equiv 0$ or $4 \pmod 5$, and $s = \frac{4(m-1)}{3}$ if $m \equiv 1 \pmod 3$ (see [3]). We may also apply the following [27].

Theorem 2.10. *There exists a GD*$[5, 1, \{4, 8^*\}, 4m + 8]$ *where* $m \equiv 0$ *or* $2 \pmod 5$, $m \geq 7$ *with the possible exception of* $m = 10$.

Theorem 2.11. *If there exists*

1) *a RMGD*$[5, 1, 5, 5m]$;

2) *a GD*$[5, \lambda, 4, 4m]$;

3) *a* $(20 + h, 5, \lambda)$ *packing design with a hole of size* h, *and*

4) $\sigma(20 + h, 5, \lambda) = \psi(20 + h, 5, \lambda)$;

Then $\sigma(20m + h, 5, \lambda) = \psi(20m + h, 5, \lambda)$.

Proof: Take a RMGD$[5, 1, 5, 5m]$ and inflate it by a factor of 4. Replace the blocks of this design by the blocks of a GD$[5, \lambda, 4, 20]$. Add h points to the groups and on the first $m-1$ groups construct a $(20+h, 5, \lambda)$ packing design with a hole of size h and on the last group construct a $(20+h, 5, \lambda)$ optimal packing design. Finally, on the blocks of size m construct a GD$[5, \lambda, 4, 4m]$.

The proof of the next theorem is very similar to the proof of theorem 2.4 of [3].

Theorem 2.12. *If there exists* (1) *a RMGD*$[5, 1, 5, 5m]$ *and* (2) *a GD*$[5, \lambda, \{4, 8^*\}, 4m + 4]$ (3) *a* $(20 + h, 5, \lambda)$ *packing design with a hole of size* h *and* (4) *a* $(20 + h + 4, 5, \lambda)$ *packing design with a hole of size* $h + 4$. *Then there exists a* $(20m + 4u + h + 4, 5, \lambda)$ *packing design with a hole of size* $4u + h + 4$ *where* $0 \leq u \leq m - 1$.

Theorem 2.13. [3] *If there exists a* (1) *RMGD*$[5, 1, 5, 5m]$ (2) *a GD*$[5, \lambda, \{4, s^*\}, 4(m - 1) + s]$ *and* (3) *a* $(20 + h, 5, \lambda)$ *packing design with a hole of size* h. *Then there exists a* $(24(m - 1) + s + h, 5, \lambda)$ *packing design with a hole of size* $4(m - 1) + s + h$.

Theorem 2.14. *If there exists a RGD*$[5, 1, 5, 5m]$ *and a* $(20 + h, 5, \lambda)$ *packing design with a hole of size* h *then there exists a* $(20m + 4u + h, 5, \lambda)$ *packing design with a hole of size* $4u + h$ *where* $0 \leq u \leq \frac{5(m-1)}{4}$.

Proof: Generalize the proof of theorem 1.5 of [3, p. 165].

In a similar way to that of theorem 2.14 we can prove the following.

Theorem 2.15. *Let* λ *be a positive even integer. If there exists a RGD*$[5, 1, 5, 5m]$ *and a* $(10 + h, 5, \lambda)$ *packing design with a hole of size* h. *Then there* *exists a* $(10m+2u+h,5,\lambda)$ *packing design with a hole of size* $2u+h$ *where* $0 \le u \le \frac{5(m-1)}{4}$.

The following theorem is a generalization of theorem 2.6 of [6, p. 50].

Theorem 2.16. *Let* λ *be a positive even integer. If there exists a RMGD*$[5,1,5,5m]$, *a* $(10+h,5,\lambda)$ *packing design with a hole of size* h*; and either a GD*$[5,\lambda,2,2m]$ *or a GD*$[5,\lambda,2,2(m+1)]$ *exists then there exists a* $(10m+2u+h+e,5,\lambda)$ *packing design with a hole of size* $2u+h+e$ *where* $0 \le u \le m-1$ *and* $e=0$ *if a GD*$[5,\lambda,2,2m]$ *exists and* $e=2$ *if a GD*$[5,\lambda,2,2(m+1)]$ *exists.*

We also shall use the following.

Theorem 2.17. *If there exists a RMGD*$[5,1,5,5m]$ *and a B*$[2m+1,5,2]$ *then there exists a* $(10m+2u+1,5,2)$ *packing design with a hole of size* $2u+1$ *where* $0 \le u \le m-1$.

Proof: Take a RMGD$[5,1,5,5m]$ and inflate it by a factor of two. To the parallel class of size m, after inflating by two, add a new point and on each block construct a B$[2m+1,5,2]$. To u parallel classes, $0 \le u \le m-1$, add 2 points and replace their blocks by the blocks of a GD$[5,2,2,12]$. On the remaining parallel classes construct a GD$[5,2,2,10]$. See [28] for the existence of these two designs. Finally replace the groups of RMGD$[5,1,5,5m]$ by the blocks of a GD$[5,2,2,10]$ in such a way that if $\{a\,b\,c\,d\,e\}$ is a group of RMGD$[5,1,5,5m]$ then the groups of GD$[5,2,2,10]$ are $\{a_0,a_1\}$ $\{b_0,b_1\}$ $\{c_0,c_1\}$ $\{d_0,d_1\}$ $\{e_0,e_1\}$.

The following recursive construction is a special case of theorem 2.4.

Theorem 2.18. *If there exists* (1) *a GD*$[6,\lambda,5,5n]$ (2) *a* $(20+h,5,\lambda)$ *packing design with a hole of size* h *and* (3) *a* $(20+h,5,\lambda)$ *optimal packing design. Then there exists a* $(20n+h,5,\lambda)$ *optimal packing design.*

Proof: Take a GD$[6,\lambda,5,5n]$. Inflate this design by a factor of 4. Replace all the blocks of this design by the blocks of GD$[5,1,4,24]$. Finally add h points to the groups and on the first $(n-1)$ groups construct a $(20+h,5,\lambda)$ packing design with a hole of size h and on the last group construct a $(20+h,5,\lambda)$ optimal packing design.

3 The Structure of Packing and Covering Designs

Let (V,β) be a (ν,κ,λ) packing design, for each 2-subset $e=\{x,y\}$ of V define $m(e)$ to be the number of blocks in β which contain e. Note that by the definition of a packing design we have $m(e) \le \lambda$ for all e.

The complement of (V,β), denoted by $C(V,\beta)$ is defined to be the graph with vertex set V and edges e occurring with multiplicity $\lambda - m(e)$ for all e. The number of edges (counting multiplicities) in $C(V,\beta)$ is given by

$\lambda\binom{\nu}{2} - |\beta|\binom{\kappa}{2}$. The degree of the vertex x in $C(V,\beta)$ is $\lambda(\nu-1) - r_x(\kappa-1)$ where r_x is the number of blocks containing x.

In a similar way we define the excess graph of a (V,β) covering design denoted by $E(V,\beta)$, to be the graph with a vertex set V and edges e occurring with multiplicity $m(e) - \lambda$ for all e. The number of edges in $E(V,\beta)$ is given by $|\beta|\binom{\kappa}{2} - \lambda\binom{\nu}{2}$; and the degree of each vertex is $r_x(\kappa-1) - \lambda(\nu-1)$ where r_x is as before.

Lemma 3.1. *Let (V,β) be a $(\nu,5,4)$ packing design with $\psi(\nu,5,4) - e$ blocks where $e = 1$ if $\nu \equiv 3 \pmod 5$ and 0 otherwise. Then the degree of each vertex of $C(V,\beta)$ is divisible by 4, and the number of edges in the graph is 0, 4 or 12 when ν mod $5 \in \{0,1\}$, $\{2,4\}$ or $\{3\}$ respectively.*

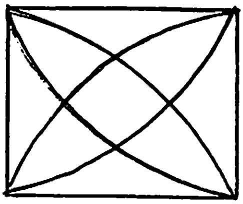
The only graph with 4 edges and every vertex of a degree divisible by 4 is the graph with four parallel edges connecting two vertices and $\nu - 2$ isolated vertices. Therefore, when $\nu \equiv 2$ or $4 \pmod 5$ a $(\nu,5,4)$ optimal packing design contains a pair of points which do not appear in any block, and all other pairs appear in precisely 4 blocks.

The situation is more complicated when $\nu \equiv 3 \pmod 5$ since there are many graphs with 12 edges satisfying the degree constraint. A particularly useful graph of this type is the one with $\nu-3$ isolated vertices and 3 vertices each connected to the other 2 by four parallel edges.

Lemma 3.2. *Let (V,β) be a $(\nu,5,4)$ minimal covering design. Then the degree of each vertex of $E(V,\beta)$ is divisible by 4 and the number of edges in the graph is 0, 6, or 8 when ν mod $5 \in \{0,1\}$, $\{2,4\}$ or $\{3\}$ respectively.*

The only graph with 6 edges and every vertex of a degree divisible by 4 is the graph with $\nu - 3$ isolated vertices and 3 vertices each connected to the other 2 by two parallel edges.

The situation is more complicated when $\nu \equiv 3 \pmod 5$ since there are many graphs with 8 edges that satisfy the degree constraint. A particularly useful graph of this type is the one with $\nu - 4$ isolated vertices and the following graph on the remaining 4 vertices.



Lemma 3.3.

(1) *Let (V,β) be a $(\nu,5,3)$ optimal packing design where $\nu \equiv 8 \pmod{20}$. Then the degree of each vertex of $C(V,\beta)$ is 1 and the number of edges in the graph is $\frac{\nu}{2}$. Hence $C(V,\beta)$ is a 1-factor.*

(2) *Let* (V,β) *be a* $(\nu,5,3)$ *minimal covering design where* $\nu \equiv 14 \pmod{20}$. *Then the degree of each vertex of* $E(\nu,\beta)$ *is 1 and the number of edges in* $E(V,\beta)$ *is* $\frac{\nu}{2}$. *Hence* $E(V,\beta)$ *is a 1-factor.*

Lemma 3.4. *Let* (V,β) *be a* $(\nu,5,2)$ *optimal packing design where* $\nu \equiv 3 \pmod{10}$. *Then the degree of each vertex of* $C(V,\beta)$ *is divisible by 4, and the number of edges in the graph is 6. Hence,* $C(V,\beta)$ *consists of* $\nu-3$ *isolated vertices and three other vertices the pair of which are connected by two edges.*

To define the complement graph of a packing design with a hole H of size h let $e=\{x,y\}$ where at least one of x or y does not lie in H and let $m(e)$ be the number of blocks in β which contain e. Then the complement graph of the packing design with a hole H of size h, denoted by $C(V\backslash H,\beta)$, is the graph with vertex set V and edges e occuring with multiplicity $\lambda-m(e)$. In a similar way the excess graph, $E(V\backslash H,\beta)$, of a (ν,κ,λ) covering design with a hole of size h is defined.

The following two theorems are very simple but most useful to us.

Theorem 3.1. *If there exists*

1) *a* $(\nu,5,\lambda)$ *covering design with* $\phi(\nu,5,\lambda)$ *blocks,*

2) *a* $(\nu,5,\lambda')$ *packing design with* $\psi(\nu,5,\lambda')$ *blocks,*

3) $\phi(\nu,5,\lambda)+\psi(\nu,5,\lambda')=\psi(\nu,5,\lambda+\lambda')$, *and*

4) *the excess graph* $E(V,\beta)$ *of the covering design is isomorphic to a subgraph* G *of the complement graph,* $C(V,\beta)$, *of the packing design.*

Then there exists a $(\nu,5,\lambda+\lambda')$ *packing design with* $\psi(\nu,5,\lambda+\lambda')$ *blocks.*

Theorem 3.2. *If there exists*

1) *a* $(\nu,5,\lambda)$ *covering design with a hole of size* h,

2) *a* $(\nu,5,\lambda')$ *packing design with a hole of size* h,

3) *the total number of blocks in these two designs is* $\psi(\nu,5,\lambda+\lambda')-\psi(h,5,\lambda+\lambda')$,

4) *the excess graph,* $E(V\backslash H,\beta)$, *of the covering design with a hole of size* h *is isomorphic to a subgraph* G *of the complement graph,* $C(V\backslash H,\beta)$, *of the packing design with a hole of size* h.

Then there exists a $(\nu,5,\lambda+\lambda')$ *packing design with a hole of size* h.

4 Notations and A Few More Designs

In this short section we discuss the notations used through this paper and construct a few more optimal packing designs for $2 \le \lambda \le 6$.

A block $\langle \kappa\ \kappa+m\ \kappa+n\ \kappa+j\ f(\kappa)\rangle \pmod{\nu}$ where $f(\kappa) = a$ if κ is even, and $f(\kappa) = b$ if κ is odd is denoted by $\langle 0mnj\rangle \cup \{a, b\} \pmod{\nu}$. Similarly, a block $\langle \kappa\ \kappa+m\ \kappa+n\ \kappa+j\ f(\kappa)\rangle \pmod{\nu}$ where $f(\kappa) = a$ if $\kappa \equiv 0 \pmod 4$; $f(\kappa) = b$ if $\kappa \equiv 1 \pmod 4$; $f(\kappa) = c$ if $\kappa \equiv 2 \pmod 4$, and $f(\kappa) = d$ if $\kappa \equiv 3 \pmod 4$ is denoted by $\langle 0mnj\rangle \cup \{a, b, c, d\} \pmod{\nu}$. Notice that a, b, c, d are not necessarily distinct.

In a similar way, a block $\langle (0, \kappa)(0, \kappa+m)(1, \kappa+n)(1, \kappa+j) f(\kappa)\rangle$ mod $(-, \nu)$ where $f(\kappa) = a$ if κ is even, and $f(\kappa) = b$ if κ is odd is denoted by $\langle (0, 0)(0, m)(1, n)(1, j)\rangle \cup \{a, b\}$ mod $(-, \nu)$.

We now improve the result of theorem 1.2

Lemma 4.1. *$\sigma(147, 5, 2) = \psi(147, 5, 2) - 1$; $\sigma(\nu, 5, 3) = \psi(\nu, 5, 3)$ for $\nu = 36, 49, 296$, $\sigma(43, 5, 6) = \psi(43, 5, 6)$.*

Proof: To prove that $\sigma(147, 5, 2) = \psi(147, 5, 2) - 1$ we show that there exists a $(147, 5, 2)$ packing design with a hole of size 17. But $\sigma(17, 5, 2) = \psi(17, 5, 2) - 1$ hence $\sigma(147, 5, 2) = \psi(147, 5, 2) - 1$. For this purpose apply theorem 2.15 with $m = 13$, $u = 8$, $h = 1$ and $\lambda = 2$.

To show that $\sigma(36, 5, 3) = \psi(36, 5, 3)$ take a T[5,3,7]. Add a point to the groups and on each group construct an (8,5,3) optimal packing design.

For $\nu = 296$ and $\lambda = 3$ apply theorem 2.9 (or theorem 2.15) with $m = 13$, $u = 5$, $h = 0$, $s = 16$, and $\lambda = 3$ gives us a (296,5,3) packing design with a hole of size 36. But $\sigma(36, 5, 3) = \psi(36, 5, 3)$. Hence, $\sigma(296, 5, 3) = \psi(296, 5, 3)$.

For $\nu = 49$ and $\lambda = 3$, again we instead construct a (49,5,3) packing design with a hole of size 9. But $\sigma(9, 5, 3) = \psi(9, 5, 3)$ hence $\sigma(49, 5, 3) = \psi(49, 5, 3)$. For this purpose let $X = Z_2 \times Z_{20} \cup \{h_i\}_{i=1}^{9}$, then the required blocks are: On $Z_2 \times Z_{20} \cup \{h_i\}_{i=1}^{5}$ construct a B[45,5,1] with a hole of size 5, say, $\{h_1, \ldots, h_5\}$. Furthermore, take the following blocks.

$\langle (0,0)(0,4)(0,8)(0,12)(0,16)\rangle + (-, i), i \in Z_4$

$\langle (0,0)(0,10)(1,0)(1,15)(1,18)\rangle \pmod{-, 20}$

$\langle (0,0)(0,3)(1,7)(1,16)\rangle \cup \{h_6, h_7\} \pmod{-, 20}$

$\langle (0,0)(0,1)(1,11)(1,12)\rangle \cup \{h_8, h_9\} \pmod{-, 20}$

$\langle (0,0)(0,1)(0,7)(0,16)\rangle \cup \{h_1, h_2\} \pmod{-, 20}$

$\langle (1,0)(1,1)(1,7)(1,12)\rangle \cup \{h_1, h_2\} \pmod{-, 20}$

$\langle (0,0)(0,3)(0,5)(1,19)\rangle \cup \{h_3, h_4\} \pmod{-, 20}$

$\langle (0,0)(1,0)(1,4)(1,7)\rangle \cup \{h_3, h_4\} \pmod{-, 20}$

$\langle (0,0)(0,7)(1,9)(1,13)h_5\rangle \pmod{-, 20}$

$\langle (0,0)(0,2)(1,1)(1,3)h_6\rangle \pmod{-, 20}$

$\langle (0,0)(0,6)(1,8)(1,18)h_7\rangle \pmod{-, 20}$

$\langle(0,0)(0,8)(1,3)(1,17)h_8\rangle$ (mod $-,20$)
$\langle(0,0)(0,9)(1,6)(1,14)h_9\rangle$ (mod $-,20$)
For $\nu = 43$ and $\lambda = 6$ the construction is as follows:

1) Take a (43,5,2) minimal covering design [31]. The excess graph of this design consists of 41 isolated vertices and two other vertices, say (42,43) joined by four edges.

2) Take two copies of a (43,5,2) optimal packing design. The complement graph of this design consists of 40 isolated vertices and three other vertices say {41 42 43} the pairs of which are joined by two edges. Hence, by theorem 3.1 there exists a (43,5,6) optimal packing design.

The following is very simple but most useful to us.

Lemma 4.2. *If there exists a* $B[\nu,5,\lambda]$ *and* $\sigma(\nu,5,\lambda') = \psi(\nu,5,\lambda) - e$ *where* e *as described in theorem 1.3 then* $\sigma(\nu,5,\lambda+\lambda') = \psi(\nu,5,\lambda+\lambda') - e$.

5 Packing with Index 7

In this section we distinguish the following cases.

5.1 $\nu \equiv 3$ (mod 20)

Lemma 5.1. $\sigma(\nu,5,7) = \psi(\nu,5,7)$ *for all positive integers* ν, $\nu \equiv 3$ (mod 20). *Furthermore, there exists a (23,5,7) packing design with a hole of size 3.*

Proof: If $\nu \equiv 3$ (mod 20) then $\sigma(\nu,5,7) = \sigma(\nu,5,2) + \sigma(\nu,5,5)$ [5,8].

For a (23,5,7) packing design with a hole of size 3, take a (23,5,4) and (23,5,3) packing design with a hole of size 3, [8], [10].

5.2 $\nu \equiv 7$ (mod 20)

Lemma 5.2. $\sigma(\nu,5,7) = \psi(\nu,5,7)$ *for* $\nu = 7, 27, 47, 67, 87$.

Proof: For $\nu = 7$ let $X = Z_7$ then the required blocks are $\langle 0\ 1\ 3\ 4\ 5\rangle$ (mod 7) $\langle 0\ 1\ 2\ 3\ 6\rangle$ (mod 7). For $\nu = 27, 47, 67, 87$ the construction is as follows:

1) Take a $(\nu,5,4)$ minimal covering design [13], [31]. This design has a triple, say, $\{a,b,c\}$ the pairs of which appear in exactly 6 blocks while each other pair appears in exactly 4 blocks.

2) Take a $(\nu,5,3)$ packing design with a hole of size 3, say, $\{a,b,c\}$.

It is clear that the above two steps yield a $(\nu, 5, 7)$ optimal packing design for $\nu = 27, 47, 67, 87$.

To complete the proof of this lemma we need to construct a $(\nu, 5, 3)$ packing design with a hole of size 3. For $\nu = 27$ see [8], for $\nu = 47, 67, 87$, the constructions are given in the following table. In general, the construction in this table, and all other tables to come, is as follows. Let $X = Z_{\nu-n} \cup H_n$ or $X = Z_2 \times Z_{\frac{\nu-n}{2}} \cup H_n$ where $H_n = \{h_1, \ldots, h_n\}$ is the hole. The blocks are constructed by taking the orbits of the tabulated base blocks.

ν	Point Set	Base Blocks
47	$Z_{44} \cup H_3$	$\langle 0\ 11\ 22\ 33\ h_3\rangle + i, i \in Z_{11}$
		$\langle 0\ 1\ 2\ 4\ 15\rangle \langle 0\ 5\ 10\ 16\ 28\rangle \langle 0\ 7\ 15\ 24\ 32\rangle$
		$\langle 0\ 1\ 3\ 7\ 19\rangle \langle 0\ 4\ 9\ 24\ 30\rangle \langle 0\ 7\ 17\ 30\rangle \cup \{h_1, h_2\}$
		$\langle 0\ 3\ 13\ 22\rangle \cup \{h_1, h_2, h_3, h_3\}$
67	$Z_{64} \cup H_3$	$\langle 0\ 16\ 32\ 48\ h_3\rangle + i, i \in Z_{16}$
		$\langle 0\ 1\ 3\ 7\ 24\rangle \langle 0\ 5\ 14\ 30\ 42\rangle \langle 0\ 8\ 18\ 38\ 53\rangle$
		$\langle 0\ 1\ 3\ 9\ 27\rangle \langle 0\ 5\ 17\ 36\ 49\rangle \langle 0\ 10\ 21\ 35\ 51\rangle$
		$\langle 0\ 1\ 3\ 10\ 52\rangle \langle 0\ 4\ 8\ 28\ 39\rangle \langle 0\ 5\ 23\ 42\rangle \cup \{h_1, h_2\}$
		$\langle 0\ 7\ 21\ 38\rangle \cup \{h_1, h_2, h_3, h_3\}$
87	$Z_{84} \cup H_3$	$\langle 0\ 21\ 42\ 63\ h3\rangle + i, i \in Z_{21}$
		$\langle 0\ 1\ 3\ 7\ 16\rangle$ twice $\langle 0\ 5\ 22\ 45\ 59\rangle \langle 0\ 8\ 27\ 51\ 63\rangle$
		$\langle 0\ 10\ 28\ 48\ 59\rangle \langle 0\ 5\ 24\ 46\ 58\rangle \langle 0\ 8\ 28\ 45\ 63\rangle$
		$\langle 0\ 10\ 33\ 44\ 58\rangle \langle 0\ 1\ 3\ 7\ 52\rangle \langle 0\ 8\ 30\ 50\ 60\rangle$
		$\langle 0\ 13\ 27\ 53\ 68\rangle \langle 0\ 5\ 17\ 28\rangle \cup \{h_1, h_2\}$
		$\langle 0\ 9\ 27\ 46\rangle \cup \{h_1, h_2, h_3, h_3\}$

Lemma 5.3. *$\sigma(\nu, 5, 7) = \psi(\nu, 5, 7)$ for all positive integers $\nu \equiv 7 \pmod{20}$.*

Proof: For $\nu = 7, 27, 47, 67, 87$ the result follows from lemma 5.2. For $\nu \geq 107$, $\nu \neq 127, 147$ simple calculations show that ν can be written in the form $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen so that

1) there exists a RMGD[5,1,5,5m];

2) $4u + h + s \equiv 7 \pmod{20}$ and $7 \leq 4u + h + s \leq 87$;

3) $0 \leq u \leq m - 1$, $s \equiv 0 \pmod 4$ and $h = 3$,

4) there exists a GD$[5, 7, \{4, s^*\}, 4m + s]$.

Now apply theorem 2.9 with $\lambda = 7$ and the result follows.

For $\nu = 127$ applying theorem 2.4 with $\lambda = 7$, $u = 1$, $h = 3$ and $n = 7$ gives a (127,5,7) packing design with a hole of size 7. But $\sigma(7, 5, 7) = \psi(7, 5, 7)$ hence $\sigma(127, 5, 7) = \psi(127, 5, 7)$.

For $\nu = 147$ applying theorem 2.13 with $m = 7$, $s = 0$, $h = 3$ and $\lambda = 3$ gives a (147,5,7) packing design with a hole of size 27. But $\sigma(27,5,7) = \psi(27,5,7)$ hence $\sigma(147,5,7) = \psi(147,5,7)$.

5.3 $\nu \equiv 4 \pmod{20}$

In this section we prove that $\sigma(\nu,5,7) = \psi(\nu,5,7)$ for all positive integers $\nu \geq 24$, $\nu \equiv 4 \pmod{20}$ with the possible exception of $\nu = 44$. But we first treat $\nu = 24, 64, 84$.

Lemma 5.4.

(1) $\sigma(\nu,5,7) = \psi(\nu,5,7)$ *for* $\nu = 24, 64, 84$.

(2) *There exists a (24,5,7) packing design with a hole of size 4.*

Proof: The required constructions are given in the next table. The last construction in the table is the (24,5,7) packing design with a hole of size 4.

Lemma 5.5. $\sigma(\nu,5,7) = \psi(\nu,5,7)$ *for all positive integers* $\nu \equiv 4 \pmod{20}$ *with the possible exception of* $\nu = 44$.

Proof: For $\nu = 24, 64, 84$ the result follows from lemma 5.4. For $\nu \geq 124$ $\nu \neq 144, 184, 224, 284, 304$ simple calculations show that ν can be written in the form $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen so that

1) there exists a RMGD[5,1,5,5m];

2) $4u + h + s \equiv 4 \pmod{20}$, $24 \leq 4u + h + s \leq 84$ and $4u + h + s \neq 44$;

3) $0 \leq u \leq m - 1$, $s \equiv 0 \pmod 4$ and $h = 4$;

4) there exists a $GD[5, 7, \{4, s^*\}, 4m + s]$.

Now apply theorem 2.9 with $\lambda = 7$ and the result follows.

For $\nu = 104, 224, 304$ apply theorem 2.11 with $h = 4$, $\lambda = 7$ and $m = 5, 11, 15$ respectively.

For $\nu = 144$ apply theorem 2.7 with $m = u = 24$, $\lambda = 7$ and $h = 0$.

For $\nu = 184$ apply theorem 2.12 with $m = 8$, $h = 0$, $\lambda = 7$ and $u = 5$ (see next lemma for the existence of a (20,5, 7) optimal packing design).

For $\nu = 284$ apply theorem 2.14 with $m = 13$, $u = 5$ and $h = 4$.

ν	Point Set	Base Blocks
24	Z_{24}	$\langle 0\ 1\ 2\ 3\ 5\rangle\langle 0\ 1\ 6\ 9\ 16\rangle\langle 0\ 2\ 7\ 11\ 18\rangle$
		$\langle 0\ 3\ 9\ 13\ 17\rangle\langle 0\ 1\ 2\ 3\ 12\rangle\langle 0\ 2\ 6\ 11\ 18\rangle$
		$\langle 0\ 3\ 7\ 13\ 19\rangle\langle 0\ 3\ 8\ 13\ 17\rangle$
64	Z_{64}	$\langle 0\ 1\ 3\ 7\ 25\rangle\langle 0\ 5\ 15\ 26\ 45\rangle\langle 0\ 8\ 17\ 31\ 44\rangle$
		$\langle 0\ 1\ 3\ 7\ 19\rangle\langle 0\ 5\ 13\ 33\ 44\rangle\langle 0\ 9\ 23\ 35\ 45\rangle$
		$\langle 0\ 1\ 3\ 7\ 15\rangle\langle 0\ 5\ 16\ 29\ 46\rangle\langle 0\ 9\ 21\ 38\ 48\rangle$
		$\langle 0\ 1\ 3\ 8\ 30\rangle\langle 0\ 4\ 14\ 25\ 51\rangle\langle 0\ 6\ 21\ 30\ 48\rangle$
		$\langle 0\ 1\ 3\ 7\ 23\rangle\langle 0\ 5\ 13\ 31\ 41\rangle\langle 0\ 9\ 20\ 34\ 49\rangle$
		$\langle 0\ 1\ 3\ 7\ 43\rangle\langle 0\ 5\ 19\ 37\ 49\rangle\langle 0\ 8\ 20\ 33\ 49\rangle$
		$\langle 0\ 9\ 19\ 30\ 47\rangle\langle 0\ 1\ 7\ 24\ 33\rangle\langle 0\ 2\ 5\ 18\ 37\rangle$
		$\langle 0\ 4\ 12\ 43\ 54\rangle$
84	$Z_{64} \cup H_{20}$	$\langle 0\ 4\ 16\ 30\ 54\rangle\langle 0\ 1\ 2\ 7\rangle \cup \{h_i\}_{i=1}^{4}\langle 0\ 2\ 11\ 21\rangle \cup \{h_i\}_{i=5}^{8}$
		$\langle 0\ 3\ 17\ 42\rangle \cup \{h_i\}_{i=9}^{12}\langle 0\ 3\ 21\ 34\rangle \cup \{h_i\}_{i=13}^{16}$
		$\langle 0\ 1\ 19\ 30\rangle \cup \{h_i\}_{i=17}^{20}$
		$\langle 0\ 2\ 8\ 20\ h_1\rangle\langle 0\ 4\ 22\ 28\ h_2\rangle\langle 0\ 2\ 10\ 30\ h_3\rangle$
		$\langle 0\ 16\ 22\ 44\ h_4\rangle\langle 0\ 4\ 18\ 30\ h_5\rangle\langle 0\ 8\ 10\ 24\ h_6\rangle$
		$\langle 0\ 4\ 11\ 44\ h_7\rangle\langle 0\ 8\ 19\ 31\ h_8\rangle\langle 0\ 15\ 35\ 42\ h_9\rangle$
		$\langle 0\ 21\ 25\ 38\ h_{10}\rangle\langle 0\ 9\ 26\ 41\ h_{11}\rangle\langle 0\ 3\ 6\ 11\ h_{12}\rangle$
		$\langle 0\ 4\ 9\ 29\ h_{13}\rangle\langle 0\ 7\ 22\ 45\ h_{14}\rangle\langle 0\ 7\ 23\ 40\ h_{15}\rangle$
		$\langle 0\ 9\ 21\ 36\ h_{16}\rangle\langle 0\ 13\ 27\ 40\ h_{17}\rangle\langle 0\ 8\ 29\ 38\ h_{18}\rangle$
		$\langle 0\ 5\ 27\ 39\ h_{19}\rangle\langle 0\ 9\ 23\ 33\ h_{20}\rangle\langle 0\ 1\ 2\ 5 \cup \{h_1, h_2\}$
		$\langle 0\ 5\ 11\ 48\rangle \cup \{h_3, h_4\}\langle 0\ 7\ 17\ 46\rangle \cup \{h_5, h_6\}$
		$\langle 0\ 8\ 27\ 39\rangle \cup \{h_7, h_8\}\langle 0\ 9\ 20\ 41\rangle \cup \{h_9, h_{10}\}$
		$\langle 0\ 13\ 28\ 45\rangle \cup \{h_{11}, h_{12}\}\langle 0\ 1\ 3\ 48\rangle \cup \{h_{13}, h_{14}\}$
		$\langle 0\ 5\ 15\ 28\rangle \cup \{h_{15}, h_{16}\}\langle 0\ 1\ 7\ 36\rangle \cup \{h_{17}, h_{18}\}$
		$\langle 0\ 3\ 16\ 49\rangle \cup \{h_{19}, h_{20}\}$
24	$Z_{20} \cup H_4$	On $Z_{20} \cup H_3$ construct a (23,5,2) packing design with a hole of size 3, say, H_3 and take the following blocks
		$\langle 0\ 5\ 10\ 15\ h_4\rangle + i, i \in Z_5$ twice
		$\langle 0\ 4\ 8\ 12\ 16\rangle + i, i \in Z_4$ twice
		$\langle 0\ 1\ 3\ 15\ h_1\rangle\langle 0\ 1\ 7\ 10\ h_2\rangle\langle 0\ 1\ 9\ 12\ h_3\rangle\langle 0\ 3\ 7\ 14\ h_4\rangle$
		$\langle 0\ 7\ 9\ 14\rangle \cup \{h_i\}_{i=1}^{4}\langle 0\ 1\ 2\ 4\ 6\rangle$

5.4 $\nu \equiv 0, 8, 12$ or 16 (mod 20)

Lemma 5.6. *$\sigma(\nu, 5, 7) = \psi(\nu, 5, 7)$ for all positive integers ν where $\nu \equiv 0, 12,$ or 16 (mod 20) with the possible exception of $\nu = 32$.*

Proof: The blocks of a $(\nu, 5, 7)$ optimal packing design for $\nu \equiv 0, 12$ or

16 (mod 20) $\nu \leq 96$ are those of a $(\nu, 5, 3)$ and $(\nu, 5, 4)$ optimal packing designs. Since a $(\nu, 5, 3)$ optimal packing design is still unknown for $\nu = 20, 32, 56$ the above method does not work for these values. For $\nu = 56$, see the next table.

ν	Point Set	Base Blocks
5	$Z_{48} \cup H_8$	$\langle 0\ 1\ 4\ 10\ 32\rangle$ twice $\langle 0\ 4\ 12\ 20\ 28\rangle\langle 0\ 1\ 5\ 15\ 27\rangle$
		$\langle 0\ 1\ 6\ 34\ 36\rangle\langle 0\ 1\ 3\ 15\ 23\rangle\langle 0\ 2\ 6\ 10\ 19\rangle\langle 0\ 4\ 10\ 24\ 35\rangle$
		$\langle 0\ 1\ 9\ 30\ h_1\rangle\langle 0\ 3\ 18\ 29\ h_2\rangle\langle 0\ 2\ 13\ 23\ h_3\rangle\langle 0\ 7\ 17\ 26\ h_4\rangle$
		$\langle 0\ 5\ 11\ 18\ h_5\rangle\langle 0\ 6\ 17\ 31\ h_6\rangle\langle 0\ 7\ 16\ 23\ h_7\rangle\langle 0\ 3\ 19\ 24\ h_8\rangle$
		$\langle 0\ 1\ 3\ 8\rangle \cup \{h_1, h_2\}\langle 0\ 3\ 12\ 33\rangle \cup \{h_3, h_4\}\langle 0\ 5\ 12\ 23\rangle$ $\cup\{h_5, h_6\}\langle 0\ 5\ 12\ 23\rangle \cup \{h_7, h_8\}\langle 0\ 2\ 15\ 29\rangle \cup \{h_i\}_{i=1}^{4}$ $\langle 0\ 2\ 15\ 29\rangle \cup \{h_i\}_{i=5}^{8}$

For $\nu = 20$ take the blocks of a (20,5,5) minimal covering design [18] together with the blocks of a (20,5,2) optimal packing design. Close observation of these two designs shows that the excess graph of the (20,5,5) minimal covering design is a 1-factor and that the complement graph of the (20,5,2) optimal packing design has a subgraph that is 1-factor. Hence, these two designs give a (20,5,7) optimal packing design.

For $\nu \geq 100$, $\nu \neq 132, 136, 140, 172, 180, 212, 272$, simple calculations show that ν can be written in the form $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen so that

1) there exists a $\text{RMGD}[5, 1, 5, 5m]$;

2) $4u+h+s \equiv 0, 12$ or $16 \pmod{20}$, $12 \leq 4u+h+s \leq 96$, $4u+h+s \neq 32$;

3) $0 \leq u \leq m-1$, $s \equiv 0 \pmod 4$ and $h = 4$ or $h = 0$;

4) there exists a $\text{GD}[5, 7, \{4, s^*\}, 4m + s]$.

Now apply theorem 2.9 with $\lambda = 7$ and the result follows.

For $\nu = 132, 136$ apply 2.4 with $n = 7$, $h = 4$, $\lambda = 7$ and $u = 2$ or 3 respectively.

For $\nu = 140$ apply theorem 2.18 with $n = \lambda = 7$ and $h = 0$.

For $\nu = 172, 180$ apply theorem 2.12 with $m = 8$, $\lambda = 7$, $h = 0$ and $u = 2$ or 4 respectively.

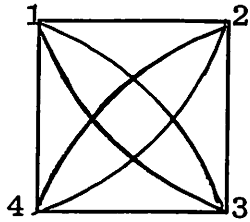
For $\nu = 212$ apply theorem 2.7 with $m = 40$, $u = 12$, $h = 0$ and $\lambda = 7$.

For $\nu = 272$ apply theorem 2.14 with $m = 13$, $h = 0$, $u = 3$ and $\lambda = 7$.

Lemma 5.7. *Let $\nu \equiv 8 \pmod{20}$ be a positive integer. Then $\sigma(\nu, 5, 7) = \psi(\nu, 5, 7)$ with the possible exception of $\nu = 28$.*

Proof: For $\nu = 8, 48, 68, 88$ the construction is as follows:

1) Take a $(\nu, 5, 4)$ minimal covering design with a hole of size 8 [13]. The excess graph of the (8,5,4) covering design consists of 4 isolated vertices and the following graph on the remaining 4 vertices say $\{1, 2, 3, 4\}$. Hence, the $(\nu, 5, 4)$ minimal covering design for $\nu = 8, 48, 68, 88$ consists of $\nu - 4$ isolated vertices and the same graph on the remaining 4 vertices. Furthermore assume we have the block $\langle 1\ 2\ 3\ 4\ 5 \rangle$ where 5 is an arbitrary number. Delete this block.



2) Take a $(\nu, 5, 3)$ optimal packing design [9]. By lemma 3.3 the complement graph of this design is a 1-factor so we may assume that the pairs (1,3) and (2,4) appear at most twice.

It is readily checked that the above two steps yield the blocks of a $(\nu, 5, 7)$ optimal packing design for $\nu = 8, 48, 68, 88$.

For $\nu \geq 108$, $\nu \neq 128, 168, 208, 268$ write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen the same as in lemma 5.6 with the difference that $4u + h + s = 8, 48, 68, 88$. Now apply theorem 2.9 with $\lambda = 7$ to get the result.

For $\nu = 128$ apply theorem 2.4 with $n = \lambda = 7$, $h = 4$ and $u = 1$.

For $\nu = 168$ apply theorem 2.12 with $m = 8$, $\lambda = 7$, $h = 0$ and $u = 1$.

For $\nu = 208$ apply theorem 2.7 with $m = 40$, $u = 0$ and $h = 8$ and notice that a (48,5,7) packing design with a hole of size 8 can be constructed by taking the blocks of a (48,5,4) and a (48,5,3) packing design with a hole of size 8 [9,13].

For $\nu = 268$ apply theorem 2.14 with $m = 13$, $\lambda = 7$, $h = 4$ and $u = 1$.

5.5 $\nu \equiv 2 \pmod 4$

Lemma 5.8. *$\sigma(\nu, 5, 7) = \psi(\nu, 5, 7)$ for all positive integers ν where $\nu \equiv 2$ (mod 4).*

Proof: For $\nu \equiv 2, 6, 10$ or $14 \pmod{20}$ a $(\nu, 5, 7)$ optimal packing design can be constructed by taking the blocks of a $(\nu, 5, 4)$ [10] and $(\nu, 5, 3)$ optimal packing designs [7]. For $\nu \equiv 18 \pmod{20}$, the proof of this case is the same as lemma 5.7. So we need to show that for $\nu \equiv 18 \pmod{20}$ there exists a $(\nu, 5, 4)$ minimal covering design such that its excess graph consists of

$\nu - 4$ isolated vertices, and the same graph of lemma 5.7 on the remaining 4 vertices, say, $\{1, 2, 3, 4\}$. Instead we show that for all $\nu \equiv 18 \pmod{20}$, $\nu \neq 18, 178$, there exists a $(\nu, 5, 4)$ covering design with a hole of size 8 or 13 and since a $(\nu, 5, 4)$, $\nu = 8, 13$, minimal covering design satisfies the above it follows that each $(\nu, 5, 4)$, $\nu \equiv 18 \pmod{20}$, $\nu \neq 18, 178$ minimal covering design satisfies this condition.

We distinguish the following cases.

Case 1. $\nu \equiv 18 \pmod{100}$, $\nu \neq 18$. In this case take a $\mathrm{TT}(6, 1, m, u)$ where $m \equiv 1 \pmod{20}$ and $u = 13$. Replace the blocks of this design by the blocks of B[6, 5, 4] and B[5, 5, 4]. On the first five groups construct a $\mathrm{B}[m, 5, 4]$ and take the last group to be the hole.

Case 2. $\nu \equiv 38 \pmod{100}$. For $\nu = 38$ see [13]. For all other values of ν take a $\mathrm{TT}(6, 1, m, u)$ where $m \equiv 5 \pmod{20}$ and $u = 13$. Then this case is exactly like case 1.

Case 3. $\nu \equiv 58 \pmod{100}$. For $\nu = 58$ see [13]. For all other values of ν take a $\mathrm{TT}(6, 1, m, u)$ where $m \equiv 9 \pmod{20}$ and $u = 12$. Replace the blocks of this design by the blocks of B[6, 5, 4] and B[5, 5, 4]. Add a point to the groups and on the first five groups construct a $\mathrm{B}[m+1, 5, 4]$ and take this point with the last group to be the hole.

Case 4. $\nu \equiv 78 \pmod{100}$, $\nu \neq 178$. For $\nu = 78$ see [14]. For all other values of ν, take a $\mathrm{TT}(6, 1, m, u)$ where $m \equiv 14 \pmod{20}$ and $u = 7$. Then this case is the same as case 3.

Case 5. $\nu \equiv 98 \pmod{100}$. For $\nu = 98$ see [14]. For all other values of ν take a $\mathrm{TT}(6, 1, m, u)$ where $m \equiv 17 \pmod{20}$ and $u = 11$. Replace the blocks of this design by the blocks of B[6, 5, 4] and B[5, 5, 4]. Add two points to the groups and on the first five groups construct a $(m + 2, 5, 4)$ covering design with a hole of size 2, and take these two points with the last group to be the hole.

To complete the proof of lemma 5.8 we need to construct a $(\nu, 5, 7)$ minimal covering design for $\nu = 18, 178$.

For $\nu = 18$ see the table below.

For $\nu = 178$ apply theorem 2.12 with $m = 8$, $h = 2$, $u = 3$ and $\lambda = 7$. Applying this theorem requires the existence of a (22,5,7), (26,5,7) packing design with a hole of size 2 and 6 respectively. Such designs can be constructed by taking a (22,5,3), (26,5,3) [7] and (22,5,4), (26,5,4) packing design with a hole of size 2 and 6 respectively [10]. Notice that a (26,5,4) packing design with a hole of size 6 can be constructed by taking a T[5,4,5]. Add a point to the groups. On the first 4 groups construct a B[6,5,4] and take the last group with this point to be the hole.

ν	Point Set	Base Blocks
18	$Z_{16} \cup H_2$	$\langle 0\ 2\ 8\ 10\rangle \cup \{a,b\}$ half orbit
		$\langle 0\ 1\ 4\ 9\ 12\rangle\langle 0\ 1\ 3\ 5 10\rangle\langle 0\ 1\ 2\ 4\ 6\rangle$
		$\langle 0\ 1\ 2\ 7\ a\rangle\langle 0\ 1\ 4\ 11\ b\rangle\langle 0\ 3\ 7\ 10\rangle \cup \{a,b\}$

5.6 $\nu \equiv 9, 13$ or 17 (mod 20)

Lemma 5.9. *Let $\nu \equiv 9, 13$ or 17 (mod 20) be a positive integer. Then $\sigma(\nu,5,7) = \psi(\nu,5,7)$ if $\nu \equiv 13$ (mod 20) and $\sigma(\nu,5,7) = \psi(\nu,5,7) - 1$ if $\nu \equiv 9$ or 17 (mod 20).*

Proof: For $\nu \equiv 9, 13$ or 17 (mod 20), $\nu \neq 13$, the result follows from lemma 4.2. Since a (13,5,2) optimal packing design does not exist we need to construct a (13,5,7) optimal packing design. For this purpose let $X = Z_2 \times Z_5 \cup H_3$. Then the required blocks are
$\langle(\alpha,0)(\alpha,1)(\alpha,2)(\alpha,3)(\alpha,4)\rangle$, $\alpha = 0,1$ twice;
and the following base blocks mod $(-,5)$
$\langle(0,0)(1,3)h_1h_2h_3\rangle\langle(0,0)(0,1)(0,3)(1,0)h1\rangle$
$\langle(0,0)(1,0)(1,1)(1,2)h_1\rangle\langle(0,0)(0,2)(1,0)(1,3)h_1\rangle$
$\langle(0,0)(0,1)(1,2)(1,3)h_2\rangle\langle(0,0)(0,2)(1,1)(1,4)h_2\rangle$
$\langle(0,0)(0,1)(1,2)(1,4)h_2\rangle\langle(0,0)(0,1)(1,0)(1,1)h_3\rangle$
$\langle(0,0)(0,2)(1,0)(1,1)h_3\rangle\langle(0,0)(0,1)(1,0)(1,3)h_3\rangle$.

Corollary 5.1. *Let $\nu \equiv 9, 13$ or 17 (mod 20) and $\lambda \geq 7$ be positive integers. Then $\sigma(\nu,5,\lambda) = \psi(\nu,5,\lambda) - e$, where e is as described in theorem 1.3.*

Proof: We have shown that for all $\nu \equiv 9, 13$ or 17 (mod 20), $(\nu,\lambda) \neq (13,2)$, $2 \leq \lambda \leq 8$, $\sigma(\nu,5,\lambda) = \psi(\nu,5,\lambda) - e$ holds where e is as above and $\sigma(13,5,2) = \psi(13,5,2) - 1$ with the possible exceptions of $(\nu,\lambda) = (17,3)(29,3)(33,3)$. We also show that $\sigma(13,5,7) = \psi(13,5,7)$. On the other hand for all $\lambda \equiv 0$ (mod 5) and $\nu \equiv 9, 13$ or 17 (mod 20) there exists a B$[\nu,5,\lambda]$. Now apply lemma 4.2 to get the result.

Theorem 5.1. *$\sigma(\nu,5,7) = \psi(\nu,5,7)$ for all positive integers $\nu \geq 5$ with the possible exception of $\nu = 28, 32, 44$.*

Proof: For $\nu \equiv 15$ or 19 (mod 20), $\sigma(\nu,5,7) = \sigma(\nu,5,4) + \sigma(\nu,5,3)$ holds.
For $\nu \equiv 11$ (mod 20) apply lemma 4.2. For all other values of ν the result follows from theorem 2.2 and lemmas 5.1, 5.3, 5.5, 5.6, 5.7, 5.8, 5.9.

6 Packing with Index 9

With lemma 4.2 at our hand we can prove the following.

Lemma 6.1. $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda)$ *for all positive integers* ν, $\nu \equiv 0$ *or* 1 (mod 5) *and for all positive integers* $\lambda \geq 9$.

Proof: By theorem 2.1 there exists a $B[\nu, 5, \lambda]$ for all positive integers λ and ν such that $\lambda \equiv 0 \pmod 4$ and $\nu \equiv 0$ or $1 \pmod 5$. On the other hand $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda)$ for all $\nu \equiv 0$ or $1 \pmod 5$ and $4 \leq \lambda \leq 7$. Hence, $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda)$ for all $\lambda \geq 9$ and $\nu \equiv 0$ or $1 \pmod 5$.

Again in this section we distinguish the following cases.

6.1 $\nu \equiv 4, 8$ or $12 \pmod{20}$

Lemma 6.2. *Let* $\nu \equiv 4, 8$ *or* $12 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 9) = \psi(\nu, 5, 9)$ *with the possible exception of* $\nu = 32$.

Proof: For $\nu \equiv 4$ or $12 \pmod{20}$, $\nu \neq 32$, $\sigma(\nu, 5, 9) = \sigma(\nu, 5, 5) + \sigma(\nu, 5, 4)$ holds.

For $\nu = 8, 48, 68$ and 88, ,$\sigma(\nu, 5, 9) = 3 \cdot \sigma(\nu, 5, 3)$. Since (28,5,3) optimal packing design is still unknown we need to construct a (28,5,9) optimal packing design.

For this purpose let $X = Z_{28}$ then the required blocks can be constructed by developing, under the action of the group Z_{28}, the following base blocks.
⟨0 1 2 4 10⟩ twice ⟨0 3 8 15 19⟩ twice ⟨0 4 10 15 22⟩ twice
⟨0 1 2 3 10⟩⟨0 2 5 14 19⟩⟨0 3 7 14 20⟩ ⟨0 1 2 4 12⟩⟨0 3 8 15 21⟩⟨0 4 9 15 23⟩.
For $\nu \geq 108$, $\nu \neq 128$, write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 5.7 with $h = 0$ and $4u + h + s = 8, 28, 48, 68, 88$.

Now apply theorem 2.9 with $\lambda = 9$ and the result follows. For $\nu = 128$ apply theorem 2.4 with $\lambda = 9$, $u = 2$, $n = 7$ and $h = 0$.

6.2 $\nu \equiv 3 \pmod{20}$

In this case $\sigma(\nu, 5, 9) = \sigma(\nu, 5, 8) + \sigma(\nu, 5, 1)$, $\nu \neq 243$.

For $\nu = 243$ apply theorem 2.9 with $m = 11$, $\lambda = 9$, $h = 3$, $s = 0$ and $u = 5$. See next lemma for a (23,5,9) packing design with a hole of size 3.

6.3 $\nu \equiv 7 \pmod{20}$)

Lemma 6.3. (1) $\sigma(\nu, 5, 9) = \psi(\nu, 5, 9)$ *for* $\nu = 7, 27, 47, 67, 87$. (2) *There exists a (23,5,9) packing design with a hole of size 3.*

Proof: For a (23,5,9) packing design with a hole of size 3, take 3 copies of a (23,5,2) and one copy of a (23,5,3) packing designs with a hole of size 3 [5], [8].

For a (27,5,9) optimal packing design proceed as follows:

1) Take a (27,5,4) minimal covering design [31]. This design has a triple, say, $\{25, 26, 27\}$ the pairs of which appear in six blocks.

2) Take a (27,5,5) optimal packing design which is constructed by taking the blocks of a B[26,5,4] together with the blocks of a (31,5,1) packing design with a hole of size 7 [32]. Assume the hole is $\{25\ 26\ 27\ 28\ 29\ 30\ 31\}$, which we delete, and change 28, 29, 30 and 31 to 27.

The above two steps give a design such that (25,26) appears 10 times, (25,27) and (26,27) appear 6 times and each other pair appears at most 9 times. Now we need to reduce the number of blocks containing (25,26) from 10 to 9. To do that assume we have the block $\langle 1\ 2\ 3\ 25\ 26 \rangle$ in design (1) and $\langle 1\ 2\ 3\ 22\ 27 \rangle$ in design (2), where $\{1, 2, 3\}$ are arbitrary numbers.

In the first block change 26 to 27 and in the second block change 27 to 26. The above step reduces the appearance of (25,26) from 10 to 9 but increases the appearance of (22,26) from 9 to 10. To fix this assume in design (1) we have the block $\langle 4\ 5\ 6\ 22\ 26 \rangle$ and in design (2) we have the block $\langle 4\ 5\ 6\ 27\ 9 \rangle$ where $\{4, 5, 6\}$ are arbitrary numbers. In the first block change 26 to 9 and in the second block change 9 to 26. This step reduces the number of blocks containing (22,26) from 10 to 9 but increase the number of blocks containing (9,22) by 1. Hence, assume that (9,22) appears at most 4 times in the blocks of design (2). Now it is easy to check that the above construction gives a (27,5,9) optimal packing design.

For the other constructions see the next table.

ν	Point Set	Base Blocks
7	$Z_6 \cup H_1$	$\langle 0\ 1\ 2\ 3\ 4\rangle\langle 0\ 1\ 2\ 4\ h_1\rangle\langle 0\ 1\ 2\ 3\ h_1\rangle$
47	$Z_{40} \cup H_7$	On $Z_{40} \cup H_7$ construct a (47,5,2) packing design with a hole of size 7 [5]. Take 3 copies of this design and take the following blocks.
		$\langle 0\ 8\ 16\ 24\ 32\rangle + i, i \in Z_8$, three times
		$\langle 0\ 10\ 20\ 30\ h_7\rangle + i, i \in Z_{10}$
		$\langle 0\ 1\ 2\ 4\ 7\rangle\langle 0\ 3\ 12\ 18\ 30\rangle\langle 0\ 5\ 14\ 25\rangle \cup \{h_1, h_2\}$
		$\langle 0\ 4\ 15\ 21\rangle \cup \{h_3, h_4\}\langle 0\ 4\ 13\ 27\rangle \cup \{h_5, h_6\}\langle 0\ 2\ 7\ 21\rangle$
		$\cup\{h_i\}_{i=1}^{4}\langle 0\ 1\ 11\ 18\rangle \cup \{h_5, h_6, h_7, h_7\}$
67	$Z_{60} \cup H_7$	On $Z_{60} \cup H_7$ construct a (67,5,2) packing design with a hole of size 7 [5]. Take 3 copies of this design and take the following blocks
		$\langle 0\ 12\ 24\ 36\ 48\rangle + i, i \in Z_{12}$, three times
		$\langle 0\ 15\ 30\ 45\ h_7\rangle + i, i \in Z_{15}$
		$\langle 0\ 1\ 3\ 8\ 35\rangle\langle 0\ 4\ 13\ 23\ 44\rangle\langle 0\ 6\ 17\ 28\ 46\rangle\langle 0\ 1\ 3\ 7\ 47\rangle$
		$\langle 0\ 6\ 16\ 34\ 43\rangle\langle 0\ 5\ 19\ 30\rangle \cup \{h_1, h_2\}\langle 0\ 1\ 4\ 26\rangle \cup \{h_3, h_4\}$
		$\langle 0\ 8\ 21\ 29\rangle \cup \{h_5, h_6\}\langle 0\ 5\ 27\ 42\rangle \cup \{h_i\}_{i=1}^{4}$
		$\langle 0\ 2\ 9\ 19\rangle \cup \{h_5, h_6, h_7, h_7\}$
87	$Z_{80} \cup H_7$	On $Z_{80} \cup H_7$ construct a (87,5,2) packing design with a hole of size 7 [5]. Take 3 copies of this design and take the following blocks
		$\langle 0\ 16\ 32\ 48\ 64\rangle + i, i \in Z_{16}$, three times
		$\langle 0\ 20\ 40\ 60\ h_7\rangle + i, i \in Z_{20}$
		$\langle 0\ 1\ 5\ 13\ 51\rangle\langle 0\ 3\ 21\ 28\ 47\rangle\langle 0\ 6\ 17\ 41\ 63\rangle\langle 0\ 1\ 3\ 11\ 15\rangle$
		$\langle 0\ 1\ 7\ 9\ 59\rangle\langle 0\ 4\ 19\ 37\ 57\rangle\langle 0\ 10\ 24\ 36\ 49\rangle\langle 0\ 3\ 5\ 34\ 45\rangle$
		$\langle 0\ 5\ 18\ 33\rangle \cup \{h_1, h_2\}\langle 0\ 7\ 24\ 43\rangle \cup \{h_3, h_4\}\langle 0\ 9\ 29\ 54\rangle$
		$\cup\{h_5, h_6\}\langle 0\ 9\ 23\ 50\rangle \cup \{h_i\}_{i=1}^{4}\langle 0\ 21\ 27\ 58\rangle$
		$\cup\{h_5 h_6 h_7 h_7\}$

Lemma 6.4. *Let $\nu \equiv 7 \pmod{20}$ be a positive integer. Then $\sigma(\nu, 5, 9) = \psi(\nu, 5, 9)$.*

Proof: For $\nu = 7, 27, 47, 67, 87$ the result follows from lemma 6.3. For $\nu \geq 107$, $\nu \neq 127, 147$ simple calculations show that ν can be written in the form $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 5.3.

Now apply theorem 2.9 with $\lambda = 9$ and the result follows.

For $\nu = 127, 147$ apply theorem 2.4 with $u = 1$, $h = 3$, $\lambda = 9$ and $n = 7, 8$ respectively.

6.4 $\nu \equiv 19 \pmod{20}$

The values under 100 are treated individually.

Lemma 6.5. *$\sigma(\nu,5,9) = \psi(\nu,5,9)$ for $\nu = 19, 39, 59, 79, 99$.*

Proof: The required constructions are given in the next table.

ν	Point Set	Base Blocks
19	Z_{19}	⟨0 1 2 3 6⟩⟨0 1 5 8 13⟩⟨0 2 6 9 13⟩⟨0 2 6 11 14⟩
		⟨0 1 3 10 12⟩⟨0 1 2 3 7⟩⟨0 1 4 9 14⟩⟨0 2 6 10 13⟩
39	Z_{39}	⟨0 1 3 5 15⟩ twice ⟨0 3 12 22 28⟩ twice ⟨0 6 13 21 32⟩ twice
		⟨0 1 5 9 17⟩⟨0 1 2 5 21⟩⟨0 5 11 21 29⟩⟨0 5 13 21 30⟩
		⟨0 1 2 4 15⟩⟨0 3 10 16 22⟩⟨0 4 11 18 23⟩⟨0 1 2 4 13⟩
		⟨0 3 14 18 24⟩⟨0 6 13 22 30⟩⟨0 5 10 19 27⟩
59	Z_{59}	⟨0 1 3 7 26⟩ twice ⟨0 5 15 35 46⟩ twice ⟨0 8 17 29 45⟩ twice
		⟨0 1 3 8 29⟩ twice ⟨0 4 13 29 48⟩ twice ⟨0 6 18 28 45⟩ twice
		⟨0 4 15 27 42⟩⟨0 1 4 9 41⟩⟨0 2 14 25 38⟩⟨0 6 22 32 39⟩
		⟨0 1 3 9 19⟩⟨0 1 3 7 21⟩⟨0 5 17 27 40⟩⟨0 8 23 34 43⟩
		⟨0 1 3 15 21⟩⟨0 5 18 37 44⟩⟨0 9 21 34 45⟩⟨0 1 3 7 29⟩
		⟨0 5 13 23 40⟩⟨0 4 11 20 25⟩
79	Z_{79}	⟨0 1 3 7 23⟩ twice ⟨0 5 13 41 53⟩ twice ⟨0 9 30 41 55⟩ twice
		⟨0 10 27 45 60⟩ twice ⟨0 1 3 7 20⟩ twice ⟨0 5 21 44 55⟩ twice
		⟨0 8 18 40 49⟩ twice ⟨0 12 26 39 54⟩⟨0 1 3 7 25⟩ twice
		⟨0 1 5 21 28⟩⟨0 2 28 40 44⟩⟨0 8 18 37 52⟩⟨0 9 20 42 56⟩
		⟨0 5 13 33 49⟩⟨0 9 19 45 56⟩⟨0 12 27 48 62⟩⟨0 1 3 7 22⟩
		⟨0 5 28 38 52⟩⟨0 8 17 42 53⟩⟨0 12 30 43 59⟩⟨0 1 3 9 27⟩
		⟨0 4 14 29 62⟩⟨0 5 16 28 35⟩⟨0 5 16 33 43⟩⟨0 8 23 37 57⟩
		⟨0 9 21 40 53⟩⟨0 5 17 2454⟩
99	$Z_{80} \cup H_{19}$	On $Z_{80} \cup H_{17}$ construct a B[97,5,5] with a hole of size 17 say H_{17}. Such design can be constructed from a T[6,5,16] with an extra point added to the groups. On the first 5 groups construct a B[17,5,5] and take the last group with the points to be the hole. Take the following blocks.
		⟨0 16 32 48 64⟩ $+ i, i \in Z_{16}$, 4 times
		⟨0 10 40 50$f(\kappa)$⟩ half orbit, where $f(\kappa) = h_{18}$
		if $\kappa \equiv 0$ or $1 \pmod 4$ and $f(\kappa) = h_{19}$ otherwise
		⟨0 8 20 44 58⟩⟨0 4 22 34 37⟩⟨0 2 22 28h_{18}⟩⟨0 4 18 42 h_{19}⟩
		⟨0 1 6 9 h_1⟩⟨0 7 17 38 h_2⟩⟨0 11 30 57 h_3⟩⟨0 12 31 57 h_4⟩
		⟨0 13 33 58 h_5⟩⟨0 14 29 53 h_6⟩⟨0 1 2 5 h_7⟩⟨0 2 7 13 h_8⟩
		⟨0 7 17 36 h_9⟩⟨0 8 21 47 h_{10}⟩⟨0 9 37 46 h_{11}⟩⟨0 10 35 49 h_{12}⟩
		⟨0 11 28 55 h_{13}⟩⟨0 1 3 8 h_{14}⟩⟨0 4 13 31 h_{15}⟩⟨0 6 23 57 h_{16}⟩
		⟨0 11 29 54 h_{17}⟩⟨0 12 33 52 h_{18}⟩⟨0 15 35 56 h_{19}⟩

Lemma 6.6. *Let $\nu \equiv 19 \pmod{20}$ be a positive integer. Then $\sigma(\nu,5,9) = \psi(\nu,5,9)$.*

Proof: For $\nu = 19, 39, 59, 79, 99$ the result follows from lemma 6.5. For $\nu \geq 119$, $\nu \neq 139, 179$ simple calculations show that ν can be written in the form $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 5.3 with the difference that $4u + h + s = 19, 39, 59, 79, 99$.

Now apply theorem 2.9 with $\lambda = 9$ and the result follows.

For $\nu = 139$, apply theorem 2.4 with $u = 4$, $h = 3$, $\lambda = 9$ and $n = 7$.

For $\nu = 179$ apply theorem 2.13 with $m = 8$, $s = 8$, $\lambda = 9$ and $h = 3$.

6.5 $\nu \equiv 2 \pmod{20}$

In this case $\sigma(\nu, 5, 9) = \sigma(\nu, 5, 5) + \sigma(\nu, 5, 4)$.

6.6 $\nu \equiv 14 \pmod{20}$

Lemma 6.7. *There exists a (74,5,3) covering design with a hole of size 14.*

Proof: Let $X = Z_2 \times Z_{30} \cup H_{14}$. Then the blocks of the required design can be constructed as follows:

1) On $Z_2 \times Z_{30} \cup H_{13}$ construct a (73,5,1) covering design with a hole of size 13, say, H_{13}, [27].

2) take the following base blocks, the first two as indicated and the remaining are mod $(-, 30)$

$\langle(0,0)(0,15)(1,0)(1,15)h_{14}\rangle + (-, i), i \in Z_{15}$
$\langle(0,0)(0,6)(0,12)(0,18)(0,24)\rangle + (-, i), i \in Z_6$
$\langle(0,0)(0,2)(0,10)(0,13)(0,14)\rangle\langle(1,0)(1,2)(1,6)(1,14)(1,17)\rangle$
$\langle(0,0)(0,5)(1,1)(1,4)(1,24)\rangle\langle(0,0)(0,7)(0,17)(1,29)\rangle \cup \{h_1, h_2\}$
$\langle(0,0)(1,1)(1,8)(1,18)\rangle \cup \{h_1, h_2\}\langle(0,0)(0,3)(0,9)(1,23)\rangle \cup \{h_3, h_4\}$
$\langle(0,0)(1,2)(1,7)(1,16)\rangle \cup \{h_3, h_4\}\langle(0,0)(0,1)(1,3)(1,5)h_5\rangle$
$\langle(0,0)(0,2)(1,9)(1,28)h_6\rangle\langle(0,0)(0,4)(1,10)(1,15)h_7\rangle$
$\langle(0,0)(0,5)(1,23)(1,27)h_8\rangle\langle(0,0)(0,7)(1,13)(1,19)h_9\rangle$
$\langle(0,0)(0,8)(1,3)(1,21)h_{10}\rangle\langle(0,0)(0,9)(1,17)(1,20)h_{11}\rangle$
$\langle(0,0)(0,11)(1,27)(1,28)h_{12}\rangle\langle(0,0)(0,14)(1,5)(1,14)h_{13}\rangle$
$\langle(0,0)(0,15)(1,9)(1,10)h_{14}\rangle$.

Lemma 6.8. (a) $\sigma(\nu, 5, 9) = \psi(\nu, 5, 9)$ *for* $\nu = 14, 34, 54, 74, 94$. (b) *there exists a (26,5,9) packing design with a hole of size 6.*

Proof: (a) For $\nu = 14$ proceed as follows:

1) Take a (14,5,3) minimal covering design [9]. The repeated pairs of this design form one factor. We may rearrange the points of the design so that the repeated pairs are (1,8) (5,11) (2,6) (3,7) (10,12) (9,14)

(4,13) and so that we have the block ⟨3 5 8 4 13⟩. In this block change 13 to 9.

2) Take a (14,5,6) optimal packing design [11]. There are 16 missing pairs in this design and they are (1,5) (1,6) (2,6) (2,10) (3,4) (3,7) (4,9) (5,11) (7,12) (9,14) (10,12) (11,14) and (1,8) (1,13) each missing twice. Furthermore, in this design we have the block ⟨1 3 5 8 9⟩. In this block change 9 to 13. Now it is easy to check that these two steps yield a (14,5,9) optimal packing design.

For $\nu = 74, 94$ the construction is as follows:

1) Take a $(\nu, 5, 3)$ packing design with a hole of size 14, [7]. The missing pairs form three 1-factor on the $\nu - 14$ points.

2) Take two copies of a $(\nu, 5, 3)$ covering design with a hole of size 14, for $\nu = 94$ see [9]. In each copy the repeated pairs form a 1-factor on the $\nu - 14$ points.

Now apply theorem 3.2 to get the result.

For $\nu = 34, 54$ see next table.

(b) For a (26,5,9) packing design with a hole of size 6 take a (26,5,4) (lemma 5.8) and a (26,5,5) packing design with a hole of size 6, [8].

ν	Point Set	Base Blocks
34	$Z_{28} \cup H_6$	⟨0 5 14 19⟩ $\cup \{h_5, h_6\}$ half orbit
		⟨0 1 3 5 9⟩⟨0 2 7 13 21⟩⟨0 2 10 14 20⟩⟨0 4 10 16 20⟩
		⟨0 1 11 13 h_1⟩⟨0 1 2 3 h_2⟩⟨0 3 6 11 h_3⟩⟨0 3 7 18 h_4⟩
		⟨0 4 11 19 h_5⟩⟨0 5 12 21 h_6⟩⟨0 2 15 21 h_1⟩⟨0 1 2 5 h_2⟩
		⟨0 1 4 11 h_3⟩⟨0 1 12 17 h_4⟩⟨0 3 9 18 h_5⟩⟨0 5 13 20 h_6⟩
		⟨0 3 9 14⟩ $\cup \{h_i\}_{i=1}^4$
54	$Z_{48} \cup H_6$	⟨0 13 24 37⟩ $\cup \{h_5, h_6\}$
		⟨0 1 3 8 21⟩ twice ⟨0 4 16 26 36⟩ twice ⟨0 6 14 25 39⟩
		twice⟨0 1 3 5 22⟩⟨0 4 14 20 32⟩⟨0 6 13 31 39⟩
		⟨0 1 3 7 17⟩⟨0 4 15 28 33⟩⟨0 6 18 27 35⟩⟨0 1 3 8 17⟩
		⟨0 7 19 30 h_1⟩⟨0 6 22 30 h_2⟩⟨0 1 2 5 h_3⟩⟨0 3 8 31 h_4⟩
		⟨0 6 13 39 h_5⟩⟨0 7 19 34 h_6⟩⟨0 9 19 30 h_1⟩⟨0 1 2 5 h_2⟩
		⟨0 2 5 39 h_3⟩⟨0 4 17 29 h_4⟩⟨0 7 17 28 h_5⟩⟨0 9 22 33 h_6⟩
		⟨0 6 21 31⟩ $\cup \{h_i\}_{i=1}^4$

Lemma 6.9. *Let $\nu \equiv 14 \pmod{20}$ be a positive integer. Then $\sigma(\nu, 5, 9) = \psi(\nu, 5, 9)$.*

Proof: For $\nu = 14, 34, 54, 74, 94$ the result follows from lemma 6.8.

For $\nu \geq 114$, $\nu \neq 134$, simple calculations show that ν can be written in the form $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen so that:

1) there exists a RMGD$[5, 1, 5, 5m]$;
2) $4u + h + s \equiv 14 \pmod{20}$, $14 \leq 4u + h + s \leq 94$;
3) $0 \leq u \leq m - 1$, $s \equiv 0 \pmod 4$, $h = 6$;
4) there exists a GD$[5, 9, \{4, s^*\}, 4m + s]$.

Now apply theorem 2.9 with $\lambda = 9$ and the result follows.

For $\nu = 134$ apply theorem 2.4 with $u = 2$, $h = 6$, $\lambda = 9$ and $n = 7$.

6.7 $\nu \equiv 18 \pmod{20}$

Lemma 6.10. $\sigma(\nu, 5, 9) = \psi(\nu, 5, 9)$ *for* $\nu = 18, 38, 58, 78, 98$.

Furthermore, these designs have a hole of size 2.

Proof: For $\nu = 18, 38, 58, 78$ the constructions are given in the next table. In the construction of a (78,5,9) optimal packing design we use a B[76,5,4] with a hole of size 16. Such design can be constructed from a T[5,4,15] by adding one point to the groups. On the first 4 groups construct a B[16,5,4] and take the last group with the point to be the hole.

For $\nu = 98$ apply theorem 2.7 with $m = u = 16$, $\lambda = 9$ and $h = 2$.

Lemma 6.11. *Let* $\nu \equiv 18 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 9) = \psi(\nu, 5, 9)$.

Proof: For $\nu = 18, 38, 58, 78, 98$ the result follows from lemma 6.10. For $\nu \geq 118$, $\nu \neq 138$, simple calculations show that ν can be written in the form $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 6.9 with one difference that $4u + h + s = 18, 38, 58, 78, 98$.

Now apply theorem 2.9 with $\lambda = 9$ and the result follows.

For $\nu = 138$ apply theorem 2.4 with $\lambda = 9$, $u = 3$, $h = 6$ and $n = 7$.

Theorem 6.1. $\sigma(\nu, 5, 9) = \psi(\nu, 5, 9)$ *for all positive integers* $\nu \geq 5$ *with the possible exception of* $\nu = 32$.

ν	Point Set	Base Blocks
18	$Z_{16} \cup H_2$	$\langle 0\ 1\ 8\ 9\rangle \cup \{h_1, h_2\}$ half orbit
		$\langle 0\ 1\ 3\ 7\ 95\rangle$ twice $\langle 0\ 1\ 2\ 4\ 6\rangle\langle 0\ 1\ 2\ 3\ 6\rangle\langle 0\ 3\ 6\ 11\ h_1\rangle$
		$\langle 0\ 3\ 7\ 12\ h_2\rangle\langle 0\ 1\ 6\ 11\ h_1\rangle\langle 0\ 3\ 7\ 12\ h_2\rangle$
38	$Z_2 \times Z_{18} \cup H_2$	On $Z_2 \times Z_{18}$ construct a B[36,5,4] and take the following blocks
		$\langle(\alpha,0)(\alpha,1)(\alpha,3)(\alpha,7)(\alpha,12)\rangle, \alpha = 0,1.$
		$\langle(0,0)(0,2)(0,4)(0,8)(1,0)\rangle\langle(0,0)(0,1)(0,3)(0,8)(1,0)\rangle$
		$\langle(0,0)(0,1)(1,2)(1,3)(1,6)\rangle\langle(0,0)(0,3)(1,1)(1,7)(1,12)\rangle$
		$\langle(0,0)(0,4)(1,7)(1,15)(1,17)\rangle\langle(0,0)(1,0)(1,1)(1,4)(1,10)\rangle$
		$\langle(0,0)(0,6)(0,7)(1,11)(1,13)\rangle\langle(0,0)(0,6)(1,2)(1,12)(1,15)\rangle$
		$\langle(0,0)(0,3)(0,8)(1,11)(1,16)\rangle\langle(0,0)(0,9)(1,3)(1,5)(1,17)\rangle$
		$\langle(0,0)(0,1)(1,16)(1,17)h_1\rangle\langle(0,0)(0,2)(1,14)(1,17)h_1\rangle$
		$\langle(0,0)(0,3)(1,9)(1,14)h_1\rangle\langle(0,0)(0,4)(1,8)(1,9)h_1\rangle$
		$\langle(0,0)(0,5)(1,7)(1,9)h_2\rangle\langle(0,0)(0,6)(1,7)(1,11)h_2\rangle$
		$\langle(0,0)(0,7)(1,2)(1,13)h_2\rangle\langle(0,0)(0,8)(1,0)(1,8)h_2\rangle$
		$\langle(0,0)(0,9)(1,1)(1,12)\rangle \cup \{h_1, h_2\}$
58	$Z_{56} \cup H_2$	On Z_{56} construct a B[56,5,4] and take the following blocks
		$\langle 0\ 9\ 28\ 37\rangle \cup \{h_1, h_2\}$, half orbit
		$\langle 0\ 1\ 3\ 13\ 28\rangle\langle 0\ 4\ 18\ 29\ 34\rangle\langle 0\ 6\ 23\ 30\ 38\rangle\langle 0\ 4\ 5\ 21\ 40\rangle$
		$\langle 0\ 1\ 3\ 6\ 14\rangle\langle 0\ 4\ 13\ 34\ 46\rangle\langle 0\ 7\ 16\ 27\ 41\rangle\langle 0\ 1\ 3\ 8\ 42\rangle$
		$\langle 0\ 4\ 13\ 23\ 43\rangle\langle 0\ 6\ 16\ 27\ 38\rangle\langle 0\ 1\ 19\ 21\ 25\rangle\langle 0\ 6\ 18\ 30\ h_1\rangle$
		$\langle 0\ 2\ 5\ 49\ h_1\rangle\langle 0\ 7\ 17\ 36 h_2\rangle\langle 0\ 8\ 23\ 31\ h_2\rangle$
78	$Z_{60} \cup H_{18}$	On $Z_{60} \cup H_{16}$ construct a B[76,5,4] with a hole of size 16 and take the following blocks.
		$\langle 0\ 11\ 30\ 41\rangle \cup \{h_{17}, h_{18}\}$ half orbit.
		$\langle 0\ 5\ 22\ 40\ h_1\rangle\langle 0\ 9\ 23\ 41\ h_2\rangle\langle 0\ 10\ 26\ 39\ h_3\rangle\langle 0\ 11\ 26\ 38\ h_4\rangle$
		$\langle 0\ 1\ 3\ 10\ h_5\rangle\langle 0\ 4\ 22\ 39\ h_6\rangle\langle 0\ 6\ 26\ 37\ h_7\rangle\langle 0\ 8\ 24\ 36\ h_8\rangle$
		$\langle 0\ 4\ 24\ 36\ h_9\rangle\langle 0\ 8\ 17\ 33\ h_{10}\rangle\langle 0\ 8\ 23\ 39\ h_{11}\rangle\langle 0\ 8\ 28\ 55\ h_{12}\rangle$
		$\langle 0\ 12\ 26\ 45\ h_{13}\rangle\langle 0\ 5\ 14\ 38\ h_{14}\rangle\langle 0\ 8\ 25\ 40\ h_{15}\rangle$
		$\langle 0\ 10\ 23\ 44\ h_{16}\rangle\langle 0\ 1\ 3\ 7\ h_{17}\rangle$ twice $\langle 0\ 1\ 3\ 7\ h_{18}\rangle$
		$\langle 0\ 11\ 29\ 41\ h_{18}\rangle\langle 0\ 3\ 17\ 38\rangle \cup \{h_i\}_{i=1}^{4}\langle 0\ 1\ 10\ 15\rangle$
		$\cup\{h_i\}_{i=5}^{8}\langle 0\ 11\ 13\ 18\rangle \cup \{h_i\}_{i=9}^{12}\langle 0\ 6\ 19\ 29\rangle$
		$\cup\{h_i\}_{i=13}^{16}$

7 Packing with index 10

7.1 $\nu \equiv 4, 8$ or $12 \pmod{20}$

Lemma 7.1. *Let $\nu \equiv 4 \pmod{20}$, $\nu \geq 24$, be a positive integer. Then $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$.*

Proof: There are several methods to prove this lemma. The following one is the shortest.

1) Take a $(\nu, 5, 4)$ optimal packing design [10]. In this design there is one pair say $(\nu - 1, \nu)$ that appears in zero blocks while each other pair appears in 4 blocks. Assume in this design we have the block $\langle 4\ 5\ 6\ \nu-3\ \nu \rangle$ where $\{4, 5, 6\}$ are arbitrary numbers. In this block change ν to $\nu - 1$.

2) Take a $(\nu - 2, 5, 2)$ optimal packing design [4].

   Now we take 4 copies of a $B[\nu + 1, 5, 1]$. In two of these copies we change $\nu + 1$ to ν and in the other two we change $\nu + 1$ to $\nu - 1$. This is done as follows.

3) Take a $B[\nu + 1, 5, 1]$ and delete the block $\langle \nu-3\ \nu-2\ \nu-1\ \nu\ \nu+1 \rangle$ and in all other blocks change $\nu + 1$ to ν.

4) Take a $B[\nu + 1, 5, 1]$ and assume we have the block $\langle 1\ 2\ 3\ \nu\ \nu+1 \rangle$ where $\{1, 2, 3\}$ are arbitrary numbers. In this block change $\nu + 1$ to $\nu - 1$ and in all other blocks change $\nu + 1$ to ν.

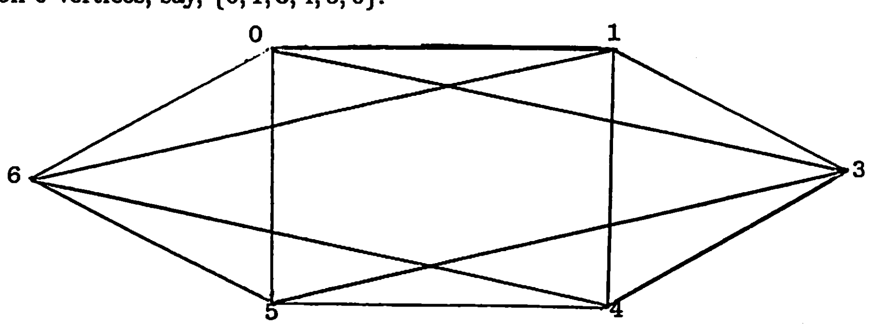
5) Take a $B[\nu+1, 5, 1]$ and assume we have the block $\langle 1\ 2\ 3\ \nu-1\ \nu+1 \rangle$. In this block change $\nu + 1$ to ν and in all other blocks change $\nu + 1$ to $\nu - 1$.

6) Take a $B[\nu+1, 5, 1]$ and assume we have the block $\langle 4\ 5\ 6\ \nu-1\ \nu+1 \rangle$. In this block change $\nu + 1$ to ν and in all other blocks change $\nu + 1$ to $\nu - 1$.

Now it is easy to check that the above six steps yield a $(\nu, 5, 10)$ optimal packing design for all $\nu \geq 24$, $\nu \equiv 4 \pmod{20}$.

Lemma 7.2. *Let $\nu \equiv 8 \pmod{20}$ be a positive integer. If there exists a $(\nu, 5, 4)$ packing design with a hole of size 8 then $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$.*

Proof: The complement graph of a $(\nu, 5, 4)$ packing design with $\psi(\nu, 5, 4) - 1$ blocks, $\nu \equiv 8 \pmod{20}$, consists of 12 edges, [10]. In the case $\nu = 8$ the complement graph consists of 2 isolated vertices and the following graph on 6 vertices, say, $\{0, 1, 3, 4, 5, 6\}$.



So if we have a $(\nu, 5, 4)$ packing design with a hole of size 8 then the complement graph of the $(\nu, 5, 4)$ packing design consists of $\nu - 6$ isolated vertices and the same graph on the other 6 vertices. In this case a $(\nu, 5, 10)$ optimal packing design, $\nu \equiv 8 \pmod{20}$ can be constructed as follows:

1) Take a $(\nu, 5, 4)$ packing design which satisfies the above.
2) Take two copies of a $(\nu, 5, 3)$ optimal packing design, [9], and assume that the pairs (1,5) and (3,6) each appears at most 5 times in the two copies of a $(\nu, 5, 3)$ optimal packing design.
3) Add the block $\langle 1\ 3\ 4\ 5\ 6 \rangle$

Corollary 7.1. $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$ *for* $\nu = 8, 48, 68, 88$.

Proof: In view of lemma 7.2 all we need to do is to show that there exists a $(\nu, 5, 4)$ packing design with a hole of size 8. For $\nu = 48, 68$ see [13]. For $\nu = 88$ take a TT(6,4,16,8). On the groups of size 16, and on the blocks construct a $B[\nu, 5, 4]$ where $\nu = 16, 6, 5$. Take the last group to be the hole. Since a $(\nu, 5, 3)$ optimal packing design exists for $\nu = 8, 48, 68, 88$, it follows that $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$ for $\nu = 8, 48, 68, 88$.

Lemma 7.3. *Let* $\nu \equiv 8 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$.

Proof: For $\nu = 28$ the construction is as follows:

1) take a (27,5,4) minimal covering design [31]. The excess graph of this design consists of 24 isolated vertices and 3 other vertices, say, $\{1, 2, 3\}$, the pairs of which are joined by two edges.
2) take a (29,5,4) optimal packing design [10]. In this design there is one pair, say, (28,29) that appears in zero blocks, so change 29 to 28.
3) take a (28,5,2) packing design with a hole of size 4, say, $\{1, 2, 3, 4\}$ [4].

For $\nu = 8, 48, 68, 88$ the result follows from the corollary.

For $\nu \geq 108$, $\nu \neq 128$, write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 5.3 with the difference that $4u + h + s = 8, 18, 48, 68, 88$ and $h = 0$.

Now apply theorem 2.9 with $\lambda = 10$ and the result follows.

For $\nu = 128$ apply theorem 2.4 with $u = 2$, $\lambda = 10$, $h = 0$ and $n = 7$.

Lemma 7.4. (a) *Let* $\nu \equiv 2 \pmod{10}$ *be a positive integer. Then* $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$. (b) *There exists a (22,5,10) packing design with a hole of size 2.*

Proof: For (a) notice that $\sigma(\nu, 5, 10) = \sigma(\nu, 5, 8) + \sigma(\nu, 5, 2)$.

For a (22,5,10) packing design with a hole of size 2 take a (22,5,2) and two copies of a (22,5,4) packing design with a hole of size 2.

7.2 $\nu \equiv 14$ or 18 (mod 20)

Lemma 7.5. *$\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$ for $\nu = 14, 34, 54, 74, 94$.*

Proof: For $\nu = 14$ the construction is as follows:

1) Take a (14,5,4) minimal covering design [13]. This design has a triple, say, $\{12, 13, 14\}$ the pairs of which appear in precisely 6 blocks while each other pair appears in precisely 4 blocks.

2) Take a (14,5,2) optimal packing design [4]. Careful inspection of this design shows that the pairs $\{(0,0),(1,1)\}$ and $\{(1,1),(1,3)\}$ appear in zero blocks. We may change the points of this design so that the pairs (12,13) and (12,14) appear in zero blocks of the (14,5,2) optimal packing design.

3) Take a (14,5,4) optimal packing design. Each pair of this design appears precisely in 4 blocks except one pair, say, $\{13, 14\}$ that appears in zero blocks. It is clear that this construction gives a (14,5,10) optimal packing design.

For $\nu = 34, 54, 74, 94$ the construction is as follows:

1) Take a $(\nu, 5, 4)$ minimal covering design [13]. This design has a triple, say, $\{\nu-2, \nu-1, \nu\}$ the pairs of which appear in precisely 6 blocks while each other pair appears in precisely 4 blocks.

2) Take a $(\nu, 5, 4)$ optimal packing design.

3) Take a $(\nu, 5, 2)$ packing design with a hole of size 4 and assume the hole is $\{\nu-3, \nu-2, \nu-1, \nu\}$. See [4] for the existence of a $(\nu, 5, 2)$, $\nu = 34, 54, 94$, packing design with a hole of size 4.

For a (74,5,2) packing design with a hole of size 4 take a T[5,2,14], add 4 points to the groups and on each group construct an (18,5,2) packing design with a hole of size 4, [4].

It is clear that these three steps yield a $(\nu, 5, 10)$ optimal packing design for $\nu = 34, 54, 74, 94$.

Lemma 7.6. *Let $\nu \equiv 14 \pmod{20}$ be a positive integer. Then $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$.*

Proof: For $\nu = 14, 34, 54, 74, 94$ the result follows from lemma 7.6. For $\nu \geq 114$, $\nu \neq 134$ we noticed that a $(\nu, 5, 2)$ optimal packing design has a hole of size 14; or a hole of size 34, 54, 74, 94 and hence a hole of size 4, [4]. Now invoke the above two constructions given in lemma 7.5 to give us the result.

For $\nu = 134$ apply theorem 2.4 with $m = 7$, $h = 2$, $\lambda = 10$ and $u = 3$.

The case $\nu \equiv 18 \pmod{20}$ is very similar to the previous case. For this purpose the following lemma is very useful.

Lemma 7.7. *There exists a $(\nu, 5, 2)$ packing design with a hole of size 4 for $\nu = 18, 38, 58, 78, 98, 138, 178$.*

Proof: For $\nu = 18, 38, 138, 178$ see [4].

For $\nu = 58, 78, 98$ the constructions are given in the next table. For $\nu = 98$ we actually construct a (98,5,2) packing design with a hole of size 18. But an (18,5,2) packing design with a hole of size 4 exists, [4], hence a (98,5,4) packing design with a hole of size 4 exists.

ν	Point Set	Base Blocks
58	$Z_{54} \cup H_4$	$\langle 0\ 1\ 3\ 9\ 31\rangle\langle 0\ 4\ 18\ 25\ 37\rangle\langle 0\ 1\ 3\ 7\ 41\rangle\langle 0\ 5\ 13\ 31\ 43\rangle$
		$\langle 0\ 5\ 15\ 32\rangle \cup \{h_1 h_2\}\langle 0\ 9\ 19\ 34\rangle \cup \{h_3, h_4\}$
78	$Z_{74} \cup H_4$	$\langle 0\ 3\ 11\ 25\ 41\rangle\langle 0\ 2\ 12\ 47\ 54\rangle\langle 0\ 6\ 21\ 30\ 49\rangle\langle 0\ 1\ 3\ 7\ 17\rangle$
		$\langle 0\ 8\ 19\ 34\ 62\rangle\langle 0\ 1\ 5\ 18\ 41\rangle\langle 0\ 5\ 29\ 42\rangle \cup \{h_1, h_2\}$
		$\langle 0\ 9\ 27\ 48\rangle \cup \{h_1, h_2\}$
98	$Z_{80} \cup H_{18}$	$\langle 0\ 16\ 32\ 48\ 64\rangle + i, i \in Z_{16}$, twice
		$\langle 0\ 4\ 12\ 26\ 50\rangle\langle 0\ 1\ 3\ 13\ 39\rangle\langle 0\ 5\ 24\ 55\rangle \cup \{h_1, h_2\}$
		$\langle 0\ 6\ 33\ 51\rangle \cup \{h_3, h_4\}\langle 0\ 9\ 20\ 37\rangle \cup \{h_5, h_6\}\langle 0\ 1\ 3\ 8\rangle$
		$\cup\{h_7, h_8\}\langle 0\ 4\ 11\ 25\rangle \cup \{h_9, h_{10}\}\langle 0\ 6\ 15\ 35\rangle$
		$\cup\{h_{11}, h_{12}\}\langle 0\ 10\ 27\ 49\rangle \cup \{h_{13}, h_{14}\}$
		$\langle 0\ 13\ 34\ 57\rangle \cup \{h_{15}, h_{16}\}\langle 0\ 15\ 33\ 52\rangle \cup \{h_{17}, h_{18}\}$

Lemma 7.8. $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$ *for* $\nu = 18, 38, 58, 78, 98, 138, 158$.

Proof: The construction of these optimal packing designs is as follows:

1) Take a $(\nu - 1, 5, 4)$ minimal covering design. This design has a triple, say, $\{\nu-3, \nu-2, \nu-1\}$ the pairs of which appear in precisely 6 blocks while each other pair appears in precisely 4 blocks [12], [13].

2) Take a $(\nu + 1, 5, 4)$ optimal packing design. This design has a pair, say, $(\nu, \nu + 1)$, that appears in zero blocks. Change $\nu + 1$ to ν.

3) Take a $(\nu, 5, 2)$ optimal packing design with a hole of size 4, say, $\{\nu - 3, \nu - 2, \nu - 1, \nu\}$.

Lemma 7.9. *Let $\nu \equiv 18 \pmod{20}$ be a positive integer. Then $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$.*

Proof: For $\nu = 18, 38, 58, 78, 98, 138, 178$ the result is given in lemma 7.9. For $\nu \geq 118$, $\nu \neq 138, 178$ write $\nu = 20m + 4u + h + s$ then the proof of this lemma is the same as lemma 6.11 with the difference that $h = 2$.

7.3 $\nu \equiv 3, 7$ or 19 (mod 20)

Lemma 7.10. *Let* $\nu \equiv 3, 7$ *or* 19 (mod 20) *be a positive integer. Then* $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$. *Furthermore,* $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda)$ *for all positive integers* λ *and* $\nu \equiv 3$ (mod 20) *with the possible exceptions of* $(\nu, \lambda) = (243, 1)$.

Proof: It is clear that if $\nu \equiv 3, 7$ or 19 (mod 20) then there exists a B$[\nu, 5, 10]$ and hence $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$. Since for all $\nu \equiv 3$ (mod 20) $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda) - e$ holds (where e as in theorem 1.3) for all $1 \leq \lambda' \leq 10$ with the possible exceptions of $(\nu, \lambda') = (243, 1)$, it follows that $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda) - e$, where e as before, for all positive integers $\lambda \geq 1$ and $\nu \equiv 3$ (mod 20) with the possible exception of $(\nu, \lambda) = (243, 10m + 1)$ where m is a nonnegative integer. We now construct a (243,5,11) optimal packing design. For a (243,5,11) optimal packing design apply theorem 2.9 with $m = 11$, $\lambda = 10$, $s = 0$, $u = 3$ and $h = 3$. The application of this theorem requires a (23,5,11) packing design with a hole of size 3. Such design can be constructed by taking 4 copies of a (23,5,2) and one copy of a (23,5,3) packing design with a hole of size 3 [5,8]. Now invoke lemma 4.2 to give us the result.

Combining all the results in this section, we have proved the following.

Theorem 7.1. *Let* $\nu \geq 5$ *be a positive integer. Then* $\sigma(\nu, 5, 10) = \psi(\nu, 5, 10)$.

8 Packing with index 11

8.1 $\nu \equiv 4$ (mod 20)

Lemma 8.1. (a) *Let* $\nu \equiv 4$ (mod 20) *be a positive integer. Then* $\sigma(\nu, 5, 11) = \psi(\nu, 5, 11)$. (b) *There exists a (24, 5, 11) packing design with a hole of size 4.*

Proof: (a) If $\nu \equiv 4$ (mod 20) then $\sigma(\nu, 5, 11) = \sigma(\nu, 5, 4) + \sigma(\nu, 5, 7)$.

Since a (44,5,7) optimal packing design is still unknown, we need to construct a (44,5,11) optimal packing design. This is done in the next table. For $\nu = 44$ the construction contains a hole of size 8 so see the next section for an (8,5,11) optimal packing design.

(b) For a (24, 5, 11) packing design with a hole of size 4 proceed as follows:

1) Take a (24, 5, 7) packing design with a hole of size 4 (lemma 5.5).

2) Take a (23, 5, 2) optimal packing design. In this design there is a triple, say, $\{21, 22, 23\}$ the pairs of which appear in zero blocks.

3) Take two copies of B[25,5,1]. Assume in both copies we have ⟨21 22 23 24 25⟩. Delete this block and in all other blocks change 25 to 24.

ν	Point Set	Base Blocks
44	$Z_2 \times Z_{18} \cup H_8$	On $Z_2 \times Z_{18} \cup H_5$ construct a B[41,5,5] such that H5 is a block, which we delete and take the following blocks.
		$\langle(0,0)(0,9)(1,0)(1,9)h_6\rangle$ half orbit
		$\langle(\alpha,0)(\alpha,2)(\alpha,4)(\alpha,6)(\alpha,10)\rangle\alpha=0,1$
		$\langle(\alpha,0)(\alpha,1)(\alpha,4)(\alpha,7)h_6\rangle\langle(\alpha,0)(\alpha,1)(\alpha,6)(\alpha,11)h_6\rangle\alpha=0,1$
		$\langle(0,0)(0,1)(0,8)(1,5)h_1\rangle\langle(0,0)(1,2)(1,3)(1,10)h_1\rangle$
		$\langle(0,0)(0,1)(1,16)(1,17)h_1\rangle\langle(0,0)(0,2)(1,14)(1,17)h_2\rangle$
		$\langle(0,0)(0,3)(1,10)(1,17)h_2\rangle\langle(0,0)(0,3)(1,10)(1,17)h_2\rangle$
		$\langle(0,0)(0,4)(1,6)(1,8)h_2\rangle\langle(0,0)(0,5)(1,8)(1,12)h_3\rangle$
		$\langle(0,0)(0,6)(1,6)(1,11)h_3\rangle\langle(0,0)(0,5)(1,5)(1,9)h_3\rangle$
		$\langle(0,0)(0,8)(1,3)(1,9)h_4\rangle\langle(0,0)(0,9)(1,0)(1,10)h_4\rangle$
		$\langle(0,0)(0,1)(1,16)(1,17)h_4\rangle\langle(0,0)(0,2)(1,11)(1,14)h_5\rangle$
		$\langle(0,0)(0,3)(1,8)(1,13)h_5\rangle\langle(0,0)(0,3)(1,6)(1,11)h_5\rangle$
		$\langle(0,0)(0,5)(1,1)(1,7)h_7\rangle\langle(0,0)(0,7)(1,2)(1,7)h_7\rangle$
		$\langle(0,0)(0,9)(1,1)(1,12)h_7\rangle\langle(0,0)(0,1)(1,16)(1,17)h_7\rangle$
		$\langle(0,0)(0,2)(1,14)(1,17)h_7\rangle\langle(0,0)(0,3)(1,8)(1,10)h_8\rangle$
		$\langle(0,0)(0,4)(1,5)(1,8)h_8\rangle\langle(0,0)(0,5)(1,6)(1,14)h_8\rangle$
		$\langle(0,0)(0,6)(1,6)(1,13)h_8\rangle\langle(0,0)(0,7)(1,11)(1,13)h_8\rangle$
		$\langle(0,0)(0,8)(1,2)(1,11)h_6\rangle\langle(0,0)(0,7)(1,2)(1,11)\rangle \cup \{h_7,h_8\}$

8.2 $\nu \equiv 8 \pmod{20}$

Lemma 8.2. *Let* $\nu \equiv 8 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu,5,11) = \psi(\nu,5,11)$ *with the possible exception of* $\nu = 28$.

Proof: A $(\nu,5,11)$ optimal packing design for $\nu = 8, 48, 68, 88$, can be constructed as follows:

1) Take a $(\nu-1,5,4)$ optimal packing design, [10].

2) Take a $(\nu+1,5,4)$ optimal packing design. In this design each pair appears in 4 blocks except one pair, say, $(\nu,\nu+1)$ which appears in zero blocks. Change $\nu+1$ to ν.

3) Take a $(\nu,5,3)$ optimal packing design, [9].

For $\nu \geq 108$, $\nu \neq 128, 168, 208, 268$ simple calculations show that ν can be written in the form $\nu = 20m+4u+h+s$ where m, u, h and s are chosen as in lemma 5.7.

Now apply theorem 2.9 with $\lambda = 11$ and the result follows.

For $\nu = 128$ apply theorem 2.4 with $n = 7$, $h = 0$, $\lambda = 11$ and $u = 2$.

For $\nu = 168$ apply theorem 2.12 with $m = 8$, $h = 0$, $\lambda = 11$ and $u = 1$.

For $\nu = 208$ apply theorem 2.7 with $m = 40$, $h = 0$, $\lambda = 11$ and $u = 8$. The application of this theorem requires a (48,5,11) packing design with a hole of size 8. Such design can be constructed as follow:

a) Take a (47,5,4) packing design with a hole of size 7 [10].

b) Take a (49,5,4) packing design with a hole of size 9 [10] and change 49 to 48.

c) Take a (48,5,3) packing design with a hole of size 8 [9].

For $\nu = 268$ apply theorem 2.14 with $m = 13$, $h = 0$, $\lambda = 11$ and $u = 2$.

8.3 $\nu \equiv 12 \pmod{20}$

Lemma 8.3. $\sigma(\nu, 5, 11) = \psi(\nu, 5, 11)$ *for* $\nu = 12, 32, 52, 72, 92$.

Proof: The required constructions are given in the next table.

ν	Point Set	Base Blocks
12	$Z_2 \times Z_5 \cup H_2$	On $Z_2 \times Z_5 \cup \{h_1\}$ construct a B[11,5,4] and take the following blocks
		$\langle(0,0)(0,1)(0,2)h_1h_2\rangle\langle(1,0)(1,2)(1,3)h_1h_2\rangle$
		$\langle(0,0)(0,1)(0,2)(1,2)(1,3)\rangle\langle(0,0)(0,2)(1,1)(1,2)(1,4)\rangle$
		$\langle(0,0)(0,1)(0,3)(1,1)h_1\rangle\langle(0,0)(1,0)(1,3)(1,4)h_1\rangle$
		$\langle(0,0)(0,1)(1,1)(1,4)h_2\rangle\langle(0,0)(0,2)(1,3)(1,4)h_2\rangle$
		$\langle(0,0)(0,1)(1,0)(1,3)h_2\rangle\langle(0,0)(0,2)(1,0)(1,4)h_2\rangle$
32	Z_{32}	$\langle 012411\rangle$ 3 times $\langle 0381521\rangle$ 3 times $\langle 04101924\rangle$ 3 times
		$\langle 0\ 1\ 2\ 7\ 17\rangle\langle 0\ 2\ 9\ 18\ 22\rangle\langle 0\ 3\ 11\ 17\ 22\rangle\langle 0\ 3\ 11\ 19\ 23\rangle$
		$\langle 0\ 1\ 3\ 7\ 19\rangle\langle 0\ 1\ 2\ 4\ 26\rangle\langle 0\ 3\ 8\ 17\ 21\rangle\langle 0\ 5\ 10\ 16\ 25\rangle$

ν	Point Set	Base Blocks
52	$Z_{40} \cup H_{12}$	On $Z_{40} \cup H_{11}$ construct a B[51,5,4] with a hole of size 11, say, H_{11}.
		Such design can be constructed from a T[5,4,10] by adding a point to the groups. On the first four groups construct a B[11,5,4] and take the last group with the point to be the hole. Take the following blocks
		$\langle 0\ 8\ 16\ 24\ 32\rangle + i, i \in Z_8$ twice
		$\langle 0\ 2\ 6\ 18\ h_1\rangle\langle 0\ 4\ 14\ 18\ h_2\rangle\langle 0\ 12\ 14\ 20\ h_3\rangle\langle 0\ 8\ 18\ 20\ h_4\rangle$
		$\langle 0\ 5\ 16\ 22\ h_5\rangle\langle 0\ 6\ 14\ 25\ h_6\rangle\langle 0\ 1\ 3\ 16\ h_7\rangle\langle 0\ 1\ 4\ 7\ h_8\rangle$
		$\langle 0\ 5\ 12\ 27\ h_9\rangle\langle 0\ 5\ 16\ 25\ h_{10}\rangle\langle 0\ 7\ 17\ 26\ h_{11}\rangle\langle 0\ 8\ 17\ 27\ h_{12}\rangle$
		$\langle 0\ 1\ 2\ 5\ h_{12}\rangle\langle 0\ 2\ 3\ 13\rangle \cup \{h_i\}_{i=1}^{4}\langle 0\ 5\ 14\ 23\rangle \cup \{h_i\}_{i=5}^{8}$
		$\langle 0\ 7\ 13\ 18\rangle \cup \{h_i\}_{i=9}^{12}\langle 0\ 1\ 2\ 5\rangle \cup \{h_1, h_2\}$
		$\langle 0\ 3\ 9\ 24\rangle \cup \{h_3, h_4\}\langle 0\ 4\ 15\ 27\rangle \cup \{h_5, h_6\}$
		$\langle 0\ 7\ 15\ 26\rangle \cup \{h_7, h_8\}\langle 0\ 7\ 17\ 30\rangle \cup \{h_9, h_{10}\}$
		$\langle 0\ 7\ 19\ 28\rangle \cup \{h_{11}, h_{12}\}$
72	$Z_{60} \cup H_{12}$	On $Z_{60} \cup H_{10}$ construct a B[70,5,8] with a hole of size 10. Such design is constructed by applying theorem 2.7 with $m = 12$, $u = 8$ and $h = 2$. Notice that a (14,5,8) packing design with a hole of size 2 is constructed by taking 2 copies of (14,5,4) packing design with a hole of size two.
		$\langle 0\ 3\ 7\ 18\ 26\rangle\langle 0\ 1\ 18\ 32\ h_{11}\rangle\langle 0\ 4\ 10\ 26\ h_{11}\rangle\langle 0\ 12\ 20\ 36\ h_{12}\rangle$
		$\langle 0\ 2\ 14\ 34\ h_{12}\rangle\langle 0\ 2\ 11\ 41\rangle \cup \{h_i\}_{i=1}^{4}\langle 0\ 5\ 15\ 38\rangle \cup \{h_i\}_{i=5}^{8}$
		$\langle 0\ 6\ 13\ 31\rangle \cup \{h_i\}_{i=9}^{12}\langle 0\ 1\ 2\ 5\rangle \cup \{h_1, h_2\}\langle 0\ 3\ 8\ 43\rangle \cup \{h_3, h_4\}$
		$\langle 0\ 6\ 17\ 27\rangle \cup \{h_5, h_6\}\langle 0\ 7\ 19\ 46\rangle \cup \{h_7, h_8\}\langle 0\ 9\ 24\ 37\rangle$
		$\cup\{h_9, h_{10}\}\langle 0\ 9\ 25\ 38\rangle \cup \{h_{11}, h_{12}\}$
92	$Z_{80} \cup H_{12}$	On $Z_{80} \cup H_{11}$ construct a B[91,5,8] with a hole of size 11. Such a design is constructed by taking a T[6,1,16], remove five points from last group then on the blocks which are of size 5, 6 and on the first 5 groups construct a BIBD with index 8 and take the last group to be the hole. Take a (80,5,1) minimal covering design [30] and take the following blocks
		$\langle 0\ 3\ 13\ 37\ 41\rangle\langle 0\ 8\ 20\ 38\ h_{12}\rangle\langle 0\ 10\ 14\ 36\ h_{12}\rangle\langle 0\ 2\ 17\ 31\rangle$
		$\cup\{h_i\}_{i=1}^{4}\langle 0\ 5\ 6\ 39\rangle \cup \{h_i\}_{i=5}^{8}\langle 0\ 7\ 18\ 37\rangle \cup \{h_i\}_{i=9}^{12}$
		$\langle 0\ 1\ 3\ 16\rangle \cup \{h_1, h_2\}\langle 0\ 5\ 17\ 26\rangle \cup \{h_3, h_4\}$
		$\langle 0\ 7\ 29\ 52\rangle \cup \{h_5, h_8\}\langle 0\ 9\ 32\ 53\rangle \cup \{h_7, h_8\}$
		$\langle 0\ 11\ 31\ 56\rangle \cup \{h_9, h_{10}\}\langle 0\ 6\ 25\ 33\rangle \cup \{h_{11}, h_{12}\}$

Lemma 8.4. *Let $\nu \equiv 12 \pmod{20}$ be a positive integer. Then $\sigma(\nu, 5, 11) = \psi(\nu, 5, 11)$.*

Proof: For $\nu = 12, 32, 52, 72, 92$ the result follows from lemma 8.3 For $\nu \geq 112$, $\nu \neq 132$, write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 5.3 with the difference that $4u+h+s = 12, 32, 52, 72, 92$ and $h = 0$.

Now apply theorem 2.9 with $\lambda = 11$ to give the result.

For $\nu = 132$ apply theorem 2.4 with $u = 3$, $h = 0$, $\lambda = 11$ and $n = 7$.

8.4 $\nu \equiv 2$ or 18 (mod 20)

Lemma 8.5. *Let $\nu \equiv 2$ or 18 (mod 20) be a positive integer. Then $\sigma(\nu, 5, 11) = \psi(\nu, 5, 11)$. Furthermore, there exists a (22,5,11) packing design with a hole of size 2.*

Proof: For $\nu \equiv 2$ (mod 20), $\sigma(\nu, 5, 11) = \sigma(\nu, 5, 8) + \sigma(\nu, 5, 3)$.

For a (22,5,11) packing design with a hole of size 2 take a (22,5,3) and two copies of a (22,5,4) packing design with a hole of size 2, [7], [10].

For $\nu \equiv 18$ (mod 20) the construction is as follows:

1) Take a $(\nu - 1, 5, 4)$ optimal packing design, [10].

2) Take a $(\nu + 1, 5, 4)$ optimal packing design. This design has a pair, say, $(\nu, \nu + 1)$ that appears in zero blocks. Change $\nu + 1$ to ν.

3) Take a $(\nu, 5, 3)$ optimal packing design, [7].

Since a (38,5,3) optimal packing design is still unknown, the above construction does not work for $\nu = 38$. For $\nu = 38$ let $X = Z_{32} \cup H_6$ then the required blocks are the following, developed under the action of the group Z_{32}

$\langle 0\ 7\ 16\ 23 \rangle \cup \{h_5, h_6\}$ half orbit

$\langle 0\ 1\ 2\ 4\ 11 \rangle$ twice $\langle 0\ 3\ 8\ 15\ 21 \rangle$ twice $\langle 0\ 4\ 10\ 19\ 24 \rangle$ twice

$\langle 0\ 2\ 6\ 12\ 16 \rangle \langle 0\ 1\ 4\ 12\ h_1 \rangle \langle 0\ 1\ 7\ 16\ h_2 \rangle \langle 0\ 3\ 14\ 19\ h_3 \rangle \langle 0\ 1\ 3\ 12\ h_4 \rangle$

$\langle 0\ 5\ 10\ 18\ h_5 \rangle 0\ 3\ 8\ 20\ h_6 \rangle 0\ 4\ 11\ 21\ h_1 \rangle \langle 0\ 5\ 13\ 19\ h_2 \rangle \langle 0\ 1\ 3\ 5\ h_3 \rangle$

$\langle 0\ 4\ 11\ 20\ h_4 \rangle \langle 0\ 6\ 13\ 23\ h_5 \rangle \langle 0\ 6\ 14\ 24\ h_6 \rangle \langle 0\ 1\ 2\ 7 \rangle \cup \{h_i\}_{i=1}^{4}$

$\langle 0\ 2\ 9\ 21 \rangle \cup \{h_1, h_2\} \langle 0\ 3\ 14\ 17 \rangle \cup \{h_2, h_3\} \langle 0\ 5\ 13\ 22 \rangle \cup \{h_4, h_5\}$

8.5 $\nu \equiv 14$ (mod 20)

Lemma 8.6. $\sigma(\nu, 5, 11) = \psi(\nu, 5, 11)$ *for* $\nu = 14, 34, 54, 74, 94$.

Proof: We construct a $(\nu, 5, 11)$ optimal packing design such that the complement graph is a three 1-factor. (To be used in lemma 12.4). This is done in two steps. In the first step we take three copies of a $(\nu, 5, 3)$ minimal covering design [9]. By lemma 3.3, the excess graph of each one of these designs is a 1-factor. In the second step we construct a $(\nu, 5, 2)$ packing design (not optimal) such that the complement graph is a six 1-factor, then apply theorem 3.1 to get the result.

For these constructions see the following table.

ν	Point Set	Base Blocks
14	Z_{14}	$\langle 0\ 1\ 2\ 4\ 10\rangle$
34	Z_{34}	$\langle 0\ 1\ 2\ 5\ 12\rangle\langle 0\ 2\ 5\ 20\ 26\rangle\langle 0\ 4\ 11\ 20\ 26\rangle$
54	Z_{54}	$\langle 0\ 1\ 4\ 11\ 32\rangle\langle 0\ 5\ 17\ 25\ 41\rangle\langle 0\ 1\ 3\ 5\ 15\rangle$ $\langle 0\ 6\ 13\ 22\ 36\rangle\langle 0\ 6\ 15\ 35\ 43\rangle$
74	Z_{74}	$\langle 0\ 1\ 5\ 11\ 29\rangle\langle 0\ 2\ 17\ 33\ 53\rangle\langle 0\ 8\ 27\ 40\ 52\rangle$ $\langle 0\ 1\ 3\ 6\ 14\rangle\langle 0\ 4\ 20\ 48\ 55\rangle\langle 0\ 7\ 22\ 40\ 49\rangle$ $\langle 0\ 10\ 24\ 36\ 53\rangle$
94	Z_{94}	$\langle 0\ 8\ 19\ 62\ 80\rangle\langle 0\ 1\ 3\ 7\ 24\rangle\langle 0\ 5\ 35\ 53\ 61\rangle$ $\langle 0\ 9\ 29\ 45\ 60\rangle\langle 0\ 10\ 37\ 50\ 62\rangle\langle 0\ 1\ 3\ 7\ 16\rangle$ $\langle 0\ 5\ 27\ 57\ 71\rangle\langle 0\ 10\ 31\ 59\ 70\rangle\langle 0\ 12\ 29\ 48\ 68\rangle$

Lemma 8.7. *Let $\nu \equiv 14 \pmod{20}$ be a positive integer. Then $\sigma(\nu, 5, 11) = \psi(\nu, 5, 11)$.*

Proof: For $\nu = 14, 34, 54, 74, 94$ the result follows from lemma 8.6. For $\nu \geq 114$, $\nu \neq 134$, write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 5.3 with the differences that $4u+h+s = 14, 34, 54, 74, 94$ and $h = 2$.

Apply theorem 2.9 with $\lambda = 11$ and the result follows.

For $\nu = 134$ apply theorem 2.4 with $u = 3$, $h = 2$, $n = 7$ and $\lambda = 11$.

8.6 $\nu \equiv 7 \pmod{20}$

Lemma 8.8. *Let $\nu \equiv 7 \pmod{20}$ be a positive integer. Then $\sigma(\nu, 5, 11) = \psi(\nu, 5, 11)$. Furthermore, there exists a (23,5,11) packing design with a hole of size 3.*

Proof: If $\nu \equiv 7 \pmod{20}$, then $\sigma(\nu, 5, 11) = \sigma(\nu, 5, 7) + \sigma(\nu, 5, 4)$. Since a (7,5,4) optimal packing design does not exist we need to construct a (7,5,11) optimal packing design. Let $X = Z_5 \cup \{a, b\}$. Then the required blocks are $\langle 0\ 1\ 2\ 3\ 4\rangle$ twice, $\langle 0\ 1\ 2\ a\ b\rangle \pmod 5 \langle 0\ 1\ 3\ a\ b\rangle \pmod 5$
$\langle 0\ 1\ 2\ 3\ a\rangle \pmod 5 \langle 0\ 1\ 3\ 4\ b\rangle \pmod 5$.

For a (23,5,11) packing design with a hole of size 3 take a (23,5,4) and a (23,5,7) packing design with a hole of size 3, [10] and lemma 5.1.

Corollary 8.1. *Let $\nu \equiv 7 \pmod{20}$ be a positive integer then for all positive integers $\lambda > 1$ we have $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda) - e$ where $e = 1$ if $\lambda \equiv 2 \pmod{10}$ and $e = 0$ otherwise with the possible exceptions of $(\nu, \lambda) = (27, 2)$.*

Proof: We have shown that $\sigma(\nu,5,\lambda) = \psi(\nu,5,\lambda) - e$ where e is as above for all positive integers $\nu \equiv 7 \pmod{20}$ and $2 \le \lambda \le 12$ with the possible exception of $(\nu,\lambda) = (27,2)$. (See theorem 1.2 for $2 \le \lambda \le 6$ and lemmas 5.3, 6.4, 7.11, 8.8 for $7 \le \lambda \le 11$ and [12] for $\lambda = 12$).

But for $\nu \equiv 7 \pmod{20}$ there exists a $B[\nu,5,10]$. Now apply lemma 4.2 to give the result.

8.7 $\nu \equiv 19 \pmod{20}$

In this section we use different designs to obtain our result. For this purpose the following is most useful to us.

Lemma 8.9. *There exists a $(\nu,5,2)$ packing designs with a hole of size 7 for all positive integers ν, $\nu \equiv 17 \pmod{20}$, $\nu \ge 57$.*

Proof: For $\nu = 57,77,97$ the result was established in [7]. For $\nu = 117,157,197,217,257$ apply theorem 2.16 with $h = 1$, $u = 3$, $\lambda = 2$ and $m = 11,15,19,21,25$ respectively. The application of this theorem requires the existence of a $GD[5,2,2,2m]$ where $m = 11,15,20,21,25$. For $m = 11,21,25$ see [6] and for $m = 20$ see [5]. For $m = 15$ let $X = Z_{28} \cup \{a,b\}$. Groups are $\{i, i+14\} \cup \{a,b\}$ where $i = 0,\dots,13$. Then the blocks are the following:

$\langle 0\ 1\ 2\ 4\ 10\rangle \pmod{28}$ $\langle 0\ 3\ 8\ 15\ 19\rangle \pmod{28}$ $\langle 0\ 5\ 11\ 18\rangle \cup \{a,b\}$ mod 28.

For $\nu = 137$ apply theorem 2.15 with $m = 13$, $u = 3$, $h = 1$ and $\lambda = 2$.

For $\nu = 177,277$ apply theorem 2.17 with $\lambda = 2$, $u = 3$, and $m = 17,27$ respectively.

For $\nu = 237,337$ take a TT(6,1,23,1) and a TT(6,1,33,1) respectively and inflate them by 2 and replace their blocks, which are of size 5 and 6, by the blocks of GD[5,2,2,10] and GD[5,2,2,12] respectively. Add five points to the groups and on the first five groups construct a B[51,5,2] and B[71,5,2] with a hole of size 5 respectively. (These two designs are constructed in the next table). Take these five points with the last group to be the hole of size 7.

For $\nu = 517$, take a TT(6,1,45,31) and inflate it by a factor of 2. Replace the blocks of TT(6,1,45,31), which are of size 5 and 6, by the blocks of a GD[5,2,2,10] and GD[5,2,2,12] respectively. Add 5 points to the groups and on the first 5 groups construct a B[95,5,2] with a hole of size 5, and on the last group construct a (67,5,2) packing design with a hole of size 7 [5]. Notice that a B[95,5,2] with a hole of size 5 can be constructed by applying theorem 2.16 with $m = 9$, $h = 1$, $\lambda = 2$ and $u = 1$.

For all other values of ν write $\nu = 20m + 4u + h + s$ where m, u, h, s are chosen so that

1) there exists a $RMGD[5,1,5,5m]$;

2) $4u + h + s = 57, 77, 97$;

3) $0 \leq u \leq m-1$, $s \equiv 0 \pmod 4$, $h = 1$ or 5;

4) there exists a GD$[5, 2, \{4, s^*\}, 4m + s]$.

Now apply theorem 2.9 with $\lambda = 2$ to give a $(\nu, 5, 2)$ packing design with a hole of size 57, 77, 97 and hence a $(\nu, 5, 2)$ packing design with a hole of size 7.

ν	Point Set	Base Blocks
51	$Z_2 \times Z_{23} \cup H_5$	$\langle(0,0)(0,1)(0,4)(0,11)(1,2)\rangle\langle(0,0)(0,2)(0,8)(0,17)(1,7)\rangle$
		$\langle(0,0)(0,5)(0,12)(1,11)(1,20)\rangle\langle(0,0)(0,2)(1,0)(1,12)(1,18)\rangle$
		$\langle(0,0)(0,10)(1,3)(1,4)(1,19)\rangle\langle(1,0)(1,1)(1,3)(1,5)(1,9)\rangle$
		$\langle(0,0)(0,1)(1,2)(1,15)h_1\rangle\langle(0,0)(0,3)(1,9)(1,20)h_2\rangle$
		$\langle(0,0)(0,4)(1,8)(1,11)h_3\rangle\langle(0,0)(0,5)(1,5)(1,18h_4\rangle$
		$\langle(0,0)(0,9)(1,12)(1,19)h_5\rangle$
71	$Z_2 \times Z_{33} \cup H_5$	$\langle(1,0)(1,3)(1,7)(1,12)(1,25)\rangle\langle(0,0)(0,1)(0,3)(0,14)(1,0)\rangle$
		$\langle(0,0)(0,4)(0,9)(0,19)(1,2)\rangle\langle(0,0)(0,6)(0,13)(0,21)(1,9)\rangle$
		$\langle(0,0)(1,0)(1,1)(1,3)(1,26)\rangle\langle(0,0)(0,4)(0,16)(1,22)(1,28)\rangle$
		$\langle(0,0)(0,9)(0,16)(1,13)(1,14)\rangle\langle(0,0)(0,3)(1,10)(1,23)(1,32)\rangle$
		$\langle(0,0)(0,10)(1,11)(1,25)(1,27)\rangle\langle(0,0)(0,1)(1,5)(1,11)(1,23)\rangle$
		$\langle(0,0)(0,2)(1,17)(1,27)h_1\rangle\langle(0,0)(0,5)(1,7)(1,24)h_2\rangle$
		$\langle(0,0)(0,6)(1,14)(1,18)h_3\rangle\langle(0,0)(0,8)(1,16)(1,21)h_4\rangle$
		$\langle(0,0)(0,11)(1,6)(1,20)h_5\rangle$

Lemma 8.10. *Let $\nu \equiv 19 \pmod{20}$ be a positive integer. Then $\sigma(\nu, 5, 11) = \psi(\nu, 5, 11)$ with the possible exception of $\nu = 39$.*

Proof: For $\nu = 19$ the construction is as follows.

1) Take a (19,5,3) optimal packing design [8]. Close observation of this design shows that the pair (1,10) appears only once and the pairs (1,2) (1,3) (2,16) appear only twice. Furthermore, in this design we have the block ⟨4 5 14 1 2⟩. In this block change 2 to 3.

2) Take a (19,5,4) optimal packing design [10]. This design has a pair say (2,10) that appears in zero blocks. Assume in this design we have the block ⟨4 5 14 16 3⟩. In this block change 3 to 2.

3) Take a (19,5,4) minimal covering design [31]. In this design each pair appears exactly four times except a triple, say, {1,2,10} the pairs of which appear in six blocks.

Now it is readily checked that the above three steps yield the blocks of a (19,5,11) optimal packing design.

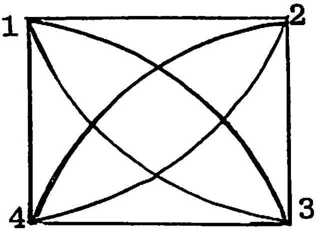
For $\nu = 239$ apply theorem 2.9 with $\lambda = m = 11$, $h = 3$ and $u = 4$.

For all other values the construction is as follows:

1) Take a $(\nu, 5, 4)$ optimal packing design [10]. In this design there is a pair, say, $\{1, 2\}$ that appears in zero blocks.

2) Take a $(\nu, 5, 4)$ minimal covering design which exists for all $\nu \equiv 19 \pmod{20}$ [13], [31]. In this design there is a triple, say, $\{1, 2, a\}$ the pairs of which appear in precisely six blocks.

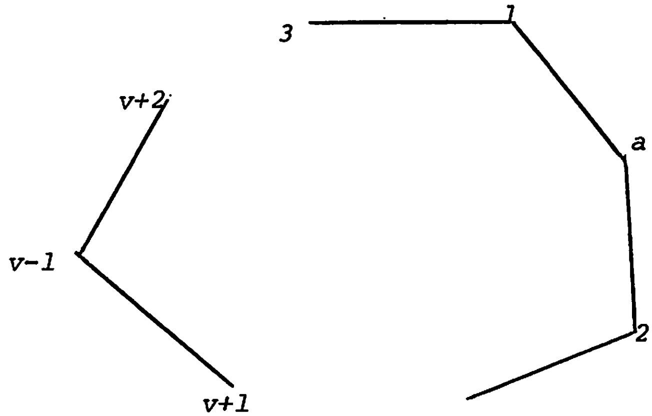
3) Take a $(\nu - 2, 5, 2)$ covering design with a hole of size 7. Careful inspection of the (7,5,2) covering design [33], shows that its excess graph consists of 3 isolated vertices and the following graph on the remaining 4 vertices, say, $\{1, 2, 3, 4\}$.



Since there exists a $(\nu - 2, 5, 2)$ covering design with a hole of size 7, $\nu - 2 \geq 57$, $\nu \equiv 19 \pmod{20}$ it follows that there exists a $(\nu - 2, 5, 2)$ minimal covering design such that its excess graph consists of $\nu - 4$ isolated vertices and the same graph on the remaining four vertices. Assume in this design we have the block $\langle 1\ 2\ 3\ 4\ a \rangle$. Delete this block.

4) take a $(\nu + 4, 5, 1)$ optimal packing design [32], $\nu \neq 243$. Careful inspection of these designs shows that their complement graph contains a circuit graph C_n where $n \geq 23$, that is, a regular connected graph of degree 2 on n vertices.

Assume in this design we have the block $\langle \nu\ \nu{+}1\ \nu{+}2\ \nu{+}3\ \nu{+}4 \rangle$, delete this block. Furthermore, we may label the points of this design so that the circuit looks as follows.



Now change $\nu+3$ and $\nu+4$ to ν, and change $\nu+1$, $\nu+2$ to $\nu-1$.

It is easy to check that the above four steps yield a $(\nu, 5, 11)$ optimal packing design for $\nu \equiv 19 \pmod{20}$, $\nu \geq 59$.

Corollary 8.2. *Let* $\nu \equiv 19 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda) - e$ *for all* $\lambda \geq 2$ *with the possible exception of* $(\nu, \lambda) = (19, 2)$ $(39, 10m+1)$, *where* $m \geq 1$ *is a positive integer where* $e = 1$ *if* $\lambda \equiv 2 \pmod{10}$ *otherwise* $e = 0$.

Proof: We have shown that $\sigma(\nu, 5, \lambda) = \psi(\nu, 5, \lambda) - e$ where e is as above for all positive integers $\nu \equiv 19 \pmod{20}$ and $2 \leq \lambda \leq 12$ with the possible exception of $(\nu, \lambda) = (19, 2)(39, 11)$.

But for $\lambda \equiv 0 \pmod{10}$ there exists a $B[\nu, 5, 10]$ for all positive integers $\nu \equiv 19 \pmod{20}$. Now apply lemma 4.2 to give the result.

Theorem 8.1. *Let* $\nu \geq 5$ *be a positive integer. Then* $\sigma(\nu, 5, 11) = \psi(\nu, 5, 11)$ *with the possible exceptions of* $\nu = 28, 39$.

9 Packing with index 13

9.1 $\nu \equiv 4 \pmod{20}$

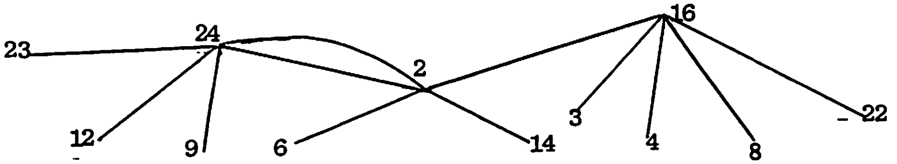
Lemma 9.1. $\sigma(24, 5, 13) = \psi(24, 5, 13)$

Proof: A (24,5,13) optimal packing design can be constructed as follows:

1) Take two copies of a (24,5,4) optimal packing design [10]. The complement graph of this design consists of 22 isolated vertices and another two vertices connected by 4 parallel edges. Assume in the first copy these two vertices are (8,22) and in the second copy they are (6,22). Furthermore, assume in one copy of these designs we have the block ⟨1 2 3 12 9⟩ where $\{1, 2, 3\}$ are arbitrary numbers. In this block change 9 to 24.

2) Take a (23,5,1) optimal packing design [32]. The complement graph of this design is C_{23}, the circuit graph. So we may assume that (2,22) (2,8) (8,6) (6,16) are edges in C_{23}.

3) Take a (24,5,3) optimal packing design [9]. The complement graph of this design consists of 18 edges. We may relabel the points of this design by interchanging 17 and 24. After relabeling the points of this design, the complement graph will have the following subgraph.



4) Take a B[25,5,1] and assume we have the block ⟨1 2 3 24 25⟩. In this block change 25 to 9 and in all other blocks change 25 to 24.

5) Add the block ⟨2 6 8 16 22⟩.

Now it is easy to check that the above five steps yield the blocks of a (24,5,13) optimal packing design.

Lemma 9.2.

(a) $\sigma(\nu, 5, 13) = \psi(\nu, 5, 13)$ *for* $\nu = 44, 64, 84$.

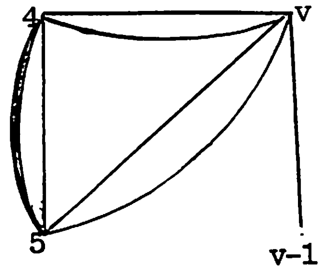
(b) *There exists a (24, 5, 13) packing design with a hole of size 4.*

Proof: (a) A $(\nu, 5, 13)$ optimal packing design for $\nu = 44, 64, 84$ can be constructed as follows:

1) Take a $(\nu, 5, 4)$ optimal packing design. The complement graph of this design consists of $\nu - 2$ isolated vertices and two other vertices, say, $\{\nu - 2, \nu\}$ joined by four parallel edges.

2) Take a $(\nu, 5, 4)$ minimal covering design [13]. The excess graph of this design consists of $\nu - 3$ isolated vertices and three other vertices, say, $\{4, 5, \nu\}$ the pairs of which are connected by two edges.

3) Take a $(\nu, 5, 3)$ optimal packing design. Careful inspection of these designs ($\nu = 44, 64, 84$) shows that their complement graph contains the following subgraph on the four vertices, say, $\{4, 5, \nu - 1, \nu\}$.



4) Take a $(\nu - 1, 5, 1)$ optimal packing design, and assume we have the block $\langle 1\ 2\ 3\ \nu-2\ \nu-1\rangle$. In this block change $\nu - 1$ to ν.

5) Take a $B[\nu + 1, 5, 1]$ and assume we have the block $\langle 1\ 2\ 3\ \nu\ \nu+1\rangle$ where $\{1, 2, 3\}$ are arbitrary numbers. In this block change $\nu + 1$ to $\nu - 1$ and in all other blocks change $\nu + 1$ to ν.

Now it is easy to check that the above 5 steps give us the blocks of a $(\nu, 5, 13)$ optimal packing design for $\nu = 44, 64, 84$.

(b) For a (24, 5, 13) packing design with a hole of size 4 proceed as follows:

1) Take a (24, 5, 5) packing design with a hole of size 4 [8].

2) Take two copies of a (23,5,2) packing design, [5]. This design has a hole of size 3, say, $\{21, 22, 23\}$.

3) Take four copies of a B[25,5,1]. Assume in each copy we have the block $\langle 21\ 22\ 23\ 24\ 25\rangle$. Delete this block, from each copy, and change 25 to 24.

It is readily checked that these three steps yield a (24,5,13) packing design with a hole of size 4.

Lemma 9.3. *Let* $\nu \equiv 4 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 13) = \psi(\nu, 5, 13)$.

Proof: For $\nu = 24, 44, 64, 84$ the result follows from lemmas 9.1 and 9.2.

For $\nu \equiv 124$, $\nu \neq 144, 184, 224$ simple calculations show that ν can be written in the form $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 5.3 with the difference that $4u + h + s = 24, 44, 64, 84$ and $h = 4$.

Now apply theorem 2.9 with $\lambda = 13$ and the result follows.

For $\nu = 144$ apply theorem 2.18 with $n = 7$, $h = 4$ and $\lambda = 13$.

For $\nu = 104, 224$ apply theorem 2.11 with $h = 4$, $\lambda = 13$ and $m = 5, 11$ respectively.

For $\nu = 184$ apply theorem 2.12 with $h = 0$, $\lambda = 13$, $u = 5$ and $m = 8$.

9.2 $\nu \equiv 8$ or 12 (mod 20)

Lemma 9.4. *Let* $\nu \equiv 8$ *or* 12 (mod 20) *be a positive integer. Then* $\sigma(\nu, 5, 13) = \psi(\nu, 5, 13)$ *with the possible exception of* $\nu = 28$.

Proof: For $\nu \equiv 12 \pmod{20}$, $\sigma(\nu, 5, 13) = \sigma(\nu, 5, 11) + \sigma(\nu, 5, 2)$ holds, and for $\nu = 8, 48, 68, 88$, $\sigma(\nu, 5, 13) = \sigma(\nu, 5, 10) + \sigma(\nu, 5, 3)$ holds.

For $\nu \geq 108$ $\nu \neq 128, 168, 208, 268$ write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen the same as in lemma 8.2.

Now apply theorem 2.9 with $\lambda = 13$ and the result follows.

The cases $\nu = 128, 168, 208, 268$ are treated the same as in lemma 8.2.

Notice that when applying theorem 2.7 for $\nu = 208$ we require a (48,5,13) packing design with a hole of size 8. Such design can be constructed by taking three copies of a (48,5,3) packing design with a hole of size 8 [9] together with the blocks of a (48,5,4) packing design with a hole of size 8 [13].

9.3 $\nu \equiv 2$ (mod 20)

Lemma 9.5. (a) *There exists a (26,5,13) packing design with a hole of size 6.* (b) $\sigma(\nu, 5, 13) = \psi(\nu, 5, 13)$ *for all* $\nu \equiv 2 \pmod{20}$, $\nu \geq 22$.

Proof: For a (26,5,13) packing design with a hole of size six take a (26,5,5) packing design with a hole of size 6 [11] and two copies of (26,5,4) packing design with a hole of size 6, lemma 5.8.

For a $(\nu, 5, 13)$ optimal packing design $\nu \equiv 2 \pmod{20}$, $\nu \neq 62, 82$ proceed as follows:

1) Take a $(\nu, 5, 5)$ optimal packing design. This design has a pair, say, $(\nu - 1, \nu)$ that appears in zero blocks [8]. Furthermore, assume that the pairs (5,9) and (4,10) appear at most 4 times.

2) Take a $(\nu + 1, 5, 2)$ optimal packing design [5]. This design has a triple, say, $\{\nu - 1, \nu, \nu + 1\}$ the pairs of which appear in zero blocks. Change $\nu + 1$ to ν

3) Take four copies of a B$[\nu - 1, 5, 1]$.

4) Take a $(\nu + 1, 5, 2)$ minimal covering design $\nu + 1 \neq 63, 83$. In this design each pair appears exactly twice except one pair, say, $(\nu - 1, \nu)$ that appears 6 times [31]. Furthermore, assume in this design we have the two blocks $\langle a\ b\ c\ \nu\ \nu + 1\rangle$ $\langle d\ e\ f\ \nu\ \nu + 1\rangle$ where $\{a, b, c, d, e, f\}$ are arbitrary numbers not necessarily disjoint. In these two blocks change $\nu + 1$ to $\nu - 1$ and in all other blocks change $\nu + 1$ to ν.

Assume in (3) we have the two blocks $\langle a\ b\ c\ 9\ \nu-1\rangle\ \langle d\ e\ f\ 10\ \nu-1\rangle$. In these two blocks change $\nu-1$ to ν.

The above 4 steps gives us a design such that each pair appears at most 13 times except $(9,\nu)$ and $(10,\nu)$ which appear at most 14 times and $(\nu-1,\nu)$ which appears at most 10 times. To reduce the number of appearances of $(9,\nu)$ and $(10,\nu)$ from 14 to 13, assume in (1) we have the block $\langle 11\ 12\ 13\ 9\ \nu\rangle$, $\langle 14\ 15\ 16\ 10\ \nu\rangle$. In the first block change ν to 5 and in the second change ν to 4. Furthermore, assume in (3) we have the blocks $\langle 11\ 12\ 13\ \nu-1\ 5\rangle$, $\langle 14\ 15\ 16\ \nu-1\ 4\rangle$ in these two blocks change 5 and 4 to ν.

Now it is easy to check that the above construction gives the blocks of a $(\nu,5,13)$ optimal packing design for all $\nu\equiv 2 \pmod{20}$ $\nu\geq 22$, $\nu\neq 62,82$.

For $\nu=62,82$ see next table.

ν	Point Set	Base Blocks
62	$Z_{56}\cup H_6$	On $Z_{56}\cup H_5$ construct a B[61,5,8] with a hole of size 5, say, H_5 and take the following blocks.
		$\langle 0\ 16\ 28\ 44\ h_1\rangle$half orbit
		$\langle 0\ 1\ 3\ 7\ 27\rangle\langle 0\ 5\ 13\ 23\ 47\rangle\langle 0\ 8\ 19\ 34\ 44\rangle\langle 0\ 1\ 3\ 5\ 14\rangle$
		$\langle 0\ 4\ 19\ 27\ 39\rangle\langle 0\ 6\ 16\ 32\ 38\rangle\langle 0\ 1\ 5\ 15\ 33\rangle\langle 0\ 6\ 19\ 26\ 35\rangle$
		$\langle 0\ 2\ 19\ 31\ h_6\rangle\langle 0\ 3\ 11\ 34\ h_2\rangle\langle 0\ 1\ 2\ 11\ h_3\rangle\langle 0\ 3\ 16\ 21\ h_4\rangle$
		$\langle 0\ 4\ 9\ 21\ h_5\rangle\langle 0\ 6\ 13\ 33\ h_6\rangle\langle 0\ 7\ 22\ 37\rangle\cup\{h_1,h_1,h_5,h_6\}$
		$\langle 0\ 8\ 11\ 25\ h_6\rangle\langle 0\ 7\ 21\ 38\rangle\cup\{h_i\}_{i=1}^{4}$
82	$Z_{76}\cup H_6$	On $Z_{76}\cup H_5$ construct a B[81,5,8] with a hole of size 5, say, H_5 and take the following blocks
		$\langle 0\ 20\ 38\ 58\ h_1\rangle$ half orbit
		$\langle 0\ 1\ 3\ 9\ 25\rangle$ twice $\langle 0\ 4\ 14\ 27\ 46\rangle$ twice $\langle 0\ 11\ 26\ 47\ 59\rangle$
		twice $\langle 0\ 1\ 5\ 40\ 45\rangle\langle 0\ 2\ 24\ 31\ 44\rangle\langle 0\ 6\ 16\ 39\ 64\rangle$
		$\langle 0\ 7\ 17\ 31\ 35\rangle\langle 0\ 1\ 3\ 8\ 27\rangle\langle 0\ 6\ 16\ 3851\rangle\langle 0\ 8\ 26\ 47\ 56\rangle$
		$\langle 0\ 7\ 18\ 37\rangle\cup\{h_i\}_{i=1}^{4}\langle 0\ 15\ 35\ 49 h_5\rangle\langle 0\ 3\ 15\ 36\ h_2\rangle$
		$\langle 0\ 5\ 9\ 22\ h_3\rangle\langle 0\ 1\ 3\ 30\ h_4\rangle\langle 0\ 5\ 11\ 26\rangle\cup\{h_1,h_1,h_5,h_6\}$
		$\langle 0\ 7\ 19\ 60\ h_6\rangle\langle 0\ 8\ 25\ 39\ h_6\rangle\langle 0\ 9\ 20\ 43\ h_6\rangle$

9.4 $\nu\equiv 14 \pmod{20}$

Lemma 9.6. *There exists a $(\nu,5,6)$ optimal packing design, for $\nu=14,54,74,94$, such that their complement graphs contains a 1-factor subgraph on $\nu-2$ vertices.*

Proof: For $\nu=14$ see [11]. A close observation of this design shows that the edges of the 1-factor are $\{(1,5)(2,6)(3,7)(4,9)(10,12)(11,14)\}$.

For $\nu = 54, 74, 94$ the $(\nu, 5, 6)$ optimal packing design, [11], was constructed by taking a $(\nu, 5, 2)$ packing design with a hole of size 4, say, $\{a, b, c, d\}$ and a $(\nu, 5, 4)$ minimal covering design. Since the $(\nu, 5, 4)$ minimal covering design, $\nu \equiv 2$ or $4 \pmod 5$, has each pair appears in exactly 4 blocks except the pairs of a triple, say, $\{a, b, c\}$, that appearing in 6 blocks, it follows that these two steps yield the blocks of a $(\nu, 5, 6)$ optimal packing design. It is clear from the above construction of the $(\nu, 5, 6)$ optimal packing design that if the complement graph of the $(\nu, 5, 2)$ packing design with a hole of size 4 has a subgraph which is 1-factor on the $\nu - 4$ vertices then the complement graph of the $(\nu, 5, 6)$ optimal packing design has a subgraph which is 1-factor on $\nu - 2$ vertices.

For a $(\nu, 5, 2)$, $\nu = 54, 94$, packing design with a hole of size 4 that satisfies the above see [4].

For $\nu = 74$ let $X = Z_2 \times Z_{35} \cup H_4$ then the required blocks are
$\langle(0,0)(0,7)(0,14)(0,21)(0,28)\rangle + (-, i)$, $i \in Z_7$
$\langle(0,0)(0,1)(0,5)(0,17)(1,0)\rangle$ (mod $-, 35$)
$\langle(0,0)(0,3)(0,11)(0,20)(1,1)\rangle$ (mod $-, 35$)
$\langle(0,0)(0,4)(0,13)(1,15)(1,27)\rangle$ mod $-, 35$
$\langle(0,0)(0,3)(0,13)(1,9)(1,30)\rangle$ (mod $-, 35$)
$\langle(0,0)(0,7)(1,10)(1,19)(1,29)\rangle$ (mod $-, 35$)
$\langle(0,0)(0,11)(1,4)(1,24)(1,32)\rangle$ (mod $-, 35$)
$\langle(0,0)(0,2)(0,10)(0,16)(1,7)\rangle$ (mod $-, 35$)
$\langle(0,0)(0,12)(1,2)(1,8)(1,20)\rangle$ (mod $-, 35$)
$\langle(0,0)(0,15)(1,6)(1,9)(1,22)\rangle$ (mod $-, 35$)
$\langle(0,0)(1,0)(1,1)(1,3)(1,5)\rangle$ (mod $-, 35$)
$\langle(0,0)(0,1)(1,14)(1,34)h_1\rangle$ (mod $-, 35$)
$\langle(0,0)(0,2)(1,12)(1,21)h_2\rangle$ (mod $-, 35$)
$\langle(0,0)(0,5)(1,20)(1,28)h_3\rangle$ (mod $-, 35$)
$\langle(0,0)(0,6)(1,17)(1,24)h_4\rangle$ (mod $-, 35$)
$\langle(1,0)(1,1)(1,5)(1,11)(1,22)\rangle$ (mod $-, 35$)

Lemma 9.7. $\sigma(\nu, 5, 13) = \psi(\nu, 5, 13)$ *for* $\nu = 14, 54, 74, 94$.

Proof: For $\nu = 14, 54, 74, 94$ the construction is as follows:

1) take a $(\nu, 5, 6)$ optimal packing design on Z_ν. By the previous lemma the complement graph of these designs contains a 1-factor on $\nu - 2$ vertices say $Z_{\nu-2}$.

2) take a $(\nu, 5, 4)$ optimal packing design on Z_ν. The complement graph of this design consists of $\nu - 2$ isolated vertices and two vertices, say, $(\nu - 1, \nu)$ joined by 4 edges.

   So the two complement graphs in (1) and (2) contains a subgraph that is 1-factor on Z_ν.

3) take a $(\nu, 5, 3)$ minimal covering design [9]. The excess graph of this design is a 1-factor.

Now apply theorem 3.1 to give the result.

Lemma 9.8. *Let* $\nu \equiv 14 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 13) = \psi(\nu, 5, 13)$.

Proof: For $\nu = 14, 54, 74, 94$ the result follows from the previous lemma. For $\nu = 34$, let $X = Z_{28} \cup H_6$ then the required blocks can be constructed by developing, under the action of Z_{28}, the following base blocks.
$\langle 0\ 5\ 14\ 19\rangle \nu\{h_1, h_2\}$ half orbit
$\langle 0\ 1\ 2\ 4\ 8\rangle\langle 0\ 3\ 12\ 19\ h_1\rangle$ $\langle 0\ 5\ 13\ 18\ h_1\rangle\langle 0\ 1\ 2\ 4\ h_2\rangle$
$\langle 0\ 3\ 9\ 18\ h_2\rangle\langle 0\ 4\ 12\ 18\ h_2\rangle$ $\langle 0\ 5\ 11\ 21\ h_3\rangle\langle 0\ 5\ 12\ 20\ h_3\rangle$
$\langle 0\ 1\ 2\ 6\ h_3\rangle\langle 0\ 2\ 11\ 18\ h_4\rangle$ $\langle 0\ 3\ 11\ 19\ h_4\rangle\langle 0\ 3\ 13\ 17\ h_4\rangle$
$\langle 0\ 4\ 11\ 17\ h_5\rangle\langle 0\ 1\ 2\ 3\ h_5\rangle$ $\langle 0\ 2\ 5\ 10\ h_5\rangle\langle 0\ 3\ 13\ 17\ h_6\rangle$
$\langle 0\ 4\ 11\ 20\ h_6\rangle\langle 0\ 6\ 12\ 19\ h_6\rangle$ $\langle 0\ 5\ 11\ 18\ \nu\{h_i\}_{i=3}^{6}$

For $\nu \equiv 114$, $\nu \neq 134$, write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 6.9. Apply theorem 2.9 with $\lambda = 13$ to give the result.

For $\nu = 134$ apply theorem 2.4 with $n = 7$, $u = 2$, $h = 6$ and $\lambda = 13$.

9.5 $\nu \equiv 18 \pmod{20}$

Lemma 9.9. $\sigma(\nu, 5, 13) = \psi(\nu, 5, 13)$ *for* $\nu = 18, 38, 58, 78, 98$.

Proof: The required constructions are given in the next table.

ν	Point Set	Base Blocks
18	Z_{18}	Take the blocks of a (18,5,6) optimal packing design on Z_{18}. Careful inspection of this design shows that each pair appears precisely 6 times except the pairs $(i, i+9)$, $i = 0, \ldots, 8$, each of them appears 4 times. Take also the following blocks
		$\langle 0\ 1\ 2\ 5\ 9\rangle\langle 0\ 1\ 6\ 10\ 12\rangle\langle 0\ 2\ 7\ 10\ 13\rangle\langle 0\ 1\ 2\ 3\ 6\rangle$
		$\langle 0\ 1\ 4\ 9\ 11\rangle\langle 0\ 2\ 6\ 9\ 14\rangle$
38	$Z_2 \times Z_{19}$	Take a (38,5,6) optimal packing design on Z_{38}. Careful inspection of the design shows that its complement graph consists of two 1-factor. We now take more blocks such that each pair appears precisely 7 times except of 19 pairs each of them appears in 8 blocks, that is, they form 1-factor. Apply theorem 3.1 to give the result. Notice that these blocks are precisely the blocks of a (38,5,7) minimal covering design.

		$\langle(0,0)(0,1)(0,2)(0,3)(0,8)\rangle\langle(0,0)(0,2)(0,6)(0,10)(0,14)\rangle$
		$\langle(1,0)(1,1)(1,3)(1,7)(1,11)\rangle\langle(0,0)(0,3)(0,7)(0,10)(1,0)\rangle$
		$\langle(0,0)(0,3)(0,9)(1,13)(1,18)\rangle\langle(0,0)(0,4)(0,11)(1,5)(1,7)\rangle$
		$\langle(0,0)(0,5)(0,11)(1,7)(1,17)\rangle\langle(0,0)(0,1)(1,0)(1,1)(1,4)\rangle$
		$\langle(0,0)(0,2)(1,4)(1,8)(1,15)\rangle\langle(0,0)(0,3)(1,0)(1,1)(1,3)\rangle$
		$\langle(0,0)(0,4)(1,1)(1,6)(1,12)\rangle\langle(0,0)(0,5)(1,10)(1,14)(1,18)\rangle$
		$\langle(0,0)(0,6)(1,8)(1,10)(1,15)\rangle\langle(0,0)(0,7)(1,5)(1,11)(1,14)$
		$\langle(0,0)(0,8)(1,3)(1,10)(1,15)\rangle\langle(0,0)(0,9)(1,7)(1,8)(1,17)\rangle$
		$\langle(0,0)(0,3)(0,8)(0,13)(1,0)\rangle\langle(0,0)(0,1)(0,2)(1,3)(1,6)\rangle$
		$\langle(0,0)(0,1)(1,0)(1,2)(1,8)\rangle\langle(0,0)(0,2)(1,5)(1,11)(1,12)\rangle$
		$\langle(0,0)(0,3)(1,12)(1,14)(1,17)\rangle\langle(0,0)(0,4)(1,13)(1,15)(1,18)\rangle$
		$\langle(0,0)(0,5)(1,10)(1,15)(1,16)\rangle\langle(0,0)(0,6)(1,3)(1,12)(1,13)\rangle$
		$\langle(0,0)(0,9)(1,6)(1,13)(1,17)\rangle\langle(0,0)(0,2)(0,9)(1,1)(1,14)\rangle$
58	Z_{58}	Take the blocks of a (58,5,6) optimal packing design. Careful inspection of this design shows that its complement graph consists of two 1-factors. Take also the following blocks. In these blocks each pair appears precisely 7 times except the pairs $(i, i+29)$ $(= 0, \ldots, 28)$ which appear 8 times, that is, they form a 1-factor. Now apply theorem 3.1 to give the result
		$\langle 0\ 1\ 3\ 20\ 33\rangle$ twice $\langle 0\ 4\ 9\ 16\ 31\rangle$ twice $\langle 0\ 6\ 14\ 24\ 35\rangle$ twice
		$\langle 0\ 2\ 6\ 13\ 43\rangle\langle 0\ 3\ 12\ 22\ 38\rangle\langle 0\ 1\ 3\ 7\ 21\rangle\langle 0\ 5\ 17\ 30\ 39\rangle$
		$\langle 0\ 8\ 18\ 32\ 43\rangle\langle 0\ 1\ 3\ 7\ 15\rangle\langle 0\ 5\ 13\ 23\ 39\rangle\langle 0\ 5\ 21\ 30\ 41\rangle$
		$\langle 0\ 1\ 3\ 20\ 33\rangle\langle 0\ 4\ 9\ 16\ 31\rangle\langle 0\ 6\ 14\ 24\ 35\rangle\langle 0\ 1\ 2\ 8\ 19\rangle$
		$\langle 0\ 3\ 12\ 32\ 36\rangle\langle 0\ 5\ 15\ 28\ 42\rangle$
78	$Z_{60} \cup H_{18}$	On $Z_{60} \cup H_{16}$ construct a B[76,5,4] with a hole of size 16, say, H_{16}, Such design can be constructed by taking a T[5,4,15]. Add a point to the groups. On the first group construct a B[16,5,4] and consider the last group with the point to be the hole. We also take the following blocks.
		$\langle 0\ 18\ 30\ 48\rangle \cup \{h_{17}, h_{18}\}$ half orbit. $\langle 0\ 3\ 9\ 26\rangle \cup \{h_i\}_{i=1}^{4}$
		$\langle 0\ 5\ 27\ 46\rangle \cup \{h_i\}_{i=5}^{8}\langle 0\ 7\ 17\ 38\rangle \cup \{h_i\}_{i=9}^{12}\langle 0\ 2\ 15\ 29\rangle \cup \{h_i\}_{i=13}^{16}$
		$\langle 0\ 4\ 8\ 16\ 32\rangle\langle 0\ 1\ 11\ 35\ h_1\rangle\langle 0\ 2\ 20\ 34\ h_2\rangle\langle 0\ 6\ 24\ 40\ h_3\rangle$
		$\langle 0\ 2\ 10\ 22\ h_4\rangle\langle 0\ 6\ 14\ 26\ h_5\rangle\langle 0\ 1\ 3\ 10\ h_6\rangle\langle 0\ 4\ 15\ 29\ h_7\rangle$
		$\langle 0\ 5\ 18\ 37\ h_8\rangle\langle 0\ 6\ 23\ 39\ h_9\rangle\langle 0\ 11\ 24\ 45\ h_{10}\rangle\langle 0\ 1\ 2\ 5\ h_{11}\rangle$
		$\langle 0\ 5\ 11\ 32\ h_{12}\rangle\langle 0\ 7\ 22\ 43\ h_{13}\rangle\langle 0\ 9\ 25\ 34\ h_{14}\rangle\langle 0\ 10\ 23\ 41\ h_{15}\rangle$
		$\langle 0\ 1\ 3\ 14\ h_6\rangle\langle 0\ 5\ 25\ 34\ h_{17}\rangle\langle 0\ 7\ 22\ 39\ h_{17}\rangle\langle 0\ 7\ 23\ 42\ h_{17}\rangle$
		$\langle 0\ 8\ 27\ 37\ h_{18}\rangle\langle 0\ 1\ 3\ 6\ h_{18}\rangle\langle 0\ 4\ 11\ 19\ h_{18}\rangle\langle 0\ 6\ 20\ 24\ h_1\rangle$
		$\langle 0\ 8\ 12\ 28\ h_2\rangle\langle 0\ 8\ 10\ 32\ h_3\rangle\langle 0\ 4\ 16\ 22\ h_4\rangle\langle 0\ 9\ 21\ 38\ h_5\rangle$
		$\langle 0\ 10\ 23\ 37\ h_6\rangle\langle 0\ 12\ 25\ 45\ h_7\rangle\langle 0\ 1\ 2\ 5\ h_8\rangle\langle 0\ 3\ 16\ 33\ h_9\rangle$
		$\langle 0\ 7\ 24\ 35\ h_{10}\rangle\langle 0\ 9\ 20\ 39\ h_{11}\rangle\langle 0\ 1\ 6\ 13\ h_{12}\rangle\langle 0\ 3\ 14\ 33\ h_{13}\rangle$
		$\langle 0\ 5\ 15\ 36\ h_{14}\rangle\langle 0\ 7\ 15\ 38\ h_{15}\rangle\langle 0\ 9\ 18\ 35\ h_{16}\rangle$

98	$Z_{80} \cup H_{18}$	On $Z_{80} \cup H_{17}$ construct a B[97,5,10] with a hole of size 17. Such design can be constructed by taking a T[6,5,8]. Inflate this design by a factor of two, that is, replace all the blocks of T[6,5,8] by the blocks of a GD[5,2,2,12], [28]. Add a point to the groups and on the first five groups construct a B[17,5,10]. Take the last group with this point to be the hole. We also take the following blocks.
		$\langle 0\ 16\ 32\ 48\ 64\rangle + i,\ i \in Z_{16}$
		$\langle 0\ 6\ 22\ 30\ 50\rangle\langle 0\ 8\ 20\ 32\ 54\rangle\langle 0\ 4\ 17\ 18\rangle \cup \{h_i\}_{i=1}^{8}$
		$\langle 6\ 21\ 31\ 59\rangle \cup \{h_i\}_{i=1}^{8}\langle 0\ 4\ 17\ 18\rangle \cup \{h_i\}_{i=9}^{16}\langle 6\ 21\ 23\ 59\rangle \cup \{h_i\}_{i=9}^{16}$
		$\langle 0\ 2\ 5\ 43\ h_{18}\rangle\langle 0\ 6\ 25\ 35\ h_{18}\rangle\langle 0\ 7\ 16\ 47\ h_{18}\rangle\langle 0\ 7\ 19\ 48\rangle \cup \{h_1, h_2\}$
		$\langle 0\ 9\ 20\ 43\rangle \cup \{h_3, h_4\}\langle 0\ 21\ 31\ 66\rangle \cup \{h_5, h_6\}\langle 0\ 1\ 4\ 57\rangle \cup \{h_7, h_8\}$
		$\langle 0\ 2\ 7\ 69\rangle \cup \{h_9, h_{10}\}\langle 0\ 5\ 26\ 55\rangle \cup \{h_{11}, h_{12}\}\langle 0\ 6\ 15\ 37\rangle \cup \{h_{13}, h_{14}\}$
		$\langle 0\ 8\ 19\ 47\rangle \cup \{h_{15}, h_{16}\}\langle 0\ 3\ 26\ 47\rangle \cup \{h_{17}, h_{17}, h_{17}, h_{18}\}$.

Lemma 9.10. *Let* $\nu \equiv 18 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 13) = \psi(\nu, 5, 13)$.

Proof: For $\nu = 18, 38, 58, 78, 98$ the result follows from lemma 9.12. For $\nu \geq 118$ the proof is the same as lemma 6.11.

In this section we have proved.

Theorem 9.1. *Let* $\nu \geq 5$ *be a positive integer. Then* $\sigma(\nu, 5, 13) = \psi(\nu, 5, 13)$ *with the possible exception of* $\nu = 28$.

10 Packing with index 14

10.1 $\nu \equiv 4 \pmod{20}$

Lemma 10.1. *Let* $\nu \equiv 4 \pmod{10}$ *be a positive integer and assume that there exists*

1) *a* $(\nu, 5, 4)$ *minimal covering design;*

2) *a* $(\nu, 5, 4)$ *optimal packing design;*

3) *a* $(\nu, 5, 2)$ *packing design with a hole of size 4.*

Then there exists a $(\nu, 5, 14)$ *optimal packing design.*

Proof: A $(\nu, 5, 14)$ optimal packing design for $\nu \equiv 4 \pmod{10}$ can be constructed as follows:

1) Take a $(\nu, 5, 4)$ minimal covering design [13]. In this design each pair appears in precisely 4 blocks except a triple, say, $\{a, b, c\}$ the pairs of which appear in 6 blocks.

2) Again take a $(\nu, 5, 4)$ minimal covering design. In this design assume the triple is $\{a, c, d\}$.

3) Take a $(\nu, 5, 4)$ optimal packing design [10]. In this design each pair appears in precisely 4 blocks except one pair, say, $\{a, c\}$ which appears in zero blocks.

4) Take a $(\nu, 5, 2)$ packing design with a hole of size 4, say, $\{a, b, c, d\}$.

Now it is easily checked that the above four steps yield a $(\nu, 5, 14)$ optimal packing design.

Lemma 10.2. *Let $\nu \equiv 4 \pmod{20}$ be a positive integer. Then there exists a $(\nu, 5, 2)$ packing design with a hole of size 4.*

Proof: For $\nu = 24, 44$ the result is given in [4].

For $\nu = 64, 84$ the constructions are given in the next table.

ν	Point Set	Base Blocks
64	$Z_2 \times Z_{30} \cup H_4$	$\langle(0,0)(0,6)(0,12)(0,18)(0,24)\rangle + (-,i),\ i \in Z_6$
		$\langle(i,0)(i,2)(i,5)(i,13)(i,20)\rangle i = 0,1$
		$\langle(0,0)(0,2)(1,1)(1,10)(1,24)\rangle\langle(0,0)(0,1)(0,4)(1,7)(1,13)\rangle$
		$\langle(0,0)(0,9)(0,14)(1,11)(1,28)\rangle\langle(0,0)(0,13)(1,0)(1,4)(1,18)\rangle$
		$\langle(0,0)(0,8)(0,14)(1,13)(1,16)\rangle\langle(0,0)(0,4)(1,0)(1,1)(1,10)\rangle$
		$\langle(0,0)(0,11)(1,4)(1,23)(1,28)\rangle\langle(0,0)(0,10)(1,25)(1,26)h_1\rangle$
		$\langle(0,0)(0,1)(1,21)(1,25)h_2\rangle\langle(0,0)(0,7)(1,14)(1,22)h_3\rangle$
		$\langle(0,0)(0,9)(1,18)(1,20)h_4\rangle$
84	$Z_2 \times Z_{40} \cup H_4$	$\langle(0,0)(0,8)(0,16)(0,24)(0,32)\rangle + (-,i),\ i \in Z_8$
		$\langle(i,0)(i,2)(i,7)(i,19)(i,29)\rangle i = 0,1$
		$\langle(0,0)(0,18)(1,0)(1,14)(1,39)\rangle\langle(0,0)(0,2)(1,1)(1,4)(1,21)\rangle$
		$\langle(0,0)(0,4)(1,10)(1,32)(1,38)\rangle\langle(0,0)(0,8)(1,13)(1,24)(1,37)\rangle$
		$\langle(0,0)(0,5)(0,15)(1,23)(1,27\rangle\langle(0,0)(0,6)(0,19)(1,26)(1,35)\rangle$
		$\langle(0,0)(0,6)(0,20)(1,15)(1,31)\rangle\langle(0,0)(0,1)(0,4)(0,15)(1,24)\rangle$
		$\langle(0,0)(1,0)(1,1)(1,3)(1,7)\rangle\langle(0,0)(0,9)(0,16)(1,27)(1,35)\rangle$
		$\langle(0,0)(0,17)(1,5)(1,10)(1,31)\rangle\langle(0,0)(0,1)(1,30)(1,38)h_1\rangle$
		$\langle(0,0)(0,9)(1,2)(1,17)h_2\rangle\langle(0,0)(0,3)(1,6)(1,15)h_3\rangle$
		$\langle(0,0)(0,12)(1,16)(1,25)h_4\rangle$

For $\nu \geq 104$, $\nu \neq 144$, write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen the same as in lemma 5.6. Applying theorem 2.9 with the appropriate parameters gives that for all $\nu \geq 104$, $\nu \neq 144$, there exists a $(\nu, 5, 2)$ packing design with a hole of size $h = 4, 24, 44, 64$, or 84. But since there exists a $(h, 5, 2)$ packing design with a hole of size 4 for $h = 4, 24, 44, 64, 84$, it follows that for all $\nu \equiv 4 \pmod{20}$, $\nu \neq 144$, there exists a $(\nu, 5, 2)$ packing design with a hole of size 4. For $\nu = 144$ apply theorem 2.13 with $m = 7$, $s = h = 0$ and $\lambda = 2$ gives us a (144,5,2) packing design with a hole of size 24 and hence with a hole of size 4.

Lemma 10.3. *Let* $\nu \equiv 4 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 14) = \psi(\nu, 5, 14)$.

Proof: We have shown, lemma 10.2, that for all $\nu \equiv 4 \pmod{20}$ there exists a $(\nu, 5, 2)$ packing design with a hole of size 4. We also have shown that for all $\nu \equiv 4 \pmod{20}$, there exists a $(\nu, 5, 4)$ minimal covering design, [13], [17] and for all $\nu \equiv 4 \pmod{20}$ there exists a $(\nu, 5, 4)$ optimal packing design [10]. Now apply lemma 10.1 to get the result.

10.2 $\nu \equiv 8 \pmod{10}$

Lemma 10.4. *Let* $\nu \equiv 8 \pmod{10}$ *be a positive integer. Then* $\sigma(\nu, 5, 14) = \psi(\nu, 5, 14)$.

Proof: In this case $\sigma(\nu, 5, 14) = \sigma(\nu, 5, 8) + \sigma(\nu, 5, 6)$.

10.3 $\nu \equiv 12 \pmod{20}$

Lemma 10.5. *Let* $\nu \equiv 12 \pmod{20}$ *be a positive integer. If there exists a* $(\nu, 5, 2)$ *packing design with a hole of size 2 then there exists a* $(\nu, 5, 14)$ *optimal packing design.*

Proof: The blocks of a $(\nu, 5, 14)$ optimal packing design, $\nu \equiv 12 \pmod{20}$, can be constructed as follows:

1) Take a $(\nu, 5, 4)$ minimal covering design. In this design there is a triple, say, $\{a, b, c\}$ the pairs of which appear in 6 blocks while each other pair appears in precisely 4 blocks.

2) Take two copies of a $(\nu, 5, 4)$ optimal packing design [10]. This design has a pair that appears in zero blocks while each other pair appears in 4 blocks. Assume that in the first copy the missing pair is (a, b) and in the second copy the missing pair is (a, c).

3) Take a $(\nu, 5, 2)$ packing design with a hole of size 2 and assume the hole is $\{b, c\}$ [4].

It is easily checked that these three steps yield the blocks of a $(\nu, 5, 14)$ optimal packing design for $\nu \equiv 12 \pmod{20}$.

Lemma 10.6. *There exists a* $(\nu, 5, 2)$ *packing design with a hole of size 2 for all positive integers* $\nu \equiv 12 \pmod{20}$.

Proof: For $\nu = 12, 52, 92$ the result is given in [4].

For $\nu = 32$ let $X = Z_{30} \cup H_2$ then the blocks are.
$\langle 0\ 6\ 12\ 18\ 24 \rangle + i,\ i \in Z_6$ $\langle 0\ 1\ 2\ 4\ 10 \rangle \pmod{30}$
$\langle 0\ 3\ 11\ 16\ 20 \rangle \pmod{30}$ $\langle 0\ 5\ 12\ 19 \rangle \pmod{30} \cup \{h_1, h_2\}$.

For $\nu = 72$ take a T[6,2,3] and inflate it by a factor of 4, that is, replace each block by the blocks of a GD[5,1,4,24]. Finally, on the first five groups construct a (12,5,2) optimal packing design and on the last group construct a (12,5,2) packing design with a hole of size 2.

For $\nu \geq 112$, $\nu \neq 132$, write $\nu = 20m+4u+h+s$ where m, u, h and s are chosen the same as in lemma 8.4. Applying theorem 2.9 with $\lambda = 2$ gives us that for all $\nu \geq 112$, $\nu \neq 132$, there exists a $(\nu, 5, 2)$ packing design with a hole of size 12, 32, 52, 72 or 92. But a $(\nu, 5, 2)$ packing design with a hole of size 2 exists for $\nu = 12, 32, 52, 72, 92$. Hence for all $\nu \equiv 12 \pmod{20}$, $\nu \neq 132$, there exists a $(\nu, 5, 2)$ packing design with a hole of size 2.

For $\nu = 132$ applying theorem 2.4 with $n = 7$, $u = 2$, $h = 0$ and $\lambda = 2$ gives a (132,5,2) packing design with a hole of size 12 and hence a (132,5,2) packing design with a hole of size 2.

Corollary 10.1. *$\sigma(\nu, 5, 14) = \psi(\nu, 5, 14)$ for all positive integers ν, $\nu \equiv 12 \pmod{20}$.*

Proof: By lemma 10.6 there exists a $(\nu, 5, 2)$ packing design with a hole of size 2. There is also a $(\nu, 5, 4)$ optimal packing design, [10], and a $(\nu, 5, 4)$ minimal covering design, for all $\nu \equiv 12 \pmod{20}$, [13]. Now apply lemma 10.5 and the result follows.

10.4 $\nu \equiv 2 \pmod{20}$

Lemma 10.7. *Let $\nu \equiv 2 \pmod{20}$ be a positive integer. Furthermore, assume*

1) *There exists a $(\nu, 5, 2)$ packing design with a hole of size 2.*

2) *There exists a $(\nu, 5, 4)$ optimal packing design.*

3) *There exists a $(\nu, 5, 4)$ minimal covering design.*

Then there exists a $(\nu, 5, 14)$ optimal packing design.

Proof: The proof of this lemma is the same as lemma 10.5.

Lemma 10.8. *There exists a $(\nu, 5, 2)$ packing design with a hole of size 2 for all positive integers ν, $\nu \equiv 2 \pmod{20}$.*

Proof: For $\nu = 22, 62, 82$ see [4].

For $\nu = 42$ let $X = Z_{40} \cup H_2$ then the blocks are
$\langle 0\ 8\ 16\ 24\ 32\rangle + i,\ i \in Z_8$ $\langle 0\ 1\ 2\ 5\ 11\rangle \pmod{40}$
$\langle 0\ 4\ 12\ 23\ 30\rangle \pmod{40}$ $\langle 0\ 3\ 18\ 23\rangle \pmod{40} \cup \{h_1, h_2\}$.

For $\nu \geq 102$, $\nu \neq 142, 182$ write $\nu = 20m+4u+h+s$ where m, u, h and s are chosen as in lemma 5.3 with the difference that $4u+h+s = 22, 42, 62, 82$ and $h = 6$.

Now applying theorem 2.9 with $\lambda = 2$ gives us that there exists a $(\nu, 5, 2)$ packing design with a hole of size 2, 22, 42, 62 or 82 (see [4] for the existence of a (26,5,2) packing design with a hole of size 6). But a $(\nu, 5, 2)$ packing design with a hole of size 2 exists for $\nu = 22, 42, 62, 82$. Hence there exists a $(\nu, 5, 2)$ packing design with a hole of size 2 for all positive integers ν, $\nu \equiv 2 \pmod{20}$, $\nu \neq 142, 182$.

For $\nu = 142$ applying theorem 2.4 with $n = 7$, $h = 2$, $u = 5$ and $\lambda = 2$ gives a (142,5,2) packing design with a hole of size 22 and hence a (142,5,2) packing design with a hole of size 2.

For $\nu = 182$ applying theorem 2.13 (or theorem 2.12) with $m = s = 8$ and $h = 6$ gives us a (182,5,2) packing design with a hole of size 42 and hence a (182,5,2) packing design with a hole of size 2.

Corollary 10.2. *Let* $\nu \equiv 2 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 14) = \psi(\nu, 5, 14)$.

Proof: By lemma 10.8 there exists a $(\nu, 5, 2)$ packing design with a hole of size 2 for all $\nu \equiv 2 \pmod{20}$. There is also a $(\nu, 5, 4)$ optimal packing design, [10], for all $\nu \equiv 2 \pmod{20}$.

On the other hand, for all $\nu \equiv 2 \pmod{20}$ there exists a $(\nu, 5, 4)$ minimal covering design [13], [17]. Now apply lemma 10.7 and the result follows.

10.5 $\nu \equiv 14 \pmod{20}$

Lemma 10.9. *Let* $\nu \equiv 14 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 14) = \psi(\nu, 5, 14)$.

Proof: We first prove the lemma for $\nu = 14, 34, 54, 74, 94$. But for $\nu = 34, 54, 74, 94$ there exists (1) a $(\nu, 5, 4)$ minimal covering design, [13] (2) a $(\nu, 5, 4)$ optimal packing design, [10]; (3) a $(\nu, 5, 2)$ packing design with a hole of size 4: see [4] for $\nu = 34, 54, 94$ and lemma 9.6 for $\nu = 74$. Now apply lemma 10.1 and the result follows.

For $\nu = 14$ let $X = Z_{14}$ then the required blocks are
$\langle 0\ 1\ 2\ 3\ 6 \rangle \pmod{14}$, 3 times $\langle 0\ 1\ 4\ 8\ 10 \rangle \pmod{14}$ twice
$\langle 0\ 2\ 5\ 7\ 10 \rangle \pmod{14}$ twice $\langle 0\ 1\ 3\ 7\ 9 \rangle \pmod{14}$
$\langle 0\ 1\ 4\ 8\ 9 \rangle \pmod{14}$.

We now show that a (22,5,14) packing design with a hole of size two exists. The blocks of this design are the blocks of a (22,5,8), (22,5,4) and (22,5,2) packing designs with a hole of size 2.

For $\nu \geq 114$, $\nu \neq 134$, write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen the same as in lemma 9.8 with the difference $h = 2$. Now apply theorem 2.9 with $\lambda = 14$ and the result follows.

For $\nu = 134$ apply theorem 2.4 with $m = 7$, $h = 2$, $u = 3$ and $\lambda = 14$ to give the result.

In this section we have shown.

Theorem 10.1. *Let $\nu \geq 5$ be a positive integer then $\sigma(\nu, 5, 14) = \psi(\nu, 5, 14)$.*

11 Packing with index 15

11.1 $\nu \equiv 4, 8$ or 12 (mod 20)

Lemma 11.1. *Let $\nu \equiv 4, 8$ or 12 (mod 20) be a positive integer. Then $\sigma(\nu, 5, 15) = \psi(\nu, 5, 15)$ with the possible exception of $\nu = 28$.*

Proof: If $\nu \equiv 4 \pmod{20}$ then $\sigma(\nu, 5, 15) = \sigma(\nu, 5, 8) + \sigma(\nu, 5, 7)$.

Since a (44,5,7) optimal packing design is still unknown, we need to construct a (44,5,15) optimal packing design. For this purpose we first show that a (44,5,3) packing design with a hole of size 4 exists. Such design can be constructed as follows:

1) Take a (42,5,1) optimal packing design, [9], and assume the pair (41,42) appears in zero blocks.

2) Take a B[45,5,1] and assume we have the block ⟨41 42 43 44 45⟩. Delete this block and in the remaining blocks change 45 to 44.

3) One more time take a B[45,5,1] and assume we have the block ⟨41 42 43 44 45⟩. Delete this block and in the remaining blocks change 45 to 43. It is easy to check that the above three steps yield a (44,5,3) packing design with a hole of size 4.

We now construct a (44,5,15) optimal packing design as follows.

1) Take a (44,5,4) minimal covering design. In this design each pair appears in exactly four blocks except a triple, say, $\{a, b, c\}$ the pairs of which appear in 6 blocks.

2) Again take a (44,5,4) minimal covering design. In this case assume the triple is $\{a, c, d\}$.

3) Take a (44,5,4) optimal packing design. In this design there is exactly one pair that appears in zero blocks. Assume the pair is (a, c).

4) Take a (44,5,3) packing design with a hole of size 4, say, $\{a, b, c, d\}$.

Now it is easy to check that the above four steps yield a (44,5,15) optimal packing design.

For $\nu \equiv 8 \pmod{20}$, $\sigma(\nu, 5, 15) = \sigma(\nu, 5, 8) + \sigma(\nu, 5, 7)$.

For $\nu \equiv 12 \pmod{20}$, $\sigma(\nu, 5, 15) = \sigma(\nu, 5, 11) + \sigma(\nu, 5, 4)$.

11.2 $\nu \equiv 2 \pmod{20}$

Lemma 11.2. a) *Let* $\nu \equiv 2 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 15) = \psi(\nu, 5, 15)$. b) *There exists a (22,5,15) packing design with a hole of size 2.*

Proof: (a) The blocks of a $(\nu, 5, 15)$ optimal packing design for all positive integers $\nu \equiv 2 \pmod{20}$, can be constructed as follows:

1) Take a $(\nu, 5, 4)$ minimal covering design which exists for all positive integers $\nu \equiv 2 \pmod{20}$ [13],[17]. This design contains a triple, say, $\{a, b, c\}$ the pairs of which appear in 6 blocks while each other pair appears in exactly 4 blocks.

2) Take two copies of a $(\nu, 5, 4)$ optimal packing designs. This design contains a pair that appears in zero blocks. Assume that the pair in the first copy is (a, b) and in the second copy (b, c).

3) Take a $(\nu, 5, 3)$ packing design with a hole of size 2, [7]. Assume the hole is (a, c).

Now it is easy to check that the above three steps yield the blocks of a $(\nu, 5, 15)$ optimal packing design for all positive integers $\nu \equiv 2 \pmod{20}$.
(b) For a (22,5,15) packing design with a hole of size two take three copies of a (22,5,4) and one copy of a (22,5,3) packing design with a hole of size 2, [10] [7].

11.3 $\nu \equiv 14 \pmod{20}$

Lemma 11.3. *Let* $\nu \equiv 14 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 15) = \psi(\nu, 5, 15)$.

Proof: If $\nu \equiv 14 \pmod{20}$ then $\sigma(\nu, 5, 15) = \sigma(\nu, 5, 11) + \sigma(\nu, 5, 4)$.

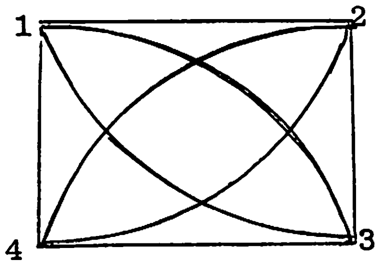
11.4 $\nu \equiv 18 \pmod{20}$

Lemma 11.4. *Let* $\nu \equiv 18 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 15) = \psi(\nu, 5, 15)$.

Proof: The blocks of a $(\nu, 5, 15)$ optimal packing design for all $\nu \equiv 18 \pmod{20}$ can be constructed as follows:

1) Take a $(\nu, 5, 4)$ minimal covering design, [13], [14]. In lemma 5.8 we have shown that for all $\nu \equiv 18 \pmod{20}$, $\nu \neq 18, 178$ the excess graph

of this design consists of $\nu-4$ isolated vertices and the following graph on the remaining four vertices, say, $\{1,2,3,4\}$.



2) Take a $(\nu,5,8)$ optimal packing design. In this design each pair appears in exactly 8 blocks except one pair say (1,3) which appears in 4 blocks.

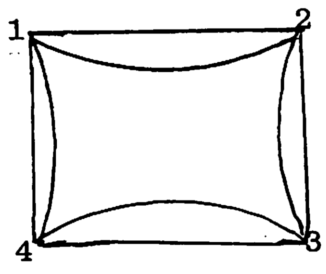
3) Take a $(\nu-1,5,2)$ optimal packing design and assume that the pair (2,4) appears in at most one block.

4) Take a $(\nu+2,5,1)$ optimal packing design and assume that the pairs of the quadruples $\{1,2,3,4\}$ and $\{\nu-1,\nu,\nu+1,\nu+2\}$ appears in zero blocks. Now change $\nu+1$ and $\nu+2$ to ν.

Now it is easily checked that the above 4 steps give a $(\nu,5,15)$ optimal packing design for all positive integers $\nu \equiv 18 \pmod{20}$ $\nu \neq 18, 178$.

For $\nu = 18$, an (18,5,15) optimal packing design can be constructed as follows:

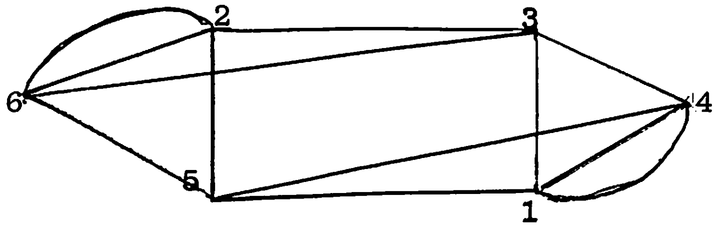
1) Take an (18,5,4) minimal covering design [12]. The excess graph of this design consists of $\nu-4$ isolated vertices and the following graph on the remaining four vertices, say, $\{1,2,3,4\}$.



2) Take an (18,5,8) optimal packing design. In this design each pair appears in exactly 8 blocks except one pair, say, (1,2) which appears in 4 blocks.

3) Take a (17,5,2) optimal packing design. The complement graph of this design consists of 11 isolated vertices and the following graph on

the remaining 6 vertices.



4) Take a (20,5,1) optimal packing design and assume that the pairs of the quadruples $\{1,2,3,4\}$ and $\{17,18,19,20\}$ appears in zero blocks. Now change 19 and 20 to 18.

It is easy to check that the above construction yields an (18,5,15) optimal packing design.

For a (178,5,15) optimal packing design apply theorem 2.9 with $m = 7$, $u = 6$, $s = 8$ and $h = 6$ to give a (178,5,15) with a hole of size 38. The application of this theorem requires a (26,5,15) packing design with a hole of size 6. Such design can be constructed by taking the blocks of a (26,5,3) packing design with a hole of size 6 [7] together with four copies of a B[26,5,4] with a hole of size 6 (see lemma 5.9, case 5 for a B[26,5,4] with a hole of size 6).

In this section we have shown.

Theorem 11.1. *Let $\nu \geq 5$ be a positive integer. Then $\sigma(\nu,5,15) = \psi(\nu,5,15)$ with the possible exception of $\nu = 28$.*

12 Packing with index 17

12.1 $\nu \equiv 4, 8$ or 12 (mod 20)

Lemma 12.1. *Let $\nu \equiv 4, 8$ or 12 (mod 20) be a positive integer. Then $\sigma(\nu,5,17) = \psi(\nu,5,17)$.*

Proof: If $\nu \equiv 4$ or 8 (mod 20) then $\sigma(\nu,5,17) = \sigma(\nu,5,10) + \sigma(\nu,5,7)$.

Since a (28,5,7) and a (44,5,7) optimal packing designs are still unknown, the above method does not work for $\nu = 28, 44$.

For $\nu = 28$ see the next table.

For $\nu = 44$, $\sigma(\nu,5,17) = \sigma(\nu,5,14) + \sigma(\nu,5,3)$.

For $\nu = 12, 52, 72, 92$, $\sigma(\nu,5,17) = \sigma(\nu,5,14) + \sigma(\nu,5,3)$.

For $\nu = 32$ see the next table.

For $\nu \geq 112$ the proof of this case is the same as lemma 8.4.

ν	Point Set	Base Blocks
28	Z_{28}	On Z_{28} construct a (28,5,8) optimal packing design and take the following blocks.
		⟨0 1 2 3 8⟩ 3 times ⟨0 2 6 14 19⟩ twice ⟨0 3 10 14 19⟩ twice
		⟨0 3 10 16 20⟩ twice ⟨0 2 7 15 18⟩⟨0 3 9 13 19⟩⟨0 4 9 17 21⟩
32	Z_{32}	On Z_{32} construct a (32,5,4) optimal packing design and take the following blocks
		On Z_{32} construct a (32,5,4) optimal packing design and take the following blocks.
		⟨0 1 2 4 11⟩ 4 times ⟨0 3 8 15 21⟩ 4 times
		⟨0 4 10 19 24⟩ 4 times ⟨0 1 4 15 20⟩⟨0 2 10 16 23⟩
		⟨0 1 2 4 20⟩⟨0 3 8 17 27⟩⟨0 4 10 16 25⟩⟨0 1 2 4 10⟩
		⟨0 3 7 15 22⟩⟨0 5 11 16 23⟩

12.2 $\nu \equiv 2 \pmod{20}$

Lemma 12.2. (a) *Let* $\nu \equiv 2 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 17) = \psi(\nu, 5, 17)$. (b) *There exists a (26,5,17) packing design with a hole of size 6.*

Proof: (a) For a $(\nu, 5, 17)$ optimal packing design $\nu \equiv 2 \pmod{20}$, $\nu \geq 22$ proceed as follows:

1) take a $(\nu, 5, 4)$ minimal covering design [13], [17]. The excess graph of this design consists of $\nu - 3$ isolated vertices and 3 other vertices, say, $\{a, b, c\}$ the pairs of which are connected by two edges.

2) take two copies of $(\nu, 5, 4)$ optimal packing design. The complement graph of this design consists of $\nu - 2$ isolated vertices and two other vertices connected by 4 edges. Assume in the first copy the two vertices are (a,b) and in the second copy the two vertices are (a,c).

3) take a $(\nu, 5, 5)$ packing design with a hole of size two, say, (b,c), [8]. (Close observation of these designs shows that these designs have a hole of size 2). It is easy to check that these three steps give a $(\nu, 5, 17)$ optimal packing design for all positive integers $\nu \equiv 2 \pmod{20}$.

(b) For a (26,5,17) packing design with a hole of size 6 take a (26,5,5) packing design with a hole of size 6, [8], and a B[26,5,12] with a hole of size 6. To construct a B[26,5,12] with a hole of size 6 take a T[5,12,5], add a new point to the groups. On the first four groups construct a B[6,5,12] and take the last group with the point to be the hole.

12.3 $\nu \equiv 14 \pmod{20}$

Lemma 12.3. *Let $\nu \equiv 14 \pmod{20}$ be a positive integer. If there exists a $(\nu, 5, 11)$ optimal packing design such that its complement graph consists of three 1-factor then $\sigma(\nu, 5, 17) = \psi(\nu, 5, 17)$.*

Proof: A $(\nu, 5, 17)$ optimal packing design, $\nu \equiv 14 \pmod{20}$, can be constructed as follows:

1) Take two copies of a $(\nu, 5, 3)$ minimal covering design. The excess graph of each $(\nu, 5, 3)$ minimal covering is a 1-factor.

2) Take a $(\nu, 5, 11)$ optimal packing design such that its complement graph consists of three 1-factor. Now apply theorem 3.1 and the result follows.

Lemma 12.4. $\sigma(\nu, 5, 17) = \psi(\nu, 5, 17)$ *for* $\nu = 14, 34, 54, 74, 94$.

Proof: In view of the previous lemma we need to show that there exists a $(\nu, 5, 3)$ minimal covering design such that its excess graph is a 1-factor, for this purpose see [9]. We also need to show that there exists a $(\nu, 5, 11)$ optimal packing such that its complement graph is a three 1-factor for this purpose see lemma 8.6.

Lemma 12.5. *Let $\nu \equiv 14 \pmod{20}$ be a positive integer. Then $\sigma(\nu, 5, 17) = \psi(\nu, 5, 17)$.*

Proof: For $\nu = 14, 34, 54, 74, 94$ the result follows from lemma 12.4.

For $\nu \geq 114$, $\nu \neq 134$ write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 6.9. Apply theorem 2.9 with $\lambda = 17$ to give the result.

For $\nu = 134$ apply theorem 2.4 with $u = 2$, $h = 6$, $n = 7$, and $\lambda = 17$.

12.4 $\nu \equiv 18 \pmod{20}$

Lemma 12.6. $\sigma(\nu, 5, 17) = \psi(\nu, 5, 17)$ *for* $\nu = 18, 38, 58, 78, 98$.

Proof: For an (18,5,17) optimal packing design proceed as follows:

1) Take an (18,5,5) optimal packing design [8]. In this design the pair (6,12) appears exactly three times and the pairs (5,12), (7,12) (2,6) (6,16) appear exactly 4 times.

2) Take an (18,5,12) minimal covering design. In this design each pair appears in precisely 12 blocks except one pair say, (6,12) that appears in 16 blocks [17].

The above two steps give a design where (5,12), (2,6), (6,16) and (7,12) appear exactly 16 times and the pair (6,12) appears 19 times. To reduce the appearance of (6,12) from 19 to 17 assume in design (2) we have the following two blocks

$$\langle a\ b\ c\ 6\ 12\rangle \qquad \langle d\ e\ f\ 6\ 12\rangle$$

where $\{a, b, c, d, e, f\}$ are arbitrary numbers not necessarily disjoint. In the first block change 12 to 16 and in the second change 12 to 2.

Furthermore, assume in this design we have the two blocks

$$\langle a\ b\ c\ 7\ 16\rangle \qquad \langle d\ e\ f\ 5\ 2\rangle$$

In the first block change 16 to 12 and in the second block change 2 to 12.

It is readily checked that the above construction yields the block of an (18,5,17) optimal packing design.

For the other values see next table.

v	Point Set	Base Blocks
38	$Z_{32} \cup H_6$	On $Z_{32} \cup H_2$ construct a (34,5,8) packing design with a hole of size 2, [12] say, $\{h_1, h_2\}$ and take the following blocks
		$\langle 0\ 3\ 16\ 19\rangle \cup \{h_1, h_2\}$ half orbit $\langle 0\ 2\ 6\ 14\ 24\rangle$
		$\langle 0\ 2\ 8\ 14\ h_1\rangle\langle 0\ 2\ 10\ 14\ h_2\rangle$*twice*$\langle 0\ 1\ 2\ 11\ h_1\rangle$
		$\langle 0\ 3\ 15\ 18\ h_3\rangle\langle 0\ 4\ 11\ 17\ h_4\rangle\langle 0\ 5\ 11\ 24\ h_5\rangle\langle 0\ 5\ 9\ 12\ h_6\rangle$
		$\langle 0\ 1\ 7\ 16\ h_3\rangle\langle 0\ 1\ 2\ 5\ h_4\rangle\langle 0\ 2\ 5\ 13\ h_5\rangle\langle 0\ 5\ 12\ 23\ h_6\rangle$
		$\langle 0\ 6\ 13\ 23\ h_3\rangle\langle 0\ 1\ 2\ 9\ h_4\rangle\langle 0\ 3\ 7\ 16\ h_5\rangle\langle 0\ 5\ 11\ 22\ h_6\rangle$
		$\langle 0\ 3\ 13\ 20\ h_3\rangle\langle 0\ 4\ 12\ 21\ h_4\rangle\langle 0\ 5\ 14\ 21\ h_6\rangle\langle 0\ 3\ 13\ 18\rangle \cup \{h_1\}_{i=3}^{6}$
58	$Z_{52} \cup H_6$	On Z_{52} construct a (52,5,11) optimal packing design. The complement graph of this design is a 1-factor. Assume the 1-factor is $(i, i+26)$ $i = 0, \ldots, 25$ and take the following blocks.
		$\langle 0\ 9\ 22\ 27\rangle \cup \{h_i\}_{i=3}^{6}\langle 0\ 11\ 26\ 37\rangle \cup \{h_1, h_2\}$ half orbit
		$\langle 0\ 2\ 4\ 6\ h_1\rangle\langle 0\ 4\ 10\ 20\ h_2\rangle\langle 0\ 6\ 16\ 34\ h_3\rangle\langle 0\ 8\ 20\ 32\ h_4\rangle$
		$\langle 0\ 8\ 22\ 34\ h_5\rangle\langle 0\ 8\ 22\ 36\ h_6\rangle\langle 0\ 1\ 2\ 5\ h_1\rangle\langle 0\ 3\ 9\ 22\ h_2\rangle$
		$\langle 0\ 7\ 21\ 32\ h_3\rangle\langle 0\ 7\ 17\ 36\ h_4\rangle\langle 0\ 8\ 23\ 35\ h_5\rangle\langle 0\ 1\ 3\ 16\ h_6\rangle$
		$\langle 0\ 5\ 23\ 32\ h_1\rangle\langle 0\ 7\ 17\ 38\ h_2\rangle\langle 0\ 8\ 19\ 31\ h_3\rangle\langle 0\ 1\ 4\ 19\ h_4\rangle$
		$\langle 0\ 5\ 20\ 29\ h_5\rangle\langle 0\ 6\ 19\ 27\ h_6\rangle\langle 0\ 7\ 24\ 35\ h_1\rangle\langle 0\ 1\ 2\ 5\ h_2\rangle$
		$\langle 0\ 3\ 12\ 25\ h_3\rangle\langle 0\ 5\ 15\ 26\ h_4\rangle\langle 0\ 6\ 17\ 39\ h_5\rangle\langle 0\ 7\ 14\ 23\ h_6\rangle$
78	$Z_{72} \cup H_6$	On $Z_{72} \cup \{h_1, h_2\}$ construct a (74,5,12) packing with a hole of size 2, say, $\{h_1, h_2\}$, and take the following blocks.
		$\langle 0\ 17\ 36\ 53\rangle \cup \{h_1, h_2\}$ half orbit $\langle 0\ 17\ 30\ 35\rangle \cup \{h_i\}_{i=3}^{6}$
		$\langle 0\ 2\ 6\ 14\ 34\rangle$ twice $\langle 0\ 7\ 23\ 33\ 48\rangle$ twice $\langle 0\ 9\ 22\ 43\ 54\rangle$ twice

ν	Point Set	Base Blocks
		$\langle 0\ 2\ 6\ 26\ h_1\rangle\langle 0\ 8\ 22\ 50\ h_2\rangle\langle 0\ 12\ 28\ 54\ h_3\rangle\langle 0\ 1\ 2\ 5\ h_4\rangle$
		$\langle 0\ 3\ 9\ 16\ h_5\rangle\langle 0\ 8\ 33\ 43\ h_6\rangle\langle 0\ 11\ 30\ 45\ h_1\rangle\langle 0\ 14\ 31\ 51\ h_2\rangle$
		$\langle 0\ 1\ 4\ 9\ h_3\rangle\langle 0\ 1\ 7\ 52\ h_4\rangle\langle 0\ 10\ 23\ 53\ h_5\rangle\langle 0\ 11\ 23\ 58\ h_6\rangle$
		$\langle 0\ 15\ 31\ 48\ h_1\rangle\langle 0\ 1\ 3\ 24\ h_2\rangle\langle 0\ 3\ 18\ 29\ h_3\rangle\langle 0\ 5\ 12\ 45\ h_4\rangle$
		$\langle 0\ 5\ 22\ 41\ h_5\rangle\langle 0\ 9\ 19\ 44\ h_6\rangle$
98	$Z_{92} \cup H_6$	On $Z_{92} \cup H_2$ construct a (94,5,12) packing design with a hole of size 2, say, $\{h_1, h_2\}$. Take also the following blocks
		$\langle 0\ 2\ 46\ 48\ h_1\rangle$ half orbit $\langle 0\ 7\ 21\ 54\rangle \cup \{h_i\}_{i=3}^{6}$
		$\langle 0\ 2\ 5\ 16\ 40\rangle$ twice $\langle 0\ 4\ 19\ 52\ 69\rangle$ twice $\langle 0\ 6\ 26\ 47\ 55\rangle$ twice
		$\langle 0\ 10\ 28\ 58\ 70\rangle$ twice $\langle 0\ 7\ 36\ 46\ 68\rangle\langle 0\ 9\ 13\ 25\ 43\rangle$
		$\langle 0\ 7\ 25\ 39\ 45\rangle\langle 0\ 10\ 29\ 59\rangle \cup \{h_1, h_1, h_1, h_2\}\langle 0\ 8\ 14\ 36\ h_2\rangle$
		$\langle 0\ 1\ 2\ 5\ h_3\rangle$twice$\langle 0\ 8\ 17\ 39\ h_4\rangle\langle 0\ 13\ 47\ 67\ h_5\rangle\langle 0\ 11\ 35\ 66\ h_6\rangle$
		$\langle 0\ 13\ 28\ 69\ h_4\rangle\langle 0\ 7\ 15\ 24\ h_5\rangle\langle 0\ 9\ 19\ 40\ h_6\rangle\langle 0\ 11\ 37\ 62\ h_3\rangle$
		$\langle 0\ 12\ 33\ 65\ h_4\rangle\langle 0\ 13\ 36\ 63\ h_5\rangle\langle 0\ 7\ 16\ 35\ h_6\rangle\langle 0\ 1\ 6\ 17\ h_3\rangle$
		$\langle 0\ 3\ 23\ 58\ h_4\rangle\langle 0\ 12\ 27\ 53\ h_5\rangle\langle 0\ 13\ 31\ 56\ h_6\rangle$

Lemma 12.7. *Let* $\nu \equiv 18 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu, 5, 17) = \psi(\nu, 5, 17)$.

Proof: For $\nu = 18, 38, 58, 78, 98$ the result follows from lemma 12.6. For $\nu \geq 118$, $\nu \neq 138$ write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 6.11. Now apply theorem 2.9 with $\lambda = 17$ to give the result.

For $\nu = 138$ apply theorem 2.4 with $h = 6$, $\lambda = 17$, $n = 7$ and $u = 3$.

To summarize this section, we have proved:

Theorem 12.1. *Let* $\nu \geq 5$ *be a positive integer. Then* $\sigma(\nu, 5, 17) = \psi(\nu, 5, 17)$.

13 Packing with index 18

Theorem 13.1. *Let* $\nu \geq 5$ *be a positive integer. Then* $\sigma(\nu, 5, 18) = \psi(\nu, 5, 18)$.

Proof: It is clear that we only need to do the cases $\nu \equiv 2, 4$ or $8 \pmod{10}$. But for $\nu \equiv 2$ or $4 \pmod{10}$, $\sigma(\nu, 5, 18) = \sigma(\nu, 5, 14) + \sigma(\nu, 5, 4)$ and for $\nu \equiv 8 \pmod{10}$ we have $\sigma(\nu, 5, 18) = \sigma(\nu, 5, 12) + \sigma(\nu, 5, 6)$.

14 Packing with index 19

14.1 $\nu \equiv 4 \pmod{20}$

Lemma 14.1. *Let* $\nu \equiv 4 \pmod{20}$ *be a positive integer. If there exists* (1) *a* $(\nu, 5, 4)$ *minimal covering design* (2) *a* $(\nu, 5, 4)$ *optimal packing design* (3) *a* $(\nu - 2, 5, 1)$ *optimal packing design. Then there exists a* $(\nu, 5, 19)$ *optimal packing design.*

Proof: A $(\nu, 5, 19)$ optimal packing design, $\nu \equiv 4 \pmod{20}$, can be constructed as follows:

1) Take a $(\nu, 5, 4)$ minimal covering design. The excess graph of this design consists of $\nu - 3$ isolated vertices and 3 other vertices, say, $(\nu - 2, \nu - 1, \nu)$ the pairs of which are connected by two edges.

2) Take three copies of a $(\nu, 5, 4)$ optimal packing design. The complement graph of this design consists of $\nu - 2$ isolated vertices and two vertices connected by 4 edges. Assume in the first copy the two vertices are $(\nu - 2, \nu - 1)$, in the second the two vertices are $(\nu - 1, \nu)$, and in the third the two vertices are $(\nu - 2, \nu)$.

3) Take a $(\nu - 2, 5, 1)$ optimal packing design and assume (4,9) appears in zero blocks.

4) Take a $B[\nu + 1, 5, 1]$. Assume we have the block $\langle 1\ 2\ 3\ \nu\ \nu + 1\rangle$ where $\{1, 2, 3\}$ are arbitrary numbers. In this block change $\nu + 1$ to $\nu - 1$ and in all other blocks change $\nu + 1$ to ν

5) Again take a $B[\nu+1, 5, 1]$. Assume we have the block $\langle 1\ 2\ 3\ \nu-1\ \nu+1\rangle$.

   In this block change $\nu + 1$ to ν and in all other blocks change $\nu + 1$ to $\nu - 1$.

The above 5 steps give us a design such that each pair appear at most 19 times except the pairs $(\nu - 2, \nu - 1)$, $(\nu - 2, \nu)$ which appear at most 17 times and the pair $(\nu - 1, \nu)$ which appears 20 times. To reduce it to 19 assume in (1) we have the block $\langle 1\ 2\ 3\ \nu - 1\ \nu\rangle$ and in (2) we have the block $\langle 1\ 2\ 3\ 4\ \nu - 2\rangle$ where $\{1, 2, 3\}$ are arbitrary numbers. In the first block change ν to $\nu - 2$, and in the second block change $\nu - 2$ to ν. Now the pair $(4, \nu)$ appears 20 times. To reduce it, assume in (4) we have the block $\langle 5\ 6\ 7\ 4\ \nu\rangle$ and in (5) we have the block $\langle 5\ 6\ 7\ \nu - 2\ 9\rangle$ where $\{5, 6, 7, 9\}$ are arbitrary numbers. In the first block change ν to 9 and in the second block change 9 to ν. Now it is easy to check that the above construction yields a $(\nu, 5, 19)$ optimal packing design.

Corollary 14.1. $\sigma(\nu, 5, 19) = \psi(\nu, 5, 19)$ *for* $\nu = 44, 64, 84$.

Proof: In view of the previous lemma we need to show that $\alpha(\nu,5,4) = \phi(\nu,5,4)$ for $\nu = 44,64,84$ for this purpose see [13]. We also need to show that $\sigma(\nu,5,4) = \psi(\nu,5,4)$ for $\nu = 44,64,84$, for this purpose see [10]. We also need to show that $\sigma(\nu,5,1) = \psi(\nu,5,1)$ for $\nu = 42,62,82$, see [9]. And since there exists a B$[\nu+1,5,1]$ for all $\nu+1 \equiv 5 \pmod{20}$ it follows $\sigma(\nu,5,19) = \psi(\nu,5,19)$ for $\nu = 44,64,84$.

Lemma 14.2. (a) *There exists a (24,5,19) optimal packing design.* (b) *There exists a (24,5,19) packing design with a hole of size 4.*

Proof: (a) A (24,5,19) optimal packing design can be constructed as follows:

1) Take a (24,5,3) optimal packing design [9]. A close observation of this design shows that the following pairs are missing: (1,11) (2,6) (2,14) (2,16) (2,17) (3,16) (4,16) (5,19) (7,13) (8,16) (9,17) (10,20) (12,17) (15,18) (16,22) (17,23) (21,24) each is missing exactly once except (2,17) which is missing twice.

2) Take two copies of a (24,5,4) minimal covering design [17]. In this design there is a triple the pairs of which appear in 6 blocks while each other pair appears in four blocks. Assume that in the first copy the triple is $\{2,16,17\}$ and in the second copy the triple is $\{2,3,16\}$.

3) Take two copies of a (24,5,4) optimal packing design [10]. In this design there is a pair that appears in zero blocks. Assume that in the first copy the pair is (2,16) and in the second copy the pair is (2,3).

Since (2,17) is missing twice and (3,16) is missing once in the blocks of (24,5,3) optimal packing design, it is easy to see that the above three steps give us a design such that each pair appears at most 19 times except (16,17) that appears in at most 21 blocks and (3,16) that appears in at most 20 blocks. To fix this, assume in step 2 we have the following three blocks $\langle 1\ 2\ 3\ 16\ 17\rangle$ $\langle 4\ 5\ 6\ 16\ 17\rangle$ $\langle 7\ 8\ 9\ 16\ 3\rangle$. In the first block change 17 to 22, in the second block change 17 to 8 and in the third block change 3 to 4. Furthermore, assume in step 3 we have the following three blocks $\langle 1\ 2\ 3\ 12\ 22\rangle$ $\langle 4\ 5\ 6\ 9\ 8\rangle$ $\langle 7\ 8\ 9\ 2\ 4\rangle$. In the first block change 22 to 17, in the second block change 8 to 17 and in the third block change 4 to 3. After this manipulation of blocks the pairs (16,17) and (3,6) appear at most 19 times. But the pairs (16,22) (8,16) (4,16) (12,17) (9,17) and (2,3) will appear one more time. But (16,22) (8,16) (4,16) (12,17) (9,17) are missing once in the (24,5,3) optimal packing design and since we assumed that (2,3) appears zero times in the blocks of (24,5,4) optimal packing design then these pairs will appear at most 19 times and as a result there exist a (24,5,19) optimal packing design.

(b) Let $X = Z_{20} \cup H_4$. Then the blocks of a (24,5,19) packing design with a hole of size 4 can be constructed as follows:

1) Take 3 copies of a (23,5,2) packing design with a hole of size 3, [5].
2) Take 6 copies of a B[25,5,1]. Assume in each copy we have the block ⟨21 22 23 24 25⟩, which we delete, and in all other blocks change 25 to 24.
3) Take also the following base blocks developed under the action of Z_{20}.
   $\langle 0\ 4\ 8\ 12\ 16\rangle + i$, $i \in Z_4$, twice
   $\langle 0\ 3\ 5\ 14\rangle \cup \{h_i\}_{i=1}^{4}$ $\langle 0\ 3\ 6\ 13\rangle \cup \{h_1, h_2\}$ $\langle 0\ 2\ 7\ 11\rangle \cup \{h_3, h_4\}\rangle$
   $\langle 0\ 1\ 2\ 3\ h_1\rangle$ $\langle 0\ 2\ 6\ 12\ h_2\rangle$ $\langle 0\ 3\ 8\ 15\ h_3\rangle$ $\langle 0\ 3\ 9\ 13\ h_4\rangle$
   $\langle 0\ 1\ 2\ 3\ 8\rangle$ $\langle 0\ 1\ 5\ 9\ 14\rangle$

Lemma 14.3. *Let $\nu \equiv 4 \pmod{20}$ be a positive integer. Then $\sigma(\nu, 5, 19) = \psi(\nu, 5, 19)$.*

Proof: For $\nu = 24, 44, 64, 84$ the result is given in the corollary and lemma 14.2. For $\nu \geq 124$, $\nu \neq 144, 184, 224$, write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen the same as in lemma 5.6 then apply theorem 2.9 with $\lambda = 19$, to give the result.

For $\nu = 144$ apply theorem 2.18 with $n = 7$, $h = 4$ and $\lambda = 19$.

For $\nu = 104, 224$ apply theorem 2.11 with $h = 4$, $\lambda = 19$ and $m = 5, 11$ respectively.

For $\nu = 184$ apply theorem 2.12 with $m = 8$, $h = 0$ and $u = 5$.

14.2 $\nu \equiv 2, 8, 12, 14$ or 18 (mod 20)

Lemma 14.4. *Let $\nu \equiv 2, 8, 12, 14$ or 18 (mod 20) be a positive integer. Then $\sigma(\nu, 5, 19) = \psi(\nu, 5, 19)$ with the possible exception of $\nu = 28$.*

Proof: We distinguish the following two cases.
Case 1: $\nu \equiv 2$ or 14 (mod 20). In this case the blocks of a $(\nu, 5, 19)$ optimal packing design are those of a $(\nu, 5, 16)$ and a $(\nu, 5, 3)$ optimal packing design.
Case 2: $\nu \equiv 18 \pmod{20}$. In this case the blocks of a $(\nu, 5, 19)$ optimal packing design are those of a $(\nu, 5, 13)$ and $(\nu, 5, 6)$ optimal packing design.
Case 3: $\nu \equiv 8$ or 12 (mod 20). In this case the blocks of a $(\nu, 5, 19)$ optimal packing design are those of a $(\nu, 5, 11)$ and a $(\nu, 5, 8)$ optimal packing design.

Since a (28,5,11) optimal packing design is still unknown, the above construction does not work for $\nu = 28$.

In this section we have shown:

Theorem 14.1. *Let ν be a positive integer. Then $\sigma(\nu, 5, 19) = \psi(\nu, 5, 19)$ with the possible exception of $\nu = 28$.*

15 Packing with index 21

Since $\sigma(\nu,5,1)$ is far from being settled and since $\sigma(\nu,5,1) \neq \psi(\nu,5,1)$ for $\nu = 9,10,\ldots,19$ and 22, [19], its worth looking at optimal packing designs with index 21. It is clear that the only cases we need to consider are $\nu \equiv 2,4$ or 8 (mod 10) and $\nu = 39$.

15.1 $\nu \equiv 2 \pmod{20}$

Lemma 15.1. a) *There exists a (22,5,21) and (26,5,21) packing design with a hole of size 2 and 6 respectively.*
b) $\sigma(\nu,5,21) = \psi(\nu,5,21)$ *for* $\nu = 22,42,62,82$.

Proof: For a (22,5,21) packing design with a hole of size 2 take four copies of a (22,5,4) and a (22,5,5) packing design with a hole of size 2, [10], [8].

For a (26,5,21) packing design with a hole of size 6 take 4 copies of a (26,5,4) [10] and a (26,5,5) packing design with a hole of size 6 [8].
b) To construct a $(\nu,5,21)$ optimal packing design for $\nu = 42,62,82$ take the blocks of a $B[\nu,5,20]$ and a $(\nu,5,1)$ optimal packing design, [9].

For $\nu = 22$ see the following table.

ν	Point Set	Base Blocks
22	$Z_2 \times Z_{10} \cup H_2$	On $Z_2 \times Z_{10} \cup H_2$ construct a (22,5,4) packing design with a hole of size 2, say, $\{h_1,h_2\}$. Take 4 copies of this design and take the following blocks
		$\langle(0,0)(0,2)(0,4)(0,6)(0,8)\rangle + (-,i) i \in Z_2$, twice
		$\langle(1,0)(1,2)(1,4)(1,6)(1,8)\rangle + (-,i) i \in Z_2$
		$\langle(0,0)(0,1)(0,3)(0,4)(1,0)\rangle\langle(0,0)(0,1)(0,5)(1,0)(1,1)\rangle$
		$\langle(0,0)(0,3)(0,4)(1,2)(1,7)\rangle\langle(0,0)(0,2)(1,3)(1,4)(1,7)\rangle$
		$\langle(0,0)(0,3)(1,4)(1,6)(1,8)\rangle\langle(0,0)(1,0)(1,1)(1,2)(1,3)\rangle$
		$\langle(0,0)(0,5)(1,0)(1,3)(1,6)\rangle\langle(0,0)(0,1)(1,5)(1,8)h_1\rangle$
		$\langle(0,0)(0,2)(1,4)(1,9)h_2\rangle\langle(0,0)(0,3)(1,9)h_1,h_2\rangle$
		$\langle(0,0)(1,2)(1,8)h_1,h_2\rangle$

Lemma 15.2. *Let* $\nu \equiv 2 \pmod{20}$ *be a positive integer. Then* $\sigma(\nu,5,21) = \psi(\nu,5,21)$.

Proof: For $\nu = 22,42,62,82$ the result was established in the previous lemma.

For $\nu \geq 122$, $\nu \neq 142$ write $\nu = 20m + 4u + h + s$ where m, u, h and s are chosen as in lemma 10.8. Now apply theorem 2.9 to give the result.

For $\nu = 102$ apply theorem 2.11 with $m = 5$, $h = 2$ and $\lambda = 21$.

For $\nu = 142$ apply theorem 2.18 with $n = 7$, $h = 2$ and $\lambda = 21$.

15.2 $\nu \equiv 14$ or 18 (mod 20)

Lemma 15.3. *Let* $\nu \equiv 14$ *or* 18 (mod 20) *be a positive integer. Then* $\sigma(\nu,5,21) = \psi(\nu,5,21)$.

Proof: If $\nu \equiv 14 \pmod{20}$ then $\sigma(\nu,5,21) = \sigma(\nu,5,17) + \sigma(\nu,5,4)$, and for $\nu \equiv 18 \pmod{20}$, $\sigma(\nu,5,21) = \sigma(\nu,5,13) + \sigma(\nu,5,8)$.

15.3 $\nu \equiv 4, 8$ or 12 (mod 20)

Lemma 15.4. *Let* $\nu \equiv 4, 8$ *or* 12 (mod 20) *be a positive integer. Then* $\sigma(\nu,5,21) = \psi(\nu,5,21)$.

Proof: For $\nu \equiv 4 \pmod{20}$ apply lemma 4.2.

For $\nu \equiv 8$ or 12 (mod 20), $\sigma(\nu,5,21) = \sigma(\nu,5,11) + \sigma(\nu,5,10)$. Since (28,5,11) is still unknown, we need to construct a (28,5,21) optimal packing design. The blocks of this design are the blocks of a $(\nu,5,9)$ and a $(\nu,5,12)$ optimal packing design.

In this section we have shown.

Theorem 15.1. *Let* $\nu \geq 5$ *be a positive integer. Then* $\sigma(\nu,5,21) = \psi(\nu,5,21)$ *with the possible exception of* $\nu = 39$.

16 Conclusion

To conclude our result, we have shown (theorem 5.1 - theorem 15.1) that for all positive integers ν, $\sigma(\nu,5,\nu) = \psi(\nu,5,\nu) - e$ holds where $e = 1$ if $\lambda(\nu-1) \equiv 0 \pmod{\nu-1}$ and $\lambda\nu(\nu-1)/(\kappa-1) \equiv 1 \pmod{\kappa}$ and $e = 0$ otherwise with the possible exceptions listed in theorem 1.3.

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