Baer *-Rings With Finitely Many Elements

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ABSTRACT. This paper contains a characterization of Baer * -rings with finitely many elements in terms of matrix rings over finite fields. As an application one can easily verify whether a given matrix ring over a finite field is a Baer *-ring or not.

A *-ring A is a ring A with an involution *. An element e in a *-ring A is called projection if it is self adjoint (i.e. $e = e^*$) and idempotent (i.e. $e^2 = e$). A Baer *-ring is a *-ring in which the right annihilator of any subset S of A i.e. $R(S) = \{x \epsilon A / sx = 0 \text{ for all } s \epsilon S\}$ is a right principal ideal generated by a projection i.e. of the form eA where e is a projection. A *-ring is said to have proper involution if $x^*x = 0$ implies that $x = 0$. It follows that the involution of every Baer *-ring is proper.

Matrix rings over an involutory ring are important rings for examples and counter examples. Berberian [1] concentrates on matrix rings over Baer *-rings. He discusses conditions on the underlying Baer *-ring A so that the matrix ring $M_n(A)$ becomes a Baer *-ring. The main problem is of determining conditions on a Baer *-ring A that are sufficient to convert $M_n(A)$ into a Baer *-ring. Berberian candidly warns that it is very hard to show that a matrix ring is a Baer *-ring; the problem is largely open. In the literature $M_2(Z_3)$ was the only matrix *-ring which was frequently quoted as an example of a Baer * -ring.

Firstly we quote some necessary results.

Theorem 1. (Herstein [2], p. 127). *Let F be a field, which is algebraic over a finite field. If the $n \times n$ matrix ring over F i.e. $M_n(F)$ has a proper involution, then $n = 1$ or 2. Moreover, if char(F) = 2 then $n \neq 2$.*

Theorem 2. (Berberian [1]). *Let B be any *-ring with unity, n a positive integer. Then*

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(i) *the involution of* $M_n(B)$ *is proper if and only if* B *satisfies the following condition :*

$$x_1^*x_1 + x_2^*x_2 + ... + x_n^*x_n = 0 \Rightarrow x_1 = x_2 = ... = x_n = 0; \qquad (*)$$

(ii) *for an involutory division ring* B, $M_n(B)$ *is a Baer *-ring if and only if the involution of* $M_n(B)$ *is proper (i.e.* B *satisfies the condition (*)).*

Theorem 3. (Herstein [[2], p. 128). *Let* A *be a *-ring with finitely many elements and with proper involution. Then* A *is the direct sum of finite fields and rings of* 2×2 *- matrices over finite fields.*

Firstly, we sharpen Theorem 1.

Theorem 4. *Let* F *be a field which is algebraic over a finite field* $F(p^\alpha), p$ *prime. If the* $n \times n$ *- matrix ring over* F *(i.e.* $M_n(F)$*) has a proper involution then*

(1) $n = 1$, *or*

(2) $n = 2$ *and* p *is of the form* $4k + 3$.

Proof: In the presence of Theorem 1 it remains to prove that p is of the form $4k+3$. Consider the equation $X^*X + Y^*Y = 0$ in F. It is clear by the same theorem that $p \neq 2$. Therefore, if p is not of the form $4k + 3$ then $p - 1$ is divisible by 4. We get an element $\gamma \epsilon Z_p$ such that $\gamma^2 = -1$. Since Z_p is a subfield of F and restriction of any involution on F to Z_p is the identity map, we have $X = \gamma$ and $Y = 1$ as a solution of the equation $X^*X + Y^*Y = 0$. By Theorem 2 (i) we get that the involution of $M_n(F)$ is not proper. Hence p must be of the form $4k + 3$. □

Since the involution of every Baer *-ring is proper we get the following result as a particular case of the above theorem.

Corollary 5. *Let* F *be a field which is algebraic over a finite field* $F(p^\alpha), p$ *prime. If* $M_n(F)$ *is a Baer *-ring then*

(1) $n = 1$, *or*

(2) $n = 2$ *and* p *is of the form* $4k + 3$.

Now we completely classify *-rings with finitely many elements into Baer *-rings and non-Baer *-rings.

Theorem 6. *A *-ring* A *with finitely many elements is a Baer *-ring if and only if* $A = A_1 \oplus A_2 \oplus ... \oplus A_r$ *where* A_i *is a field or* A_i *is a 2 x 2 -*

matrix ring over a finite field $F(p^\alpha)$ with α odd positive integer and p is a prime number of the form $4k+3$.

Proof: Let A be a Baer *-ring. Clearly the involution of A is proper. Hence $A = A_1 \oplus ... \oplus A_r$ where A_i is a field or A_i is a 2 x 2 - matrix ring over a finite field $F(p^\alpha)$ with $p \neq 2$ (by Theorem 3). We have also proved in Theorem 4 that p must be of the form $4k+3$. Suppose that α is even. Then we get that $q = p^\alpha$ is of the form $4k+1$. That is $q-1$ is divisible by 4. If γ is a generator of the multiplicative group $F^*(p^\alpha) = F(p^\alpha) - \{0\}$ then $\gamma^{(q-1)/2} = -1$.

Case 1. The involution of $F(p^\alpha)$ is the identity. Then $X = \gamma^{(q-1)/2}, Y = 1$ is a nontrivial solution to the equation $X^*X + Y^*Y = 0$ in $F(p^\alpha)$. Hence by Theorem 2(i) we get that the involution of A is not proper, a contradiction.

Case 2. Involution of $F(p^\alpha)$ is not identity. Then the involution must be $* : x \mapsto x^{p^{\alpha/2}}$. Now consider the equation

$$X^*X + Y^*Y = 0 \;\; i.e. \;\; X^{p^{\alpha/2}+1} + Y^{p^{\alpha/2}+1} = 0.$$

The element $\gamma_0 = \gamma^{(p^{\alpha/2}-1)/2}$ has the property that $\gamma_0^{p^{\alpha/2}+1} = \gamma^{(p^\alpha-1)/2} = -1$. Therefore if we put $X = \gamma_0$ and $Y = 1$, then we get a nontrivial solution to the equation $X^*X + Y^*Y = 0$. Thus by Theorem 2 we get that the involution of $F(p^\alpha)$ is improper, a contradiction. Therefore α must be odd.

Conversely, let A be as stated in the theorem. If A_i is a field then it is obviously a Baer *-ring. Suppose that A_i is a 2 x 2 - matrix ring over a finite field $F(p^\alpha)$ where α is an odd positive integer and p is of the form $4k+3$. We prove that A_i is a Baer *-ring. Since α is odd, the field $F(p^\alpha)$ has only the identity involution. Suppose that the equation $X^2 + Y^2 = 0$ has a nontrivial solution in $F(p^\alpha)$. That means there exists an element say γ_1 in $F(p^\alpha)$ such that $\gamma_1^2 = -1$. This means that $p^\alpha - 1$ is divisible by 4, so p^α is of the form $4k+1$. But p is of the form $4k+3$ and α is odd imply that p^α must be of the form $4k+3$, a contradiction. Hence there is no nontrivial solution for the equation $X^*X + Y^*Y = 0$ in $F(p^\alpha)$. Applying Theorem 2 (ii) we get that A_i is a Baer *-ring. Thus A is a Baer *-ring as required. □

We have the following immediate corollary.

Corollary 7. (i) *$M_n(Z_m)$ is a Baer *-ring for $n \geq 2$ iff $n = 2$ and m is a square free integer whose every prime factor is of the form $4k+3$.*

(ii) *Z_m is a Baer *-ring iff m is a square free integer.*

Proof: (ii) The identity involution is the only involution on Z_m. Hence it follows that the involution on Z_m is proper if and only if m is a square free integer. If m is a square free integer then Z_m is a direct sum of fields and hence it is a Baer *-ring.

(i) Firstly we observe that if m is not a square free integer then the involution on Z_m is not proper, and Z_m being a *-subring of $M_n(Z_m)$, the involution of $M_n(Z_m)$ is also not proper, for any $n \geq 1$. Secondly, we observe that if m is a square free integer say $m = p_1, p_2, ..., p_t$, where each p_i is a distinct prime then $M_n(Z_m) = M_n(Z_{p_1}) \oplus M_n(Z_{p_2}) \oplus ... \oplus M_n(Z_{p_t})$. Now by Corollary 5 and Theorem 6 the result follows immediately.

References

[1] S.K. Berberian, *Baer *-rings*, Springer-Verlag, Berlin, 1972.

[2] I.N. Herstein, *Rings with Involution*, Chicago Univ. Press, London, 1976.