On Surjective Semispan of Abstract Graphs*

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1 Introduction

In [2] Lelek defined the surjective semispan of a continuum, which is an analogue of the diameter of a continuum. Since the surjective semispan of a continuum depends on the metric, this could be thought of as a "geometric" span. This concept has led to interesting questions and results for such simple geometric objects as simple closed curves and simple triods [2, 3, 4, 5, 6, 7, 8]. Sam B. Nadler, Jr. has defined the surjective semispan of an abstract graph (in private communications). It is a natural combinatorial version of "geometric" span.

We calculate the surjective semispans for some basic abstract graphs. An algorithm for the surjective semispan of a tree is determined (Theorem 2). Bounds are found for the surjective semispan of the wedge of two graphs (Theorem 3) and we give corresponding examples. It is shown that in calculating the surjective semispan of a graph we can restrict our attention to homomorphisms where the domains are paths (Theorem 4). It is shown that the surjective semispan is invariant under subdivision only for paths (Theorem 5). We generalize the result for trees (Theorem 2) by determining an algorithm for the surjective semispan of a Husimi tree (Theorem 6). (Both proofs are given since the proof of Theorem 2 motivates and clarifies the proof of Theorem 6).

2 Notations and Definitions

All of the graphs in this paper are abstract graphs. All of these graphs are simple graphs (no loops or multiple edges). The following terminology from [1] is used. We denote the path of length n by P_n or by $v_0 - v_1 - \cdots - v_n$. The edge set of a graph G is denoted by $E(G)$ and its vertex

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set by $V(G)$. The cycle of order n is denoted by C_n where $V(C_n) = \{v_i\}_{i=0}^{n-1}$ and $E(C_n) = \{v_i v_{i+1}\}_{i=0}^{n-2} \cup \{v_{n-1}v_0\}$. The complete graph on n vertices is denoted by K_n. A complete n-partite graph is denoted by $K(p_1, \ldots, p_n)$ where $V_1, V_2, \ldots, V_n$ is a partition of its vertex set, $|V_i| = p_i$ for $i = 1, 2, \ldots, n$, and for $p \in V(V_i)$ and $q \in V(V_j)$, $pq \in E(K(P_1, \ldots, P_n))$ if and only if $i \neq j$. By a tree we mean a connected acyclic graph. A Husimi tree is a connected graph such that each of its blocks is a complete graph. Note that a tree is a Husimi tree where each block is K_2.

We also define the following terms. By a diameter path of G, we mean a path in G of length the diameter of G, where the distance between the endpoints of the path is also the diameter of the graph. By $G^v \vee H^w$ we mean the graph obtained by identifying the vertex v in G with the vertex w in H. We will refer to $G^v \vee H^w$ as the wedge of G and H, with v and w identified. Let G and H be two graphs. A function $f\colon V(G) \to V(H)$ is called a homomorphism if whenever $vw \in E(G)$ either $f(v) = f(w)$ or $f(v)f(w) \in E(H)$.

As mentioned in the introduction, the definition of the surjective semispan of an abstract graph is motivated by a similar definition for continua. In order to motivate our definition and clarify our notation, we give the following terminology due to Lelek ([2], [3]).

Let (X, d) be a continuum, that is a compact, connected, metric space. For any two continuous functions f and g from a continuum Z into X, let

$$\mathrm{glb}(f, g) = \mathrm{glb}\{d(f(z), g(z))\colon z \in Z\}$$

Then, the *surjective span of* X denoted by $\sigma^*(X)$, is defined as follows:

$\sigma^*(X) = \mathrm{lub}\{glb(f, g)\colon f$ and g map some continuum Z onto X, where f and g are continuous$\}$. The surjective semispan of X denoted by $\sigma_0^*(X)$ is defined in the same way but by requiring only one of the maps $f, g\colon Z \to X$ to map onto X. Now the *span of* X and the *semispan of* X denoted by $\sigma(X)$ and $\sigma_0(X)$ respectively, are defined so as to take into account the subcontinua of X namely:

$$\sigma(X) = \mathrm{lub}\{\sigma^*(Y) \mid Y \text{ is a subcontinuum of } X\};$$
$$\sigma_0(X) = \mathrm{lub}\{\sigma_0^*(Y) \mid Y \text{ is a subcontinuum of } X\}.$$

In general, note that the diameter of a continuum (X, d) may be thought of as being

$$\mathrm{lub}\{\mathrm{glb}(f, g)\colon f \text{ and } g \text{ map some space } Z \text{ into } X\}.$$

Thus the various spans of a continuum may be considered to be analogues of diameter using continuous functions on continua with equal or related ranges.

The following definition (motivated by the above definitions) is due to Sam B. Nadler, Jr. Let G be a connected abstract graph. If G_α is a connected graph and $f, g\colon V(G_\alpha) \to V(G)$ are arbitrary maps then let

$$d(f,g) = \min\{d(f(v_i), g(v_i))\}_{i=1}^{n},$$

where $\{v_i\}_{i=1}^{n} = V(G_\alpha)$ and $d(f(v_i), g(v_i))$ is the length of the shortest path in G from $f(v_i)$ to $g(v_i)$. We define the surjective semispan of G, $\sigma_0^*(G)$ by $\sigma_0^*(G) = \max\{d(f_\alpha, g_\alpha)\colon f_\alpha, g_\alpha\colon V(G_\alpha) \to V(G)$, f_α and g_α are homomorphisms, G_α is a connected graph and f_α is surjective$\}$.

It is clear from this definition that $0 \le \sigma_0^*(G) \le \operatorname{diam} G$. The diameter of an abstract graph G may be thought of as being

$$\operatorname{lub}\{\operatorname{glb}(f,g)\colon f \text{ and } g \text{ are homomorphisms from some vertex set into } V(G)\}.$$

So, the surjective semispan of G may be considered an analogue of the diameter of G.

3 Theorems and Examples

Theorem 1. *Let G be a connected graph. Then $\sigma_0^*(G) = 0$ if and only if $G = K_1$.*

Proof: It is clear that $\sigma_0^*(K_1) = 0$. Suppose that $G \ne K_1$. Let $v \in V(G)$ and let $w \in V(G) - \{v\}$ such that $d(v,w) = 1$. We define homomorphisms $f, g\colon V(G) \to V(G)$ as follows.

$$f(u) = u, \quad u \in V(G)$$

$$g(u) = \begin{cases} v & u \ne v \\ w & u = v. \end{cases}$$

Hence, $\sigma_0^*(G) \ge 1$ since $f(V(G)) = V(G)$ and $d(f,g) = 1$.

Next we give some easy examples.

Example 1. $\sigma_0^*(P_n) = 1$.

We see that $\sigma_0^*(P_n) \ge 1$ by Theorem 1. Next, we will see that $\sigma_0^*(P_n)$ is not larger than 1. Let $f, g\colon V(G) \to V(P_n)$ be homomorphisms where G is a connected graph, $f(V(G)) = V(P_n)$ and $d(f,g) \ge 1$. There are vertices $v, w \in V(G)$ such that $f(v) = v_0$ and $f(w) = v_n$. Let $v - w_1 - \cdots - w_{m-1} - w$ be a path in G from v to w. We can assume that $f(w_i) \cap \{v_0, v_n\} = \emptyset$ for $i = 1, 2, \ldots, m-1$. Let $w_0 = v$ and $w_m = w$. So, $g(w_0) = v_h$ where $h > 0$ and $g(w_m) = v_k$ where $k < n$. Let

$$i = \max\{j\colon 0 \le j \le m, f(w_j) = v_{k'}, g(w_j) = v_{h'} \text{ and } k' < h'\}.$$

Hence, $f(w_{j+1}) = v_{k''}$ and $g(w_{j+1}) = v_{h''}$ where $h'' < k''$. It must be the case that $k' + 1 = h'$, since $k' < h'$, $k'' \in \{k'-1, k', k'+1\}$, $h'' \in \{h'-1, h', h'+1\}$, and $h'' < k''$. Hence, $d(f,g) = d(v_{k'}, v_{h'}) = d(v_{k'}, v_{k'+1}) = 1$. Consequently, $\sigma_0^*(P_n) = 1$.

Example 2. $\sigma_0^*(C_n) = [\frac{n}{2}]$.

Let $f, g\colon V(C_n) \to V(C_n)$ be homomorphisms given by $f(v_i) = v_i$ and $g(v_i) = v_{k(i)}$ for $i = 0, 1, \ldots, n-1$ where $k(i) = (i + [\frac{n}{2}]) \bmod n$. So $\sigma_0^*(C_n) = [\frac{n}{2}]$, since $f(V(C_n)) = V(C_n)$, $d(f,g) = [\frac{n}{2}]$ and $\operatorname{diam}(C_n) = [\frac{n}{2}]$.

Example 3. $\sigma_0^*(K_n) = 1$.

This is clear since $\sigma_0^*(K_n) \geq 1$ by Theorem 1 and $\operatorname{diam} K_n = 1$.

Example 4. $\sigma_0^*(K(3,3)) = 2$.

Let $V_0 = \{u_0, u_1, u_2\}$, $V_1 = \{w_0, w_1, w_2\}$ be the partition of $V(K(3,3))$ such that $u_i w_j \in E(K(3,3))$ for each $i, j \in \{0,1,2\}$. Let $f, g\colon V(P_5) \to V(K(3,3))$ be homomorphisms defined as follows:

i	$g(v_i)$	$f(v_i)$
0	u_1	u_0
1	w_1	w_0
2	u_0	u_1
3	w_0	w_1
4	u_0	u_2
5	w_0	w_2

Consequently, $\sigma_0^*(K(3,3)) = 2$ since $f(V(P_5)) = V(K(3,3))$, $d(f,g) = 2$, and $\operatorname{diam} K(3,3) = 2$.

Example 5. $\sigma_0^*(K(p_1, p_2, \ldots, p_n)) = 2$ if $p_i \geq 2$ for each $i \in \{1, 2, \ldots, n\}$ and $n \geq 2$.

This can be shown by defining homomorphisms similar to those in Example 4.

Example 6. $\sigma_0^*(K(p_1, p_2, \ldots, p_n)) = 1$ if $p_i = 1$ for some $i \in \{1, 2, \ldots, n\}$ and $n \geq 2$.

Let $f, g\colon V(G) \to V(K(p_1, p_2, \ldots, p_n))$ be homomorphisms where G is a connected graph and $f(V(G)) = V(K(p_1, p_2, \ldots, p_n))$. Let $v \in V(K(p_1, p_2, \ldots, p_n))$ such that $vw \in E(K(p_1, p_2, \ldots, p_n))$ for each $w \in V(K(p_1, p_2, \ldots, p_n)) - \{v\}$. There is a vertex $u \in V(G))$ such that $f(u) = v$. Hence, $d(f(u), g(u)) \leq 1$ and $\sigma_0^*(K(p_1, p_2, \ldots, p_n)) \leq 1$. By Theorem 1 we have that $\sigma_0^*(K(p_1, p_2, \ldots, p_n)) = 1$.

Example 7. $\sigma_0^*(P) = 2$, where P is the Petersen graph (see Figure 1)

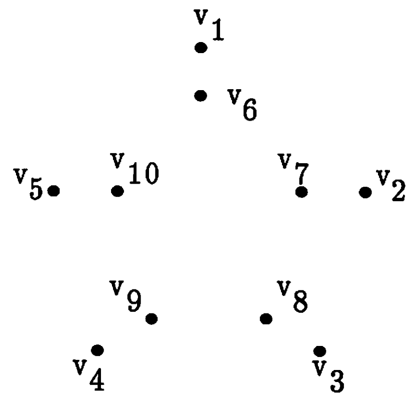

Figure 1

Let $f, g\colon V(P_{12}) \to V(P)$ be homomorphisms as defined below.

v_i	$g(v_i)$	$f(v_i)$	v_i	$g(v_i)$	$f(v_i)$
0	v_4	v_6	7	v_1	v_3
1	v_4	v_1	8	v_2	v_4
2	v_5	v_2	9	v_2	v_9
3	v_5	v_7	10	v_2	v_4
4	v_5	v_2	11	v_3	v_5
5	v_1	v_3	12	v_3	v_{10}
6	v_1	v_8			

Consequently, $\sigma_0^*(P) = 2$ since $f(V(P_{12})) = V(P)$, $d(f,g) = 2$, and $\operatorname{diam} P = 2$.

Let T be a tree which is not a path. Let $P_n = v_0 - v_1 - \cdots - v_n$ be a diameter path in T. We define graphs T_j and B_j for $j = 1, 2, \ldots, n-1$, a set $\{b_j\}_{j=1}^{n-1}$ and an integer $b(P_n)$ all based on P_n.

$(*)$ Let T_j be the subgraph of T such that $V(T_j) = V(T)$ and $E(T_j) = E(T) - \{v_{j-1}v_j, v_jv_{j+1}\}$ for $j = 1, 2, \ldots, n-1$. For each j, let B_j be the component of T_j which contains v_j. If $V(B_j) - \{v_j\} = \emptyset$ then let $b_j = 0$. If $V(B_j) - \{v_j\} \neq \emptyset$ then let w_j be a vertex in $V(B_j)$ such that $d(v_j, w_j) \geq d(v_j, w)$ for each $w \in V(B_j)$ and let $b_j = d(v_j, w_j)$. Note that $j + b_j \leq n$ and $b_j + n - j \leq n$ since $\operatorname{diam} T = n$. So $b_j \leq \min\{j, n-j\}$. Let $b(P_n) = \max\{b_j\}_{j=1}^{n-1}$. We refer to $b(P_n)$ as the branch length for T determined by P_n. Note that since T is not a path, $b(P_n) \neq 0$.

Lemma 1. *Let T be a tree which is not a path. Let P_n and P_n' be two diameter paths in T. Then $b(P_n) = b(P_n')$.*

Proof: Let $P_n = v_0 - v_1 - \cdots - v_n$ and $P'_n = v'_0 - v'_1 - \cdots - v'_n$ be diameter paths in T. Let T_j and B_j for $j = 1, 2, \ldots, n-1$, $\{b_j\}_{j=1}^{n-1}$, and $b(P_n)$ be defined as in $(*)$ above. Let T'_j and B'_j for $j = 1, 2, \ldots, n-1$, $\{b'_j\}_{j=1}^{n-1}$ and $b(P'_n)$ be defined as above but based on the path P'_n instead of P_n.

Next we will show that $V(P_n) \cap V(P'_n) \neq \emptyset$. Suppose that $V(P_n) \cap V(P'_n) = \emptyset$. The graph T is connected, so there is a path from some $v_i \in V(P_n)$ and to some $v'_k \in V(P'_n)$. Also, $1 \leq i \leq n-1$ and $1 \leq k \leq n-1$ since $\operatorname{diam} T = n$ and T is acyclic. Also notice that either $i \geq \frac{n}{2}$ or $n-i \geq \frac{n}{2}$ and either $k \geq \frac{n}{2}$ or $n-k \geq \frac{n}{2}$. Since $d(v_i, v'_k) \geq 1$ and T is acyclic we see that, $d(v_0, v'_0) \geq i+1+k$, $d(v_0, v'_n) \geq i+1+n-k$, $d(v_n, v'_0) \geq n-i+1+k$, and $d(v_n, v'_n) \geq n-i+1+n-k$. If $i \geq \frac{n}{2}$ and $k \geq \frac{n}{2}$ then $d(v_0, v'_0) \geq n+1$. If $i \geq \frac{n}{2}$ and $n-k \geq \frac{n}{2}$ then $d(v_0, v'_n) \geq n+1$. If $n-i \geq \frac{n}{2}$ and $k \geq \frac{n}{2}$ then $d(v_n, v'_0) \geq n+1$. If $n-i \geq \frac{n}{2}$ and $n-k \geq \frac{n}{2}$ then $d(v_n, v'_n) \geq n+1$. So we would have $\operatorname{diam} T \geq n+1$, contrary to our assumption that $\operatorname{diam} T = n$. So $V(P_n) \cap V(P'_n) \neq \emptyset$.

Let $q = \min\{j \mid v_j \in V(P'_n)\}$ and $k = \max\{j \mid v_j \in V(P'_n)\}$. Let $v_q = v'_p$ and $v_k = v'_m$. To simplify the proof we may assume that $p \leq m$. Since T contains no cycles, $m - p = k - q$ and $v_{q+j} = v'_{p+j}$ for $j = 0, 1, \ldots, k-q$. Since T contains no cycles and $\operatorname{diam} T = n$ we see that $d(v'_0, v_n) = p + k - q + n - k = p - q + n \leq n$. So, $p - q \leq 0$. Similarly, $d(v_0, v'_n) \leq q + m - p + n - m = q - p + n \leq n$. So $q - p \leq 0$ and $p = q$. Consequently, $k = m$ and $v_j = v'_j$ for $q \leq j \leq k$. Recalling how B_j was defined, we can see that if $k - q \geq 2$ then $B_j = B'_j$ for $q+1 \leq j \leq k-1$. So if $k - q \geq 2$ then $b_j = b'_j$ for $q+1 \leq j \leq k-1$.

Next we will show that $b_q = b'_q$ and $b_k = b'_k$. Since $v'_0 - v'_1 - \cdots - v'_q \subset B_q$ and $v_0 - v_2 - \cdots - v_q \subset B_q$, we see that $b_q, b'_q \geq q$. But $b_q, b'_q \leq \min\{q, n-q\}$ so, $q \leq n - q$ and $b_q = b'_q = q$. Since $v'_k - v'_{k+1} - v'_{k+2} - \cdots - v'_n \subset B_k$ and $v_k - v_{k+1} - v_{k+2} - \cdots - v_n \subset B'_k$ we see that $b_k, b'_k \geq n - k$. But $b_k, b'_k \leq \min\{k, n-k\}$ so, $n - k \leq k$ and $b_k = b'_k = n - k$.

Next we will show that $b(P_n) = b(P'_n)$. For j where $1 \leq j \leq q-1$, $b'_j, b_j \leq \min\{j, n-j\} \leq j \leq q \leq n-q \leq n-j$, and $b'_j, b_j \leq q = b_q$. For j where $k+1 \leq j \leq n-1$, $b'_j, b_j \leq \min\{j, n-j\} \leq n-j \leq n-k \leq k \leq j$, and $b'_j, b_j \leq n-k = b_k$. Consequently $b(P_n) = \max\{b_j\}_{j=1}^{n-1} = \max\{b_j\}_{j=q}^{k} = \max\{b'_j\}_{j=q}^{k} = b(P'_n)$. Now we can represent the branch length of a tree by b, since this value does not depend on the particular diameter path which was used to determine it.

Theorem 2. *Let T be a tree which is not a path. Then $\sigma_0^*(T) = b$ where b is the branch length of T.*

Proof: Let $P_n = v_0 - v_1 - \cdots - v_n$ be a diameter path of T. Let T_j and B_j for $j = 1, 2, \ldots, n-1$, and $\{b_j\}_{j=1}^{n-1}$ be defined as in $(*)$ above. So $\max\{b_j\}_{j=1}^{n-1} = b_k$ for some k where $1 \leq k \leq n-1$. By the definition of branch length and Lemma 1, $b = b(P_n) = b_k$. Let $w_k \in V(B_k)$ such that $d(w_k, v_k) = b_k$.

Let G'_B be the subgraph of T such that $V(G'_B) = V(T)$ and $E(G'_B) = E(T) - \{v_{k-1}v_k\}$. Let G_B be the component of G'_B which contains v_k. Let G'_T be the subgraph of T such that $V(G'_T) = V(T)$ and $E(G'_T) = E(T) - \{v_k v_{k+1}\}$. Let G_T be the component of G'_T which contains v_k.

Let $G = (G_T^{w_k} \vee Q_n^{z_0})^{z_n} \vee G_B^{w_k}$ where $Q_n = z_0 - z_1 - \cdots - z_n$. We define homomorphisms $f, g\colon V(G) \to V(T)$ as follows:

$$\begin{cases} f(v) = v \\ g(v) = v_0 \end{cases} \quad v \in V(G_B)$$

$$\begin{cases} f(z_i) = w_k \\ g(z_i) = v_1 \end{cases} \quad 0 \le i \le n$$

$$\begin{cases} f(v) = v \\ g(v) = v_n \end{cases} \quad v \in V(G_T)$$

We observe that for $v \in V(G_B)$, $d(f(v), g(v)) = d(f(v), v_0) \ge d(v_k, v_0) = k$, for $v \in V(G_T)$, $d(f(v), g(v)) = d(f(v), v_n) \ge d(v_k, v_n) = n - k$, and for i where $0 \le i \le n$ $d(f(z_i), g(z_i)) = d(w_k, v_i) \ge d(w_k, v_k) = b_k$. So $d(f, g) = \min\{k, b_k, n-k\}$ and $\min\{k, b_k, n-k\} = b_k$ since $b_k \le \min\{n, n-k\}$ as we observed previously. Also, $f(V(G)) \supseteq f(V(G_B)) \cup f(V(G_T)) = V(G_B) \cup V(G_T) = V(T)$. So $\sigma_0^*(T) \ge b_k = b$.

Next we will show that $\sigma_0^*(T) \le b_k = b$. Let $f, g\colon V(H) \to V(T)$ be homomorphisms where H is a connected graph, $f(V(H)) = V(T)$ and $d(f, g) = \sigma_0^*(T)$. We define a homomorphism $h\colon V(T) \to V(P_n)$, where P_n is a diameter path in T, in the following manner:

$$\begin{cases} h(v_0) = v_0 \\ h(v) = v_i \quad \text{for } v \in B_i \text{ where } 1 \le i \le n-1 \\ h(v_n) = v_n \end{cases}$$

So, $h \circ f, h \circ g\colon V(H) \to V(P_n)$ are homomorphisms such that $h \circ f(V(H)) = V(P_n)$. Hence there are vertices $v, w \in V(H)$ such that $h \circ f(v) = v_0$ and $h \circ f(w) = v_n$. Let $v = u_0 - u_1 - \cdots - u_d = w$ be a path from v to w in H. We can assume that $h \circ f(u_i) \ne v_0$ when $i \ne 0$ and $h \circ f(u_i) \ne v_n$ when $i \ne d$.

Let $j' = \max\{i \in \{0, 1, \ldots, d\}\colon h \circ g(u_i) = v_{j''},\ h \circ f(u_i) = v_{k''}$ and $j'' > k''\}$. Let $h \circ g(u_{j'}) = v_{j''}$ and $h \circ f(u_{j'}) = v_{k''}$. Then $j'' - k''$ is either 1 or 2. If $j'' - k'' = 2$ then $h \circ g(u_{j'+1}) = h \circ f(u_{j'+1}) = v_{k''+1}$, $g(u_{j'+1}) = f(u_{j'+1}) = v_{k''+1}$ and $d(f, g) = 0$. However, $d(f, g) = \sigma_0^*(T)$ and $\sigma_0^*(T) \ge 1$ by Theorem 1 so $j'' - k'' = 1$. If $h \circ f(u_{j'+1}) = h \circ g(u_{j'+1}) = v_{j''}$

then $f(u_{j'+1}) = v_{j''}$, $g(u_{j'+1}) \in B_{j''}$ and $d(f(u_{j'+1}), g(u_{j'+1})) \leq b_{j''} \leq b$. If $h \circ f(u_{j'+1}) = h \circ g(u_{j'+1}) = v_{k''}$ then $f(u_{j'+1}) \in B_{k''}$, $g(u_{j'+1}) = v_{k''}$ and $d(f(u_{j'+1}), g(u_{j'+1})) \leq b_{k''} \leq b$. If $h \circ f(u_{j'+1}) = v_{j''}$ and $h \circ g(u_{j'+1}) = v_{k''}$ then $f(u_{j'}) = v_{k''}$ and $g(u_{j'}) = v_{j''}$ and $d(f,g) \leq 1 \leq b$. Consequently, in all cases we have seen that $d(f,g) \leq b$. So, $\sigma_0^*(T) = b$.

We saw in Examples 5 and 6 that complete n-partite graphs have surjective semispan either 1 or 2. However for any positive integer n, there is a bipartite graph G such that $\sigma_0^*(G) = n$. This follows immediately from Theorem 2 since a tree is a bipartite graph and obviously there are trees with arbitrary branch length.

Theorem 3. *Let G and H be connected graphs. For each $v \in V(G)$ and $w \in V(H)$,*

$$\max\{\sigma_0^*(G), \sigma_0^*(H)\} \leq \sigma_0^*(G^v \vee H^w)$$
$$\leq \min\{\sigma_0^*(G) + \mathit{diam}\, H, \sigma_0^*(H) + \mathit{diam}\, G, \max\{\mathit{diam}\, G, \mathit{diam}\, H\}\}$$

Proof: Let $v \in V(G)$ and $w \in V(H)$. Let $f, g \colon V(G') \to V(G)$ be homomorphisms where G' is a connected graph, $f(V(G')) = V(G)$, and $d(f,g) = \sigma_0^*(G)$. There is a vertex $z \in V(G')$ such that $f(z) = v$. Let

$$f'g' \colon V(G'^z \vee H^w) \to V(G^v \vee H^w)$$

be homomorphisms given by

$$f'(u) = \begin{cases} f(u) & u \in V(G') \\ u & u \in V(H) \end{cases}$$

$$g'(u) = \begin{cases} g(u) & u \in V(G') \\ g(z) & u \in V(H) \end{cases}$$

So, $f(V(G'^v \vee H^w)) = V(G^v \vee H^w)$. Note that $d(f'(u), g'(u)) = d(f(u), g(u))$ when $u \in V(G')$ and that $d(f'(u), g'(u)) = d(u, g(z)) \geq d(v, g(z)) \geq d(f,g)$ when $u \in V(H)$. So it follows that $d(f', g') = \sigma_0^*(G)$. Hence, $\sigma_0^*(G^v \vee H^w) \geq \sigma_0^*(G)$. Similarly, we see that $\sigma_0^*(G^v \vee H^w) \geq \sigma_0^*(H)$. Hence, $\max\{\sigma_0^*(G), \sigma_0^*(H)\} \leq \sigma_0^*(G^v \vee H^w)$.

Next we will show that $\sigma_0^*(G^v \vee H^w) \leq \min\{\sigma_0^*(G) + \operatorname{diam} H, \sigma_0^*(H) + \operatorname{diam} G, \max\{\operatorname{diam} G, \operatorname{diam} H\}\}$. Let $f, g \colon V(G') \to V(G^v \vee H^w)$ be homomorphisms where G' is a connected graph, $f(V(G')) = V(G^v \vee H^w)$, and $d(f,g) = \sigma_0^*(G^v \vee H^w)$. We define a homomorphism $h \colon V(G^v \vee H^w) \to V(G)$ by

$$h(u) = \begin{cases} u & u \in V(G) \\ w & u \in V(H). \end{cases}$$

So, $h \circ f$ and $h \circ g$ are homomorphisms from $V(G')$ into $V(G)$ such that $h \circ f(V(G')) = V(G)$. So, $d(h \circ g, h \circ f) \leq \sigma_0^*(G)$. Let $u \in V(G')$ such that $d(h \circ g(u), h \circ f(u)) \leq \sigma_0^*(G)$.

We consider four cases.

Case 1. $h \circ f(u) = f(u)$ and $h \circ g(u) = g(u)$.
So, $d(g(u), f(u)) \leq \sigma_0^*(G)$.

Case 2. $h \circ f(u) = h \circ g(u) = w$.
So, $f(u), g(u) \in V(H)$ and $d(f(u), g(u)) \leq \operatorname{diam} H$.

Case 3. $h \circ g(u) \in V(G) - \{w\}$, $h \circ f(u) = w$.
So, $d(g(u), f(u)) \leq d(g(u), w) + d(w, f(u)) = d(h \circ g(u), h \circ f(u)) + d(w, f(u)) \leq \sigma_0^*(G) + \operatorname{diam} H$.

Case 4. $h \circ f(u) \in V(G) - \{w\}$, $h \circ g(u) = w$.
Similarly, we get that $d(g(u), f(u)) \leq \sigma_0^*(G) + \operatorname{diam} H$.

In all four cases we see that $d(g(u), f(u)) \leq \sigma_0^*(G) + \operatorname{diam} H$. So, $\sigma_0^*(G^v \vee H^w) \leq \sigma_0^*(G) + \operatorname{diam} H$. Similarly, we can show that $\sigma_0^*(G^v \vee H^w) \leq \sigma_0^*(H) + \operatorname{diam} G$.

There is a vertex $u \in V(G')$ such that $f(u) = w$ and either $g(u) \in V(G)$ or $g(u) \in V(H)$. and so

$$d(f(u), g(u)) \leq \max\{\operatorname{diam} G, \operatorname{diam} H\}$$

and

$$d(f, g) \leq \max\{\operatorname{diam} G, \operatorname{diam} H\}.$$

Hence, we have shown that

$$\sigma_0^*(G^v \vee H^w) \leq \min\{\sigma_0^*(G) + \operatorname{diam} H, \sigma_0^*(H) + \operatorname{diam} G, \max\{\operatorname{diam} G, \operatorname{diam} H\}\}$$

and we have proved Theorem 3.

In Example 8 we give two graphs such that the surjective semispan of a wedge of the two graphs is equal to the lower bound but not the upper bound given in Theorem 3. Following that is an example of two graphs for which the surjective semispan of a wedge of the two graphs is equal to the upper bound but not the lower bound given in Theorem 3. In Example 10, two graphs are given such that the lower bound and upper bound for the surjective semispan of their wedge, as given in Theorem 3, are equal.

Example 8. Let $G = I^{u_0} \vee J^{w_0} \vee K^{y_0}$ and $H = P_1$, where I, J, K, and P_1 are the paths $u_0 - u_1 - u_2$, $w_0 - w_1 - w_2$, $y_0 - y_1 - y_2$, and $v_0 - v_1$ respectively. Then $\sigma_0^*(G) = 2$ and $\sigma_0^*(G^{u_0} \vee H^{v_0}) = 2$ both by Corollary 1. So,

$$\max\{\sigma_0^*(H), \sigma_0^*(G)\} = 2 <$$
$$\min\{\sigma_0^*(G) + \operatorname{diam} H, \sigma_0^*(H) + \operatorname{diam} G, \max\{\operatorname{diam} H, \operatorname{diam} G\}\}.$$

Example 9. Let $G = I^{u_0} \vee J^{w_0} \vee K^{y_0}$ and $H = P_1$, where I, J, K and P_1 are the paths $u_0 - u_1 - u_2$, $w_0 - w_1 - w_2$, $y_0 - y_1$, and $v_0 - v_1$ respectively. Then $\sigma_0^*(G^{y_1} \vee H^{v_0}) = 2$ by Corollary 1. So,

$$\max\{\sigma_0^*(H), \sigma_0^*(G)\} = 1 < 2 =$$
$$\min\{\sigma_0^*(G) + \operatorname{diam} H, \sigma_0^*(H) + \operatorname{diam} G, \max\{\operatorname{diam} H, \operatorname{diam} G\}\}$$

Example 10. $\sigma_0^*(C_n^{v_0} \vee C_m^{w_0}) = \max\{\left[\frac{n}{2}\right], \left[\frac{m}{2}\right]\}$. It follows from Example 2 that

$$\sigma_0^*(C_r) = \left[\frac{r}{2}\right] = \operatorname{diam}(C_r),$$

for any $r \geq 3$. Thus

$$\max\{\sigma_0^*(C_n), \sigma_0^*(C_m)\} = \max\{\operatorname{diam}(C_n), \operatorname{diam}(C_m)\} = \max\{\left[\frac{n}{2}\right], \left[\frac{m}{2}\right]\}.$$

It follows from Theorem 3 that

$$\max\{\sigma_0^*(C_n), \sigma_0^*(C_m)\} \leq \sigma_0^*(C_n^{v_0} \vee C_m^{w_0}) \leq \max\{\operatorname{diam}(C_n), \operatorname{diam}(C_m)\},$$

so we get

$$\max\{\sigma_0^*(C_n), \sigma_0^*(C_m)\} =$$
$$\sigma_0^*(C_n^{v_0} \vee C_m^{w_0}) = \max\{\operatorname{diam}(C_n), \operatorname{diam}(C_m)\} = \max\{\left[\frac{n}{2}\right], \left[\frac{m}{2}\right]\}.$$

Theorem 4. *Let G be a connected graph. Then there are homomorphisms $f, g \colon V(P_m) \to V(G)$ for some m such that $f(V(P_m)) = V(G)$ and $d(f, g) = \sigma_0^*(G)$.*

Proof: Let $f', g' \colon V(G') \to V(G)$ be homomorphisms such that G' is a connected graph, $f'(V(G')) = V(G)$, and $d(f', g') = \sigma_0^*(G)$. Let $V(G') = \{a_i\}_{i=0}^n$. Since G' is connected there is a path in G', $a_0 = y_0^1 - y_1^1 - \cdots - y_{n_1}^1 = a_1$ from a_0 to a_1. Similarly, there are paths $a_{i-1} = y_{k_i+1}^i - \cdots y_{k_i+n_i}^i = a_i$ in G' from a_{i-1} to a_i for $i = 2, 3, \ldots, n$ where $k_i = \sum_{j=1}^{i-1} n_j$.

We define a homomorphism $h \colon V(P_m) \to V(G')$ where $m = \sum_{i=1}^n n_i$ by

$$h(v_j) = y_j^1 \quad 0 \leq j \leq n_1$$
$$h(v_j) = y_j^i \quad k_i + 1 \leq j \leq k_i + n_i$$

We see that $f' \circ h, g' \circ h \colon V(P_m) \to V(G)$ are homomorphisms from $V(P_m)$ into $V(G)$ such that $f' \circ h(V(P_m)) = f'(V(G')) = V(G)$ and $d(f' \circ h, g' \circ h) = d(f', g') = \sigma_0^*(G)$. So, $f = f' \circ h$ and $g = g' \circ h$ are the required homomorphisms and we have proved Theorem 4.

Theorem 5. *Let $G \neq K_0$ be a connected graph. Then $\sigma_0^*(G) = \sigma_0^*(G')$ for each subdivision G' of G if and only if G is a path.*

Proof: If $G = P_n$ for some $n \in Z^+$ then for a subdivision G' of P_n, $G' = P_m$ for some $m \geq n$ and $\sigma_0^*(P_m) = \sigma_0^*(P_n) = 1$.

Now suppose that $G \neq K_0$ is a connected graph which is not a path. So, G is either a cycle or it has a vertex of order greater than two. In both cases we can find a subdivision G' of G such that $\sigma_0^*(G') > \sigma_0^*(G)$.

If $G = C_n$ for some n then we let G' be a subdivision of G obtained by subdividing any two edges of G. Then $G' = C_{n+2}$ and $\sigma_0^*(G') = \left[\frac{n+2}{2}\right] = \left[\frac{n}{2}\right] + 1 = \sigma_0^*(G) + 1$.

If G has a vertex v of order greater than two, let v_0v, v_1v and v_2v be three edges of G. Suppose that $\sigma_0^*(G) = n$. We obtain a subdivision G' of G by replacing each edge v_iv with the edges $v_iw_{n+1}^i, w_{n+1}^i w_{n+2}^i, \ldots, w_{2n}^i v_i^1$, $v_i^1 w_1^i, w_1^i w_2^i, \ldots, w_{n-1}^i w_n^i$, and $w_n^i v$ for $i = 0, 1, 2$.

Let G_1 be the graph obtained by removing the vertices w_{n+1}^0 and w_n^0 from G'. Let D_1 be the component of G_1 which contains v. Let G_2 be the graph obtained by removing the vertices w_{n+1}^1 and w_n^1 from G'. Let D_2 be the component of G_2 which contains v.

We now define homomorphisms $f, g\colon V\left[(D_1^{v_2^1} \vee P_k^{w_0})^{w_k} \vee D_2^{v_0^1}\right] \to V(G')$ where $k = 4n + 4$ and P_k is the path $w_0 - w_1 - \cdots - w_k$ as follows:

$$\begin{cases} f(u) = u \\ g(u) = v_0^1 \end{cases} \quad u \in V(D_1)$$

$$\begin{cases} f(w_i) = v_2^1 \\ g(w_i) = w_i^0 \end{cases} \quad i = 1, \ldots, n$$

$$\begin{cases} f(w_{n+1}) = v_2^1 \\ g(w_{n+1}) = v \end{cases}$$

$$\begin{cases} f(w_i) = v_2^1 \\ g(w_i) = w_{2n-i+2}^1 \end{cases} \quad i = n+2, \ldots, 2n+1$$

$$\begin{cases} f(w_{2n+2}) = v_2^1 \\ g(w_{2n+2}) = v_1^1 \end{cases}$$

$$\begin{cases} f(w_i) = w_{i-(2n+2)}^2 \\ g(w_i) = v_1^1 \end{cases} \quad i = 2n+3, \ldots, 3n+2$$

$$\begin{cases} f(w_{3n+3}) = v \\ g(w_{3n+3}) = v_1^1 \end{cases}$$

$$\begin{cases} f(w_i) = w_{4n-k+4}^0 \\ g(w_i) = v_1^1 \end{cases} \quad i = 3n+4, \ldots, 4n+3$$

$$\begin{cases} f(w_{4n+4}) = v_0^1 \\ g(w_{4n+4}) = v_1^1 \end{cases}$$

$$\begin{cases} f(u) = u \\ g(u) = v_1^1 \end{cases} \quad u \in V(D_2)$$

We see that $f(V\left[(D_1^{v_2^1} \vee P_k^{w_0})^{w_k} \vee D_0^{v_0^1}\right]) = V(G')$ and $d(f,g) = n+1$. Consequently, $\sigma_0^*(G') \geq n+1 = \sigma_0^*(G)+1$ and we have proved Theorem 5.

4 Husimi Trees

Let T be a Husimi tree which is not a path. Let $P_n = v_0 - v_1 - \cdots - v_n$ be a diameter path in T. We define graphs T_j and B_j for $j = 1, 2, \ldots, n-1$, a set $\{w_j\}_{j \in A}$ where $A \subset \{1, 2, \ldots, n-1\}$, a set $\{b_j\}_{j=1}^{n-1}$ and an integer $b(P_n)$ based on P_n.

Let T_j for $j = 1, 2, \ldots, n-1$ be the subgraph of T such that $V(T_j) = V(T)$ and $E(T_j) = E(T) - (\{v_{j-1}w \mid v_{j-1}w \in E(B_{v_{j-1},v_j})\} \cup \{wv_{j+1} \mid v_{j+1}w \in E(B_{v_j,v_{j+1}})\})$ where $B_{v_j,v_{j+1}}$ is the block of T which contains v_j and v_{j+1} for $j = 0, 1, \ldots, n-1$.

For each j, let B_j be the component of T_j which contains v_j. By the definition of T_j it is clear that $v_i \notin V(B_j)$ for $i \neq j$. If $V(B_j) - \{v_j\} = \emptyset$ then let $b_j = 0$. If $V(B_j) - \{v_j\} \neq \emptyset$ let $w_j \in V(B_j)$ such that $d(v_j, w_j) \geq d(v_j, w)$ for each $w \in V(B_j)$ and $b_j = d(v_j, w_j)$ for each $j = 1, 2, \ldots, n-1$. Let $b(P_n) = \max\{b_j\}_{j=1}^{n-1}$.

Next we will determine an upper bound for each b_j where $b_j \neq 0$. Since T is not a path, it is clear that not all of the b_j's are equal to zero. For each j such that $b_j \neq 0$, w_j has been defined. If $d(w_j, v_i) > d(w_j, v_j)$ for $i \neq j$ (which means every path from w_j to v_i when $i \neq j$ goes through v_j) then $d(v_0, w_j) = j + b_j \leq n$ and $d(v_n, w_j) = n - j + b_j \leq n$ since the diameter of T is n. Consequently,

$*\ b_j \leq \min\{j, n-j\}$.

If there is an $i \neq j$ such that $d(v_i, w_j) = d(v_j, w_j)$ then there is a path from w_j to v_i which does not include v_j and a path from w_j to v_j which does not include v_i. The union of these two paths and the edge v_iv_j contains a cycle which includes the edge v_iv_j. Consequently, v_i and v_j are in the same block of T. Since all such blocks in a Husimi tree are complete, $d(v_i, v_j) = 1$ and $|i-j| = 1$.

If $j - i = 1$ then $d(v_0, w_j) = d(v_0, v_i) + d(v_i, w_j) = j - 1 + b_j \leq n$ and $d(v_n, w_j) = d(v_n, v_j) + d(v_j, w_j) = n - j + b_j \leq n$. So when $i = j-1$,

$**\ b_j \leq \min\{n-j+1, j\}$.

If $i-j=1$ then $d(v_0,w_j)=d(v_0,v_j)+d(v_j,w_j)\leq j+b_j\leq n$ and $d(v_n,w_j)=d(v_n,v_{j+1})+d(v_{j+1},w_j)=n-(j+1)+b_j\leq n$. So when $i=j+1$,

$*** \; b_j\leq \min\{n-j,j+1\}$.

Next we will determine upper bounds for certain subsets of $\{b_j\}_{j=1}^{n-1}$.

$****$ Let $1\leq j<i\leq \frac{n+1}{2}$. Then $b_j\leq \max\{\min\{j,n-j\},\min\{n-j,j+1\},\min\{n-j+1,j\}\}=\max\{j,j+1,j\}=j+1\leq i$.

$*****$ Let $\frac{n-1}{2}\leq i<j\leq n-1$. Then $b_j\leq \max\{\min\{j,n-j\},\min\{n-j,j+1\},\min\{n-j+1,j\}\}=\max\{n-j,n-j,n-j+1\}=n-j+1\leq n-i$.

Theorem 6. *Let T be a Husimi tree which is not a path. If P_n is a diameter path of T, then $\sigma_0^*(T)=b(P_n)$.*

Proof: Let $P_n=v_0-v_1-\cdots-v_n$ be a diameter path of T. Suppose that $b=b(P_n)=b_k$. There are three cases to consider:

Case i) $d(w_k,v_j)>b_k$ for $j\neq k$

Case ii) $d(w_k,v_{k-1})=b_k$

Case iii) $d(w_k,v_{k+1})=b_k$

Case i. $d(w_k,v_j)>b_k$ for $j\neq k$.

Let H be the connected subgraph of T where $E(H)=E(T)-E(B_k)$ and $Q=z_0-z_1-\cdots-z_m$ where $m=n+b_k$. Let $G=(H^{v_0}\vee Q^{z_0})^{z_m}\vee B_k^{v_k}$. Let $w_k=q_0-\cdots-q_t=v_n$ be the shortest path in T from w_k to v_n. Note that $t=b_k+n-k$. We define homomorphisms $f,g\colon V(G)\rightarrow V(T)$ as follows:

$$\begin{cases} f(v)=v \\ g(v)=w_k \end{cases} \quad v\in V(H)$$

$$\begin{cases} f(z_i)=v_0 \\ g(z_i)=q_i \end{cases} \quad 0\leq i\leq t=b_k+n-k$$

$$\begin{cases} f(z_i)=v_{i-(b_k+n-k)} \\ g(z_i)=v_n \end{cases} \quad b_k+n-k\leq i\leq m=n+b_k$$

$$\begin{cases} f(v)=v \\ g(v)=v_n \end{cases} \quad v\in V(B_k)$$

So, $f(V(G))=V(T)$ and $d(f,g)=\min\{k,n-k,b_k\}=b_k$.

Case ii. $b_k=d(w_k,v_k)=d(w_k,v_{k-1})$.

Let H_1' be the subgraph of T such that $E(H_1') = E(T) - \{v_{k-1}v \mid v \in B_{v_{k-1}v_k}\}$. Let H_1 be the component of H_1' which contains v_k. Let H_2' be the subgraph of T such that $E(H_2') = E(T) - \{v_k v \mid v \in B_{v_{k-1}v_k}\}$. Let H_2 be the component of H_2' which contains v_{k-1}. Let $Q = z_0 - z_1 - \cdots - z_n$ and $G = (H_1^{w_k} \vee Q^{z_0})^{z_n} \vee H_2^{w_k}$. We define homomorphisms $f, g\colon V(G) \to V(T)$ as follows:

$$\begin{cases} f(v) = v \\ g(v) = v_0 \end{cases} v \in V(H_1)$$

$$\begin{cases} f(z_i) = w_k \\ g(z_i) = v_i \end{cases} 0 \le i \le n$$

$$\begin{cases} f(v) = v \\ g(v) = v_n \end{cases} v \in V(H_2).$$

So, $f(V(G)) = V(T)$ and $d(f,g) = \min\{k, n-k+1, b_k\} = b_k$.

Case iii. $b_k = d(w_k, v_k) = d(w_k, v_{k+1})$.

The proof is similar to the proof of case (ii).

Consequently, in all three cases we have seen that $\sigma_0^*(T) \ge b$. Next we will show that $\sigma_0^*(T) = b$.

Let $f, g\colon V(H) \to V(T)$ be homomorphisms such that $f(V(H)) = V(T)$. Define a homomorphism $h\colon V(T) \to V(P_n)$, where P_n is the diameter path $v_0 - v_1 - \cdots - v_n$ in T by

$$h(v) = \begin{cases} v_i & \text{when } d(v, v_i) < d(v, v_j) \text{ for } 0 \le j \le n \text{ and } j \ne i \\ v_i & \text{when } d(v, v_i) = d(v, v_{i-1}). \end{cases}$$

Let $q_0 - q_1 - \cdots - q_r$ be a path in H such that $h \circ f(q_0) = v_0$, $h \circ f(q_r) = v_n$ and $h \circ f(q_i) \notin \{v_0, v_n\}$ for $1 \le i \le r-1$. Let $t = \max\{i \mid h \circ f(q_i) < h \circ g(q_i)\}$ where we consider P_n to be ordered by $v_0 < v_1 < \cdots < v_n$. So for ℓ where $t+1 \le \ell \le r$, $h \circ g(q_\ell) \le h \circ f(q_\ell)$. Let $h \circ f(q_t) = v_j$. There are four cases to consider:

Case i.

$h \circ g(q_t) = v_{j+1}$

and

$h \circ f(q_{t+1}) = h \circ g(q_{t+1}) = v_j$

Case ii.

$h \circ g(q_t) = v_{j+1}$,

$h \circ g(q_{t+1}) = v_j$,

and

$h \circ f(q_{t+1}) = v_{j+1}$

Case iii.

$h \circ g(q_t) = v_{j+2}$

and

$h \circ f(q_{t+1}) = h \circ g(q_{t+1}) = v_{j+1}$

Case iv.

$h \circ g(q_t) = v_{j+1}$

and

$h \circ g(q_{t+1}) = h \circ f(q_{t+1}) = v_{j+1}.$

Case i.

$h \circ g(q_t) = v_{j+1}$

and

$h \circ f(q_{t+1}) = h \circ g(q_{t+1}) = v_j$

Since $h \circ g(q_t) = v_{j+1}$ and $h \circ g(q_{t+1}) = v_j$, $g(q_t) \in V(B_{v_j,v_{j+1}})$, $d(v_j, g(q_t)) = 1$, and $g(q_{t+1}) = v_j$. Since $h \circ f(q_{t+1}) = v_j$, $f(q_{t+1}) \in V(B_j)$. Consequently, $d(f,g) \leq d(g(q_{t+1}), f(q_{t+1})) = d(v_j, f(q_{t+1})) \leq b_j \leq b$.

Case ii.

$h \circ g(q_t) = v_{j+1},$

$h \circ g(q_{t+1}) = v_j,$

and

$h \circ f(q_{t+1}) = v_{j+1}$

Since $h \circ f(q_t) = v_j$ and $h \circ f(q_{t+1}) = v_{j+1}$, $f(q_t) = v_j$. Since $h \circ g(q_t) = v_{j+1}$ and $h \circ g(q_{t+1}) = v_j$, $g(q_t) \in V(B_{v_j,v_{j+1}})$ and $d(g(q_t), v_j) = 1$. So, $d(f,g) \leq d(f(q_t), g(q_t)) \leq d(v_j, g(q_t)) = 1 \leq b$.

Case iii.

$h \circ g(q_t) = v_{j+2}$

and

$h \circ f(q_{t+1}) = h \circ g(q_{t+1}) = v_{j+1}$

Since $h \circ g(q_t) = v_{j+2}$ and $h \circ g(q_{t+1}) = v_{j+1}$, it must be the case that $g(q_t) \in V(B_{v_{j+1},v_{j+2}})$ and $g(q_{t+1}) = v_{j+1}$. Since $h \circ f(q_t) = v_j$ and $h \circ f(q_{t+1}) = v_{j+1}$, it must be the case that $f(q_t) = v_j$ and $f(q_{t+1}) \in V(B_{v_j,v_{j+1}})$. Consequently, $d(f,g) \leq d(g(q_{t+1}), f(q_{t+1})) = d(v_{j+1}, f(q_{t+1})) = 1 \leq b$.

Case iv.

$h \circ g(q_t) = v_{j+1}$

and

$h \circ g(q_{t+1}) = h \circ f(q_{t+1}) = v_{j+1}.$

Since $h \circ g(q_t) = h \circ g(q_{t+1}) = v_{j+1}$, we see that $g(q_t) \in V(B_{j+1})$ and $g(q_{t+1}) \in V(B_{j+1})$. Since $h \circ f(q_t) = v_j$ and $h \circ f(q_{t+1}) = v_{j+1}$, we see that $f(q_t) = v_j$ and $f(q_{t+1}) \in V(B_{v_j,v_{j+1}}) - \{v_j\}$.

If $f(q_{t+1}) = v_{j+1}$ then $d(f,g) \leq d(f(q_{t+1}), g(q_{t+1})) = d(v_{j+1}, g(q_{t+1})) \leq b_{j+1} \leq b$. If $g(q_{t+1}) = v_{j+1}$ then $d(f,g) \leq d(f(q_{t+1}), g(q_{t+1})) = d(f(q_{t+1}), v_{j+1}) \leq 1 \leq b$.

Next, we need to consider the case where $g(q_{t+1}) \in V(B_{j+1}) - \{v_{j+1}\}$ and $f(q_{t+1}) \in V(B_{v_j,v_{j+1}}) - \{v_{j+1}, v_j\}$. Let $r^* = \min\{i \mid t+1 < i \leq r \text{ and } f(q_i) = v_{j+1}\}$. If $g(q_r^*) \in V(B_{j+1})$ then $d(f,g) \leq d(f(q_r^*), g(q_r^*)) = d(v_{j+1}, g(q_r^*)) \leq b_{j+1} \leq b$. If $g(q_r^*) \notin V(B_{j+1})$ then let $t^* = \min\{i \mid t+1 < i < r^* \text{ and } g(q_i) = v_{j+1}\}$. So, $f(q_t^*) \in V(B_{v_j,v_{j+1}})$ since $h o f(q_t^*) \geq h o g(q_t^*)$. So $d(f,g) \leq d(g(q_t^*), f(q_t^*)) = d(v_{j+1}, f(q_t^*)) \leq b_{j+1} \leq b$.

Consequently, in all cases we have shown that $d(f,g) \leq b$. Hence, $\sigma_0^*(T) = b$. Also, since the diameter path we choose in this proof was arbitrary, we get the following corollary.

Corollary 1. *Let T be a Husimi tree which is not a path. Let P_n and P_n' be two diameter paths of T. Then $b(P_n) = b(P_n')$.*

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