Independence, domination, irredundance, and forbidden pairs

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ABSTRACT. For different properties $\mathcal{P}$ of a connected graph G, we characterize the connected graphs F (resp. the pairs (X, Y) of connected graphs) such that G has Property $\mathcal{P}$ if G does not admit F (resp. neither X nor Y) as an induced subgraph. We consider here the lower and upper independence, domination and irredundance parameters which are related by the well known inequalities $ir \leq \gamma \leq i \leq \alpha \leq \Gamma \leq IR$, and the properties $\mathcal{P}$ corresponding to the equality between some of these parameters.

1 Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$. The subgraph induced by a subset A of V is denoted by $G[A]$. The closed neighbor of a vertex x is $N[x] = N(x) \cup \{x\}$ and for $A \subseteq V$, $N[A] = \bigcup_{x \in A} N[x]$.

A set D of vertices of G is *dominating* if every vertex of $V - D$ has at least one neighbor in D. The minimum cardinality of a dominating set is denoted by $\gamma(G)$ and the maximum cardinality of a minimal (under inclusion) dominating set by $\Gamma(G)$.

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A set S of vertices of G is *independent* if no two vertices of S are adjacent. The maximum cardinality of an independent set is denoted by $\alpha(G)$ and the minimum cardinality of a maximal independent set by $i(G)$.

A vertex x of a set I of vertices of G is *irredundant* in I if $N[x] - N[I - \{x\}] \neq \emptyset$, and *redundant* otherwise. When x is irredundant in I, the vertices of $N[x] - N[I - \{x\}]$ are called the *I-private neighbors* of x. The *external* I-private neighbors of x are its I-private neighbors which are contained in $V - I$. The set I is *irredundant* if all its vertices are irredundant. Note that if I is irredundant and x is a non-isolated vertex of $G[I]$, then all the I-private neighbors of x are external. Throughout the paper, if there is no ambiguity, the accented letter x' will always be used to denote an external I-private neighbor of the vertex x of an irredundant set I. The maximum cardinality of an irredundant set is denoted by $IR(G)$ and the minimum cardinality of a maximal irredundant set by $ir(G)$.

The three notions of domination, independence and irredundance are closely related. It is clear from the definitions that a set is a maximal independent set if and only if it is both independent and dominating, and in this case it is a minimal dominating set. Also, a set is a minimal dominating set if and only if it is both dominating and irredundant, and in this case a maximal irredundant set. This leads to the following inequality chain, valid in any graph G as first observed in [3]:

$$(*)\ ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G).$$

However, a minimal dominating set is obviously not necessarily independent, and a maximal irredundant set is not necessarily dominating. The following property of the maximal irredundant sets is worth noting and of common use:

> Let I be a maximal irredundant set of G which does not dominate V. Then, for every vertex u which is not dominated by I, there exists at least one non-isolated vertex y of $G[I]$ such that u dominates the whole I-private neighborhood of y.

The reason is that, if the conclusion was not true, $I \cup \{u\}$ would be irredundant, in contradiction to the maximality of I.

Given a family $\mathcal{F} = \{H_1, H_2, ..., H_k\}$ of graphs we say that the graph G is $\mathcal{F}$-free if G contains no induced subgraph isomorphic to any H_i, $i = 1, 2, ..., k$. In particular, if $\mathcal{F} = \{F\}$, we simply say G is F-free.

Our aim is to characterize connected graphs F, or pairs of connected graphs (X, Y), such that G has a given property $\mathcal{P}$ if G is F-free or (X, Y)-free. We are interested here in properties $\mathcal{P}$ of the type "two among the six previously defined parameters are equal". Similar problems have already been considered for other properties, especially for hamiltonian properties (see e.g. [2], [5]).

For each property $\mathcal{P}$, the characterization of F (resp. of (X, Y)) contains two parts. In the direct part we prove that every F-free (resp. (X, Y)-free) graph has Property $\mathcal{P}$. The direct parts are all contained in Section 2. In the converse part we prove that if any F-free graph (resp. any (X, Y)-free graph) has Property $\mathcal{P}$, then the graph F (resp. the pair (X, Y)) belongs to a previously defined list. Sections 3 to 8 are devoted to the converse parts of the characterization of $\mathcal{F}$ for different properties $\mathcal{P}$. For some of them, the characterization is complete. For other ones, we have only partial results. Note that all the properties considered in this paper are true for G if and only if they are true for each connected component of G.

Let $\mathcal{G}$ be a family of graphs, and $\mathcal{F}$ a subfamily of $\mathcal{G}$. If $\mathcal{F}$-free implies Property $\mathcal{P}$, then obviously, $\mathcal{G}$-free implies $\mathcal{P}$. Therefore we look for minimal families of forbidden subgraphs. In particular, if we know that for any graph, F-free implies $\mathcal{P}$, then in the research of pairs (X, Y) for which (X, Y)-free implies $\mathcal{P}$, we suppose that neither of (X, Y) is an induced subgraph of F. Similarly we suppose that neither of (X, Y) is a subgraph of the other one.

If the condition G is F-free (resp. (X, Y)-free) implies G satisfies $\mathcal{P}$, then a fortiori, for any induced subgraph F' of F (resp. any induced subgraphs X' of X and Y' of Y), the condition G is F'-free (resp. G is (X', Y')-free) implies G satisfies $\mathcal{P}$. Since after we have determined F, or (X, Y), it is easy and of little interest to enumerate all the subgraphs F' of F, or all the pairs (X', Y') with X' subgraph of X and Y' subgraph of Y, we try only to determine the maximal graphs F_0 (resp. the maximal pairs (X_0, Y_0)) such that any F_0-free graph (resp. any (X_0, Y_0)-free graph) satisfies $\mathcal{P}$.

To establish the converse part of a result related for instance to one forbidden graph F, we construct several graphs M_1, M_2,..., M_t which do not satisfy $\mathcal{P}$. These graphs are thus not F-free and F is an induced subgraph of each M_i, that is a subgraph of a maximal common subgraph F_0 of all the M_i's. Moreover if the graphs M_i are arbitrarily large, the following stronger statement is proved: for a given positive integer n_0, if the condition "G is F-free" implies that G has Property $\mathcal{P}$ for any graph G of order at least n_0, then F is an induced subgraph of F_0. This is why in Examples 3.1, 4.1, 5.1, 7.1 and 8.1 given later, we describe infinite families of graphs $H_i(k)$ and $L_i(k)$ for which $\mathcal{P}$ is not satisfied. The same remark holds for pairs of forbidden subgraphs.

Figure 1 shows some special small graphs which will be used in the paper. The notation of some of them is classical such as the Claw $C = K_{1,3}$, the Bull B, the Deer D, or the Wounded W. In an extended claw $C_{i,j,k}$, i, j and k denote the respective lengths of the branches. So C is an abbreviation for $C_{1,1,1}$. When we enumerate the vertices of a claw or of an extended claw, we always begin by the center and separate it from the other vertices with a semi-colon.

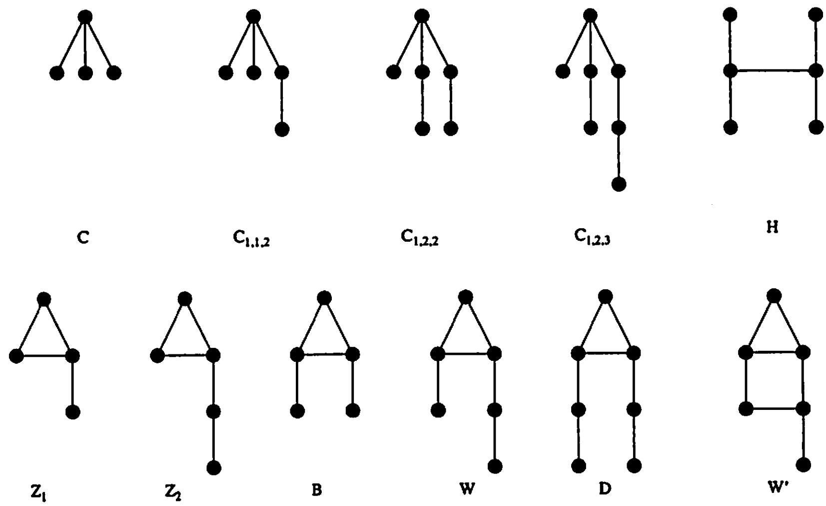

Figure 1

2 Direct results

In this section we give some forbidden subgraphs results, some of which are straightforward or already known. These results are presented here for convenience and to simplify the proofs of Theorems 3.2, 3.3, 4.2, 4.3, 6.1, 7.2 and 7.3.

Remark 2.1 : Any P_3-free graph satisfies $ir = \gamma = i = \alpha = \Gamma = IR$.

Proof : If G is P_3-free, it is the disjoint union of q cliques, and $ir = \gamma = i = \alpha = \Gamma = IR = q$.

Theorem 2.2. (Allan and Laskar [1]) *Any C-free graph satisfies $i = \gamma$.*

Theorem 2.3 (Favaron [6]) : Any (C, D)-free graph satisfies $ir = i$.

Theorem 2.4. *Any P_4-free graph satisfies $ir = \gamma$ and $\alpha = IR$.*

Proof : Let I be a maximal irredundant set of a P_4-free graph G. If there exists a vertex u which is not dominated by I, then u dominates the I-private neighborhood of a non-isolated vertex x of $G[I]$. Then, $G[u, x', x, y] \simeq P_4$ where x' is a I-private neighbor of x, and y a neighbor of x in I. So I is a dominating set and $|I| \geq \gamma$.

If we chose I to be a minimum maximal irredundant set, we find $ir \geq \gamma$ and thus $ir = \gamma$ by (*).

Let us now choose I to be a maximum irredundant set such that $G[I]$ has a minimum number of edges. If a connected component C of $G[I]$ contains a vertex x of degree at least two, let x_1 and x_2 be two neighbors of x in I, and let x', x'_1, x'_2 be respective I-private neighbors of x, x_1, x_2. If $x'_1x' \notin E$, then $G[x'_1, x_1, x, x'] \simeq P_4$, and if $x'_1x' \in E$, then $G[x'_1, x', x, x_2] \simeq P_4$. So the components of $G[I]$ are isomorphic to K_1 or K_2. Moreover, if $\{x, x_1\}$ is a component of $G[I]$, then x'_1 can be adjacent to x', but to no external I-private neighbor y' of another vertex y of I for otherwise $G[x_1, x'_1, y', y] \simeq P_4$. The set $I' = (I - \{x_1\}) \cup \{x'_1\}$, of same order as I, is irredundant (x and x'_1 are isolated in I') and $G[I']$ has fewer edges than $G[I]$, a contradiction to the choice of I. Therefore I is independent and thus $IR \leq \alpha$, which implies $IR = \alpha$ by (*). □

Note that the second part of this theorem is also a corollary of Theorem 2.8.

Theorem 2.5.

a) *Any $(P_4, K_{3,3})$-free graph satisfies $i = ir$.*

b) *Any (C_4, H)-free graph satisfies $i = \gamma$.*

Proof : Let A be a minimum dominating set such that $G[A]$ contains the minimum number of edges. If A is not independent, let x and y be two adjacent vertices of A. Since A is irredundant, the A-private neighborhood $B_x = \{x'_1, x'_2, \cdots, x'_t\}$ of x is not empty. The vertex x'_1 does not dominate B_x for otherwise $A' = (A - \{x\}) \cup \{x'_1\}$ is a minimum dominating set such that $G[A']$ contains fewer edges than $G[A]$. Hence x admits at least two non-adjacent A-private neighbors x'_1 and x'_2. Similarly, y admits two non-adjacent A-private neighbors y'_1 and y'_2.

a) If G is P_4-free, then the four edges $x'_1y'_1$, $x'_1y'_2$, $x'_2y'_1$, $x'_2y'_2$ exist and $G[x, y'_1, y'_2, y, x'_1, x'_2] \simeq K_{3,3}$. If moreover G is $K_{3,3}$-free, we get a contradiction, the dominating set A must be independent and thus $i \leq \gamma$. This implies $i = \gamma$ by (*), and $i = ir$ by Theorem 2.4.

b) If G is H-free, then at least one of the four edges $x'_1y'_1$, $x'_1y'_2$, $x'_2y'_1$, $x'_2y'_2$ exists, and G contains C_4. So, if G is also C_4-free, the dominating set A is independent and $i = \gamma$. □

In the study of the irredundance in P_5-free graphs, we use the following two lemmas.

Lemma 2.6. *Let I be an irredundant set of a P_5-free graph G.*

a) *If x_1 and x_2 are two non isolated vertices of $G[I]$ which are not in the same connected component of $G[I]$, and if x'_1 and x'_2 are respective I-private neighbors of x_1 and x_2, then x'_1 and x'_2 are not adjacent.*

b) *$G[I]$ contains no induced subgraph isomorphic to P_4, Z_1, C_4, or $C_4{+}e$ (see Figure 2).*

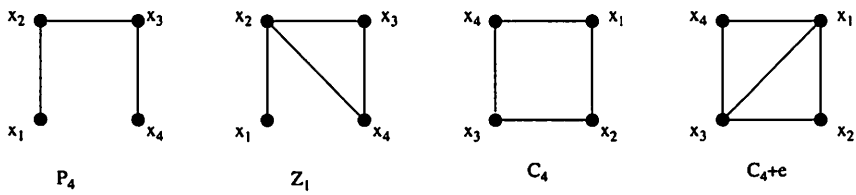

Figure 2

Proof: a) Let x_3 be a neighbor of x_1 in I. If $x'_1x'_2 \in E$, then $G[x_3, x_1, x'_1, x'_2, x_2] \simeq P_5$, a contradiction.
b) Let x_1, x_2, x_3, x_4 be four vertices of I such that $G[x_1, x_2, x_3, x_4]$ is one of the graphs of Figure 2. If $G[x_1, x_2, x_3, x_4] \simeq P_4$ or Z_1, then $x'_1x'_3 \in E$ for otherwise $G[x'_1, x_1, x_2, x_3, x'_3] \simeq P_5$, and thus $G[x_1, x'_1, x'_3, x_3, x_4] \simeq P_5$, a contradiction. If $G[x_1, x_2, x_3, x_4] \simeq C_4$ or $C_4{+}e$, then $G[x'_2, x_2, x_1, x_4, x'_4] \not\simeq P_5$ implies $x'_2x'_4 \in E$. Now, $G[x'_1, x_1, x_2, x'_2, x'_4] \not\simeq P_5$ implies $x'_1x'_2 \in E$ or $x'_1x'_4 \in E$, say without loss of generality $x'_1x'_2 \in E$. Then $G[x_4, x_3, x_2, x'_2, x'_1] \simeq P_5$, a contradiction. □

Lemma 2.7. *Let G be a P_5-free graph, and I a maximum irredundant set of G such that $G[I]$ contains the minimum number of edges. Then, for every connected component C of order $p \geq 2$ of $G[I]$, p is at least 3 and C is isomorphic to the complete graph K_p. Moreover, if $C = \{x_1, x_2, \dots, x_p\}$ and x'_i is any I-private neighbor of x_i for $1 \leq i \leq p$, then $G[C \cup \{x'_1, x'_2, \dots, x'_p\}] \simeq K_p \times K_2$.*

Proof : Let $C = \{x_1, x_2, \dots, x_p\}$ be a component of $G[I]$. If C is a star $K_{1,p-1}$ centered at x_1, then, by Lemma 2.6.a, $(I - \{x_1\}) \cup \{x'_1\}$ is an irredundant set which contradicts the choice of I. Therefore C is not a star and thus $p \geq 3$. If $p = 3$, then $C \simeq K_3$. If $p \geq 4$, then, looking at every set of four vertices of C, we see by Lemma 2.6.b that C is still isomorphic to K_p. Moreover, by Lemma 2.6.a, $I' = (I - C) \cup \{x'_1, x'_2, \dots, x'_p\}$ is an irredundant set of same order as I. If $G[x'_1, x'_2, \dots, x'_p] \not\simeq K_p$, then I' contradicts the choice of I. Therefore $G[C \cup \{x'_1, x'_2, \dots, x'_p\}] \simeq K_p \times K_2$. □

With these two lemmas we are prepared to prove the following theorem.

Theorem 2.8. *Every $(P_5, K_3 \times K_2)$-free graph satisfies $\alpha = IR$.*

Proof: Let I be a maximum irredundant set of minimum size in the graph G. If I is not independent, then by Lemma 2.7, G contains an induced $K_3 \times K_2$. Therefore I is independent, $|I| \leq \alpha$, and thus $\alpha(G) = IR(G)$ by (*). □

Theorem 2.9. *Every $(Z_1, C_{1,2,2})$-free connected graph G of order $n \geq 18$ satisfies $\alpha = IR$.*

Proof : It is known [7] that every connected Z_1-free graph is K_3-free or complete multipartite. Therefore if G contains a triangle then G is a complete multipartite graph $K_{n_1,n_2,\cdots,n_p}$ and thus α and IR are both equal to the largest n_i. Hence, we suppose G triangle-free, and we consider a maximum irredundant set I of minimum size. As usually, if x is a vertex of I, x' denotes an I-private neighbor of x. Let $\Delta(I)$ be the maximum degree of the induced subgraph $G[I]$.

If $\Delta(I) \geq 3$ then, since G is K_3-free, $G[I]$ contains a claw $(x_1; x_2, x_3, x_4)$ and $G[x'_2, x'_3, x'_4]$ is not a triangle. Suppose without loss of generality $x'_2x'_3 \notin E$. Then $G[x_1; x_4, x_2, x'_2, x_3, x'_3] \simeq C_{1,2,2}$, a contradiction. Hence $\Delta(I) \leq 2$. If $G[I]$ contains an induced path $x_1x_2x_3x_4x_5$, then $G[x_3; x'_3, x_2, x_1, x_4, x_5] \simeq C_{1,2,2}$, a contradiction. Therefore every connected component of $G[I]$ is a cycle of length 4 or 5, a path of length at most 4, or an isolated vertex.

The remainder of the proof will be broken into five cases that depend on the nature of the connected components $\mathcal{C}$ of $G[I]$.

$\mathcal{C} = P_3$. Let $\mathcal{C}$ be a connected component of $G[I]$ isomorphic to a path $x_1x_2x_3$. Suppose the vertex x'_2 is adjacent to an I-private neighbor y' of a non-isolated vertex y of $I - \mathcal{C}$, and let z be a neighbor of y in I. Since G is K_3-free and $G[x_2; x_1, x'_2, y', x_3, x'_3] \not\simeq C_{1,2,2}$, exactly one of $y'x'_3$ and $x'_2x'_3$ is an edge of G. If $x'_2x'_3 \in E$, then $G[x'_2; x'_3, y', y, x_2, x_1] \simeq C_{1,2,2}$, and if $y'x'_3 \in E$, then $G[y'; x'_3, y, z, x'_2, x_2] \simeq C_{1,2,2}$. In both cases we get a contradiction. So $(I - \mathcal{C}) \cup \{x_1, x'_2, x_3\}$ is an irredundant set contradicting the choice of I. This proves that $G[I]$ has no component isomorphic to P_3.

$\mathcal{C} = P_4$. Let $\mathcal{C}$ be a connected component of $G[I]$ isomorphic to a path $x_1x_2x_3x_4$. The properties $G[x_2; x'_2, x_1, x'_1, x_3, x_4] \not\simeq C_{1,2,2}$ and $G[x_3; x'_3, x_4, x'_4, x_2, x_1] \not\simeq C_{1,2,2}$ imply $x'_1x'_2 \in E$ and $x'_3x'_4 \in E$. The vertex x'_2 (resp. x'_3) is adjacent to no external I-private neighbor y' of any vertex y of $I - \mathcal{C}$ for otherwise , since G is K_3-free, $G[x'_2; x'_1, y', y, x_2, x_3] \simeq C_{1,2,2}$ (resp. $G[x'_3; x'_4, y', y, x_3, x_2] \simeq C_{1,2,2}$). If $x'_2x'_3 \notin E$, $(I - \mathcal{C}) \cup \{x_1, x'_2, x'_3, x_4\}$ is an irredundant set contradicting the choice of I. Therefore $x_2x'_3 \in E$. If x'_1 is adjacent to an external I-private neighbor y' of a vertex y of $I - \mathcal{C}$, then, since G is K_3-free and $x'_3y' \notin E$ as previously verified, $G[x'_1; x_1, y', y, x'_2, x'_3] \simeq C_{1,2,2}$, a contradiction. Hence $I' = (I - \mathcal{C}) \cup \{x'_1, x_2, x'_3, x_4\}$ is an irredundant set of order $|I|$ such that $G[I']$ contains fewer edges than $G[I]$. This proves that $G[I]$ has no component isomorphic to P_4.

$\mathcal{C} = C_4$. Let $\mathcal{C}$ be a connected component of $G[I]$ isomorphic to a cycle $x_1x_2x_3x_4x_1$.

Claim: For $1 \leq i \leq 4$, no vertex x'_i dominates the I-private neighborhood of a vertex y of $I - \mathcal{C}$.

Proof of the claim: First we show that the vertex x'_i is adjacent to at least one of x'_{i+1} and x'_{i-1}, where the indices are taken modulo 4 with $1 \leq i \leq 4$. Consider without loss of generality $i = 1$. Since $G[x_1; x'_1, x_2, x'_2, x_4, x'_4] \not\simeq C_{1,2,2}$, either x'_1 is adjacent to x'_2 or to x'_4, or x'_2 and x'_4 are adjacent. But in the latter case, $G[x_2; x_3, x'_2, x'_4, x_1, x'_1] \not\simeq C_{1,2,2}$ again implies $x'_1x'_2 \in E$ or

$x'_1x'_4 \in E$. Suppose now some x'_i, say x'_1, is adjacent to some external I-private neighbor y' of a vertex y of $I - C$. Then, if without loss of generality $x'_1x'_2 \in E$, and since G is K_3-free, $G[x'_1; x'_2, y', y, x_1, x_4] \simeq C_{1,2,2}$, a contradiction. Hence no x'_i dominates the I-private neighborhood of a vertex of $I - C$. ◇

The first consequence of the claim is that each x_i, $1 \leq i \leq 4$, has exactly one I-private neighbor . For, if for instance x_1 has two I-private neighbors x'_1 and x''_1 (necessarily non-adjacent by the K_3-free condition), then $(I - \{x_1, x_3\}) \cup \{x'_1, x''_1\}$ is an irredundant set contradicting the choice of I. The second consequence is that $(I - C) \cup \{x'_1, x'_2, x'_3, x'_4\}$ is an irredundant set of same order as I, and thus, by the choice of I, $G[x'_1, x'_2, x'_3, x'_4]$ contains at least four edges. Since G is K_3-free, $G[x'_1, x'_2, x'_3, x'_4]$ is isomorphic to a cycle which can be $x'_1x'_2x'_3x'_4x'_1$ or, without loss of generality, $x'_1x'_2x'_4x'_3x'_1$. In the first case, $I' = (I - C) \cup \{x_1, x_3, x'_2, x'_4\}$ is an irredundant set of same order as I such that $G[I']$ contains fewer edges than $G[I]$, a contradiction. Therefore $G[x'_1, x'_2, x'_3, x'_4]$ is the cycle $x'_1x'_2x'_4x'_3x'_1$.

Let $M = G[x_1, x_2, x_3, x_4, x'_1, x'_2, x'_3, x'_4]$. Suppose G contains a vertex w at distance two from M. This vertex w is not dominated by M and there exists a vertex v in $G - M$ such that $vw \in E$ and, say by symmetry between $\{x_1, x_2, x_3, x_4\}$ and $\{x'_1, x'_2, x'_3, x'_4\}$, $vx_1 \in E$. Since G is K_3-free, the condition $G[x_1; x'_1, v, w, x_4, x'_4] \not\simeq C_{1,2,2}$ implies $vx'_4 \in E$. By symmetry $vx'_2 \in E$ and thus $G[v, x'_2, x'_4] \simeq K_3$, a contradiction. Therefore, since G is connected, M dominates G. In particular, since a vertex v of $I - C$ cannot be adjacent to C nor to an I-private neighbor x'_i, $I = C$. If a vertex v of $G - M$ is dominated by C then, since the I-private neighborhoods of the x_i's have order one, v is adjacent to exactly two vertices of C, either x_1 and x_3, or x_2 and x_4, by the K_3-free condition. Let S_{13} (resp. S_{24}) be the set of the vertices of $G - M$ adjacent to x_1 and x_3 (resp. to x_2 and x_4). The sets S_{13} and S_{24} are independent and disjoint. Similarly, the vertices of $G - M$ which are adjacent to some vertex of $\{x'_1, x'_2, x'_3, x'_4\}$ belong to two disjoint independent sets, $S_{14} = \{v \in G - M; v \textit{ is adjacent to } x'_1 \textit{ and } x'_4\}$ and $S_{23} = \{v \in G - M; v \textit{ is adjacent to } x'_2 \textit{ and } x'_3\}$. By the hypothesis $n \geq 18$, at least one of these four independent sets, say S_{13}, contains at least two vertices, and $S_{13} \cup \{x_2, x_4\}$ is an irredundant set contradicting the choice of I. Therefore $G[I]$ has no component isomorphic to C_4.

$C = C_5$. Let C be a connected component of $G[I]$ isomorphic to a cycle $x_1x_2x_3x_4x_5x_1$. Since $G[x_2; x'_2, x_1, x'_1, x_3, x_4] \not\simeq C_{1,2,2}$, $x'_1x'_2 \in E$ and similarly $x'_ix_{i+1} \in E$ for every i, $1 \leq i \leq 5$, where the indices are taken modulo 5. Since G is K_3-free, $G[x'_1, x'_2, x'_3, x'_4, x'_5]$ is the cycle $x'_1x'_2x'_3x'_4x'_5x'_1$ and $G[x_1, x_2, x_3, x_4, x_5, x'_1, x'_2, x'_3, x'_4, x'_5] \simeq K_5 \times K_2$. Let us denote this subgraph by M. Suppose that some vertex w of G is at distance two from M. The vertex w is not dominated by M, and for some vertex $v \notin M$, $vw \in E$ and, say, $x_1v \in E$. Since G is K_3-free, the condition

$G[x_1; x_5, v, w, x_2, x_3] \not\simeq C_{1,2,2}$ implies $vx_3 \in E$. By symmetry, $vx_4 \in E$ and thus $G[v, x_3, x_4] \simeq K_3$, a contradiction. Therefore no vertex of G is at distance two from M, and since G is connected, M dominates G.

If some vertex $v \in G - M$ is adjacent to, say, x'_1, then the condition $G[x_1; x_5, x'_1, v, x_2, x_3] \not\simeq C_{1,2,2}$ implies that v is also adjacent to x_2, x_3 or x_5. In other words, the irredundant set C is dominating. C is thus a maximal irredundant set of G and $I = C$. We now prove that the case where $G[I]$ has a component isomorphic to C_5, and is thus itself isomorphic to C_5, cannot occur if G is sufficiently large.

Suppose first that x_1 has two I-private neighbors x'_1 and x''_1. Then, as seen previously, both $G[x'_1, x'_2, x'_3, x'_4, x'_5]$ and $G[x''_1, x'_2, x'_3, x'_4, x'_5]$ are isomorphic to a cycle C_5. By the K_3-free condition, x'_1 and x''_1 are independent and thus $I' = \{x_2, x_4, x'_1, x''_1, x'_3\}$ is an irredundant set (since independent) contradicting the choice of I. Similarly, each vertex x_i has exactly one I-private neighbor x'_i, and because of the K_3-free condition each of the $n-10$ vertices of $G - M$ has exactly two neighbors in $I = C$. If $2(n-10) > 15$, that is $n \geq 18$, then at least four vertices of $G - M$ have a common neighbor x_i in C. Again by the K_3-free condition, these four vertices and x'_i form an independent and thus irredundant set contradicting the choice of I. Therefore if $n \geq 18$, no component of $G[I]$ is isomorphic to C_5.

$C = P_2$ or P_1. We are now reduced to the case where $G[I]$ consists of isolated vertices and components isomorphic to K_2. Let $\{x_1, x_2\}$, $\{y_1, y_2\}$, $\{z_1, z_2\}$ be three such components. If $x'_1y'_1 \in E$ and $x'_1y'_2 \in E$, then $y'_1y'_2 \notin E$ by the K_3-free condition and $G[x'_1; y'_2, y'_1, y_1, x_1, x_2] \simeq C_{1,2,2}$, a contradiction. If $x'_1y'_1 \in E$ and $x'_1z'_1 \in E$, then $y'_1z'_1 \notin E$ by the K_3-free condition and $G[x'_1; x_1, y'_1, y_1, z'_1, z_1] \simeq C_{1,2,2}$, a contradiction. Therefore any I-private neighbor of a vertex of a K_2-component is adjacent to an I-private neighbor of at most one vertex belonging to another K_2-component. If I is not independent, we start with a K_2-component $\{x_{11}, x_{12}\}$. If x'_{12} has no neighbor in the I-private neighborhoods of the vertices of the other K_2-components, we set $I' = (I - \{x_{12}\}) \cup \{x'_{12}\}$. If x'_{12} is adjacent to, say, x'_{21}, we look at an eventual neighbor of x'_{22} in the I-private neighborhoods of the vertices of the other K_2-components, and so forth. Finally we obtain a path $x_{11}x_{12}x'_{12}x'_{21}x_{21}x_{22}x'_{22}\cdots x'_{t1}x_{t1}x_{t2}x'_{t2}$ where x'_{t2} has no neighbor in the I-private neighborhoods of the vertices of the not yet considered K_2-components of $G[I]$. The set $(I - \{x_{12}, x_{22}, \cdots, x_{t2}\}) \cup \{x'_{12}, x'_{22}, \cdots, x'_{t2}\}$ is an irredundant set contradicting the choice of I.

Hence I is independent and thus $\alpha = IR$, which completes the proof of the theorem. □

3 Equality $i = \gamma$

In this section we characterize those forbidden graphs and pairs of forbidden subgraphs that imply that $\gamma = i$. To do this we first describe five infinite classes of graphs for which $\gamma \neq i$.

Examples 3.1

The graph $H_1(k)$ of order $3k+12$ consists of a path $x_1x_2\cdots x_{3k+8}$, four extra vertices y_2, y_3, y_{3k+6}, y_{3k+7}, and the four pendent edges x_2y_2, x_3y_3, $x_{3k+6}y_{3k+6}$, $x_{3k+7}y_{3k+7}$. It satisfies $\gamma = k+4$ and $i = k+5$ ($\{x_2, x_3, x_6, x_9, \cdots, x_{3k+3}, x_{3k+6}, x_{3k+7}\}$ is a minimum dominating set and $\{x_2, y_3, x_5, x_8, x_{11}, \cdots, x_{3k+2}, x_{3k+5}, y_{3k+6}, x_{3k+7}\}$ a minimum maximal independent set).

The graph $H_2(k)$ is the complete bipartite graph $K_{3,k}$. It is P_4-free, K_3-free, and satisfies $\gamma = 2$, $i = 3$.

The graph $H_3(k)$ consists of a clique K_k with vertex set $\{x_1, x_2, \cdots, x_k\}$, four extra vertices y_1, z_1, y_2, z_2, and the four pendent edges x_1y_1, x_1z_1, x_2y_2, x_2z_2. It is P_5-free, $K_{1,4}$-free and C_4-free. It satisfies $ir = \gamma = 2$ and $i = 3$ ($\{x_1, x_2\}$ is a minimum dominating set and a minimum maximal irredundant set, $\{x_1, y_2, z_2\}$ is a minimum maximal independent set).

The graph $H_4(k)$ of order $4k+2$ consists of a complete $2k$-partite graph, each vertex class of which has two elements z_i and t_i, $1 \leq i \leq 2k$; two adjacent extra vertices x and y; and all the edges between x and the classes $\{z_i, t_i\}$, $1 \leq i \leq k$, and between y and the classes $\{z_i, t_i\}$, $k+1 \leq i \leq 2k$. The graph $H_4(k)$ is P_4-free and its only maximal induced complete bipartite subgraph is $K_{3,3}$. It satisfies $\gamma = 2$ ($\{x, y\}$ is a minimum dominating set) and $i = 3$ ($\{y, z_1, t_1\}$ is a minimum maximal independent set).

The graph $H_5(k)$ of order $2k+6$ consists of a clique $\{x, x_1, x_2, \cdots, x_k, y, y_1, \cdots, y_k\}$, four extra vertices u, v, z, t, the six edges xu, xv, yz, yt, uz, vt, and all the edges x_iv, x_iz, y_iu, y_it for $1 \leq i \leq k$. This graph is $C_{1,1,2}$-free and the only maximal induced complete bipartite subgraphs are C and $K_{2,2} \simeq C_4$. It satisfies $\gamma < i$ since $\{x, y\}$ is a dominating set and no independent set of order two is maximal.

Theorem 3.2. *Let F be a connected graph and n_0 a given positive integer. The condition "G is F-free" implies $\gamma(G) = i(G)$ for any connected graph G of order at least n_0 if and only if F is a subgraph of a claw.*

Proof **"only if"** : The graph F is an induced subgraph of all the graphs $H_1(k)$ and $H_2(k)$ since they satisfy $\gamma \neq i$. Hence F is a path or a tree of maximum degree 3. Since $H_2(k)$ is P_4-free, F is necessarily a subgraph of a claw.

"if" : By Theorem 2.2, if G is C-free then $\gamma(G) = i(G)$. □

Theorem 3.3. *Let (X, Y) be a pair of connected graphs, neither of which is a subgraph of C or a subgraph of each other, and let n_0 be a given positive integer. The condition "G is (X, Y)-free" implies $\gamma(G) = i(G)$*

for any connected graph G of order at least n_0 if and only if (X,Y) is maximally one of the two pairs $(P_4, K_{3,3})$ and (C_4, H) (cf Figure 1).

Proof **"only if"** : Suppose without loss of generality that X is a subgraph of an infinite number of graphs of the family H_1. Then X is a tree of maximum degree at most 3 not contained in a claw, and thus not contained in H_2. Hence Y is a subgraph of H_2 not contained in C, and so is a complete bipartite graph $K_{1,r}$ with $r \geq 4$, or $K_{r,s}$ with $r \geq 2$ and $s \geq 2$. Such a graph Y is not contained in H_3. Therefore X is a subtree of H_3, namely P_4, $C_{1,1,2}$ or H. If $X \simeq P_4$, then Y is a subgraph of $H_4(k)$ which is P_4-free. This gives the first possible maximal pair $(P_4, K_{3,3})$. If $X \simeq C_{1,1,2}$ or H, then Y is a subgraph of H_5 which is $C_{1,1,2}$-free and thus H-free. This gives the second possible maximal pair (C_4, H).
"if" : By Theorem 2.5, if G is $(P_4, K_{3,3})$-free or (C_4, H)-free, then it satisfies $\gamma = i$. □

4 Equality $i = ir$

In this section we characterize those forbidden graphs and pairs of forbidden subgraphs that imply that $i = ir$. To do this we describe two infinite families of graphs for which $\gamma \neq ir$, and thus $i \neq ir$. These two classes generalize the deer D.

Examples 4.1
The graph $H_6(k)$ of order $6k+7$ consists of a triangle zx_1y_1 with two pendent paths $x_1x_2\cdots x_{6k+3}$ and $y_1y_2y_3$ (note that $H_6(0) \simeq D$). The graph $H_6(k)$ is K_4-free and C_4-free. It satisfies $i = \gamma = 2k+3$ ($\{z, y_2, x_2, x_5, x_8, \cdots, x_{6k+2}\}$ is both a minimum dominating set and a minimum maximal independent set) and $ir \leq 2k+2$ ($\{x_1, y_1, x_6, x_7, \cdots, x_{6k}, x_{6k+1}\}$ is a maximal irredundant set).

The graph $H_7(k)$ of order $k+4$ consists of a clique K_k with vertex set $\{x_1, x_2, \cdots, x_k\}$, and two pendent paths $x_1y_1z_1$ and $x_2y_2z_2$ (note that $H_7(3) \simeq D$). The graph $H_7(k)$ is C-free. It satisfies $i = \gamma = 3$ and $ir = 2$ ($\{x_1, z_1, y_2\}$ is a minimum dominating and a minimum maximal independent set, and $\{x_1, x_2\}$ is a minimum maximal irredundant set).

Theorem 4.2. *Let F be a connected graph and n_0 a given positive integer. The condition "G is F-free" implies $ir(G) = i(G)$ for any connected graph G of order at least n_0 if and only if F is a subgraph of P_3.*

Proof **"only if"** : If $ir = i$ then $ir = \gamma$ and by Theorem 3.2, F is a subgraph of a claw. But $H_6(k)$ satisfies $ir \neq i$ and is C-free. So F is a subgraph of P_3.
2. "if" : By Theorem 2.1, if G is P_3-free then $ir = i$. □

Theorem 4.3. *Let n_0 be a given positive integer and (X,Y) a pair of connected graphs, neither of which is a subgraph of P_3 or a subgraph of* *each other. The condition "G is (X, Y)-free" implies $ir(G) = i(G)$ for any connected graph G of order at least n_0 if and only if (X, Y) is maximally one of the two pairs $(P_4, K_{3,3})$ and (C, D).*

Proof "only if" : If $ir = i$, then $\gamma = i$ and by Theorem 3.3, either X or Y is a claw, or (X, Y) is a subgraph of either $(P_4, K_{3,3})$ or (C_4, H). The pair $(P_4, K_{3,3})$ is a first possible maximal pair. For the pair (C_4, H), we remark that the only maximal connected subgraph of H or of C_4 which is contained in $H_7(k)$ is P_4, a subgraph of H but not of C_4. So a pair coming from (C_4, H) is necessarily contained in (C_4, P_4), which is itself already obtained by the maximal pair $(P_4, K_{3,3})$. Hence the pair (C_4, H) gives no new possibility for the property $ir = i$. Finally if $X \simeq C$, then Y is a subgraph common to $H_6(k)$ and $H_7(k)$, which are C-free, and so Y is a subgraph of D. This gives (C, D) as a second possible pair.

"if" : By Theorems 2.3 and 2.5, any (C, D)-free or $(P_4, K_{3,3})$-free graph satisfies $i = ir$. □

5 Equality $\gamma = ir$

In this section, we do not completely characterize the pairs of graphs, the exclusion of which implies $ir = \gamma$. However, we give some partial results. In addition to H_6 and H_7 (cf Examples 4.1), we describe six other infinite families of graphs for which $\gamma \neq ir$.

Examples 5.1

The graph $H_8(k)$ consists of a cycle $x_1 x_2 \cdots x_k$ plus $k-1$ pendent paths $x_i y_i z_i$, $1 \leq i \leq k-1$. Its girth is k which can be made arbitrarily large. It satisfies $ir < \gamma$, since $\{x_1, x_2, \cdots, x_{k-1}\}$ is a maximal irredundant set, and $\{y_1, y_2, \cdots, y_{k-1}, x_k\}$ a minimum dominating set.

The graph $H_9(k)$ is obtained from $H_7(k)$ by adding the edge $y_1 y_2$.

The graph $H_{10}(k)$ is obtained from $H_9(k)$ by replacing the clique $\{x_1, x_2, \cdots, x_k\}$ by a complete bipartite graph $K_{2,k-2}$ of vertex classes $\{x_1, x_2\}$ and $\{x_3, x_4, \cdots, x_k\}$, and adding the edge $x_1 x_2$.

The graph $H_{11}(k)$ is obtained from $H_9(k)$ by adding a new vertex y adjacent to y_1 and y_2, and the edge $z_1 z_2$.

The graphs H_9, H_{10} and H_{11} are P_6-free. Just as for $H_7(k)$, $\{x_1, x_2\}$ is a maximal irredundant set and $\{y_1, y_2, x_1\}$ a minimum dominating set of these three graphs. So H_9, H_{10} and H_{11} satisfy $ir < \gamma$.

The graph $H_{12}(k)$, $k \geq 4$, consists of a complete bipartite graph $K_{k,k}$ of vertex classes $\{x_1, x_2, \cdots, x_k\}$ and $\{t_1, t_2, \cdots, t_k\}$, plus two pendent paths $x_1 y_1 z_1$ and $x_2 y_2 z_2$.

The graph $H_{13}(k)$ is obtained from $H_{12}(k)$ by adding the edge $y_1 y_2$.

In $H_{12}(k)$ and $H_{13}(k)$, $\{x_1, x_2, t_1\}$ is a maximal irredundant set and $\{y_1, y_2, x_1, t_1\}$ a minimum dominating set. So H_{12} and H_{13} satisfy $ir < \gamma$.

They are both K_3-free.

Theorem 5.2. *Let F be a connected graph and n_0 a given positive integer. If the condition "G is F-free" implies $ir(G) = \gamma(G)$ for any connected graph G of order at least n_0, then F is a subgraph of P_5.*

Proof : The graph F must be a subgraph of $H_8(k)$ for any k, and thus a tree. Also, F must be a subgraph of $H_7(k)$, and thus a path P_l with $l \leq 6$. Since H_9 and H_{10} are P_6-free, $F \simeq P_l$ with $l \leq 5$. □

The direct Theorem 2.4 only implies that if G is P_4-free, then $ir = \gamma$. However, we think that the following is true.

Conjecture 5.3. *Every sufficiently large P_5-free graph satisfies $ir = \gamma$.*

If the conjecture is true, it is normal in the study of the forbidden pairs (X, Y) that imply $ir = \gamma$ to suppose X and Y not included in P_5. We do this in the next result.

Theorem 5.4. *Let (X, Y) be a pair of connected graphs, neither of which is a subgraph of P_5 or a subgraph of each other, and let n_0 be a given positive integer. If the condition "G is (X, Y)-free" implies $ir(G) = \gamma(G)$ for any connected graph G of order at least n_0, then (X, Y) is maximally one of the two pairs (P_6, W') and $(C_{1,2,3}, D)$ (cf Figure 1).*

Proof : Suppose without loss of generality X is an induced subgraph of an infinite number of graphs of the family H_7, and so X is isomorphic to P_6 or contains a triangle.

If $X \simeq P_6$, then Y is a subgraph of $H_9(k)$, $H_{10}(k)$ and $H_{11}(k)$ which are P_6-free. Thus Y is a subgraph of W', and so we get the first maximal pair (P_6, W').

Suppose now X contains a triangle. Then Y is a subgraph of $H_{12}(k)$, $H_{13}(k)$ and $H_8(k)$. Since the girth of $H_8(k)$ is arbitrarily large, Y is a tree of maximum degree at most 3. The maximal subtrees of maximum degree at most 3 of $H_{12}(k)$ have at most one vertex of degree 3 and are isomorphic to $C_{1,3,3}$. The maximal subtrees of $H_{13}(k)$ with at most one degree 3 vertex are $C_{1,2,3}$ and $C_{1,1,4}$. Hence Y is a subtree of $C_{1,2,3}$. If X contains a clique K_4, then Y is also a subgraph of $H_6(k)$ and thus is a path, and of $H_{10}(k)$ and thus is a subgraph of P_5, in contradiction to the hypothesis. Therefore X is a subgraph of $H_7(k)$ containing K_3 but not K_4, and so is a subgraph, different from a path, of D. We thus obtain the second possible maximal pair $(D, C_{1,2,3})$.

For this last pair, we can specify that if $X \simeq Z_1$ or Z_2, then Y is a subgraph of $C_{1,2,3}$, but if $X \simeq D$, W or B, then X is not contained in H_9 and H_{10} and thus Y is a subgraph of $C_{1,2,2}$. □

Note: Since this paper was written, Puech has proved one of the direct counterpart of Theorem 5.4, and as a corollary, Conjecture 5.3:

Theorem 5.5. (Puech [8]) *Every (P_6, W')-free graph G, and in particular every P_5-free graph, satisfies $ir(G) = \gamma(G)$.*

This new theorem allows us to restate Theorem 5.2 as:

Theorem 5.6. *Let F be a connected graph and n_0 a given positive integer. The condition "G is F-free" implies $ir(G) = \gamma(G)$ for any connected graph G of order at least n_0 if and only if F is a subgraph of P_5.*

6 Equality $i = \alpha$

In this section we characterize those forbidden graphs and pairs of forbidden subgraphs that imply that $i = \alpha$.

Theorem 6.1. *Let F be a connected graph and n_0 a given positive integer. The condition "G is F-free" implies $i(G) = \alpha(G)$ for any connected graph G of order at least n_0 if and only if F is a subgraph of P_3.*

Proof **"only if"** : Since both the cycle C_{6k} and the star $K_{1,k}$ satisfy $i \neq \alpha$, F is a subgraph of C_{6k} and $K_{1,k}$ for any k, and thus a subgraph of P_3.
"if" : By Remark 2.1, if G is P_3-free then $i = \alpha$. □

Theorem 6.2. *The positive integer n_0 being given, there is no pair (X, Y) of connected graphs, neither of which is a subgraph of P_3 or a subgraph of each other, such that the condition "G is (X, Y)-free" implies $i(G) = \alpha(G)$ for any connected graph of order at least n_0.*

Proof : Suppose that such a pair exists and that X is a subgraph of C_{6k} for an infinite number of values of k. Thus, $X \simeq P_l$ for some $l \geq 4$. Then Y is a subgraph of $K_{1,k}$, and so $Y \simeq K_{1,r}$ with $r \geq 3$. But the graph consisting of a clique K_k plus one pendent edge satisfies $i = 1$, $\alpha = 2$, and contains neither X nor Y, a contradiction. □

Note that the inequalities (*) and the direct Remark 2.1 implies that Theorems 6.1 and 6.2 remain valid for any equality $\lambda = \mu$, where $\lambda \in \{ir, \gamma, i\}$ and $\mu \in \{\alpha, \Gamma, IR\}$.

7 Equalities $\alpha = \Gamma$ and $\alpha = IR$

In this section we characterize those forbidden graphs and pairs of forbidden subgraphs that imply that $\alpha = \Gamma$ and $\alpha = IR$. For each of the following ten infinite families of graphs, α is strictly less than Γ.

Examples 7.1
For a given positive integer k, the graph $L_1(k)$ consists of a cycle $x_1x_2\dots x_{8k}x_1$ of order $8k$ and the two chords x_1x_{4k+1} and $x_{2k}x_{6k}$.
The graph $L_2(k)$ consists of a cycle $x_1x_2\dots x_{8k}x_1$ of order $8k$ and the four chords x_1x_{4k+1}, x_1x_{4k+2}, x_2x_{4k+1}, x_2x_{4k+2}. It is C-free and C_4-free.

The graph $L_3(k)$ consists of a cycle $x_1x_2\dots x_{8k}x_1$ of order $8k$ and the $4k$ chords x_ix_{4k+i} for $1 \le i \le 4k$. It is K_3-free.

For $1 \le i \le 3$, $L_i(k)$ satisfies $\Gamma = 4k$ ($\{x_1, x_2, x_5, x_6, x_9, x_{10}, \dots, x_{8k-3}, x_{8k-2}\}$ is a minimal dominating set) and $\alpha = 4k-1$ ($\{x_1, x_3, x_5, \dots, x_{4k-1}, x_{4k+2}, x_{4k+4}, \dots, x_{8k-2}\}$ is a maximum independent set).

The graph $L_4(k)$ consists of a cycle $x_1x_2\dots x_8x_1$ of order 8 with the three chords x_1x_5, x_2x_6 and x_4x_8, an independent set S of k vertices, and an extra vertex v joined to x_2, x_5, x_8 and to every vertex of S. The graph $L_4(k)$ is P_6-free and K_3-free. It satisfies $\alpha = k+3$ ($S \cup \{x_1, x_3, x_6\}$ is a maximum independent set) and $\Gamma = k+4$ ($S \cup \{x_1, x_2, x_5, x_6\}$ is a maximum minimal dominating set).

The graph $L_5(k)$ consists of a clique K_k, a triangle $x_1x_2x_3$ disjoint from K_k, and a perfect matching between the triangle $x_1x_2x_3$ and a triangle $y_1y_2y_3$ of the clique. The graph $L_5(k)$ is P_5-free and C-free. It satisfies $\alpha = 2$ ($\{x_1, y_2\}$ is a maximum independent set) and $\Gamma = 3$ ($\{y_1, y_2, y_3\}$ is a maximum minimal dominating set).

The graph $L_6(k)$ consists of a clique K_k, a triangle $x_1x_2x_3$ disjoint from K_k, a perfect matching between the triangle $x_1x_2x_3$ and a triangle $y_1y_2y_3$ of the clique, and all the edges between the triangle $x_1x_2x_3$ and $K_k - \{y_1, y_2, y_3\}$. The graph $L_6(k)$ is P_5-free and C-free. It satisfies $\alpha = 2$ and $\Gamma = 3$ for the same reasons as for $L_5(k)$.

The graph $L_7(k)$ consists of k graphs isomorphic to K_2 with vertex sets $\{w_i, t_i\}$, $1 \le i \le k$, one graph isomorphic to $K_3 \times K_2$ with vertex set $\{x_1, x_2, x_3, y_1, y_2, y_3\}$, and an extra vertex v joined to each vertex w_i and t_i and to x_1, x_2, y_1, y_3. The graph $L_7(k)$ is P_5-free and K_4-free. It satisfies $\alpha = k+2$ ($\{x_1, y_2, w_1, w_2, \cdots, w_k\}$ is independent and there exists a vertex covering with $k+2$ cliques) and $\Gamma \ge k+3$ ($\{x_1, x_2, x_3, w_1, w_2, \cdots, w_k\}$ is a minimal dominating set).

The graph $L_8(k)$ consists of a Petersen graph P, a clique K_k, and all the edges between the clique and P. The graph $L_8(k)$ is P_6-free and C_4-free. It satisfies $\alpha = 4$ (a maximum independent set of P is a maximum independent set of $L_8(k)$) and $\Gamma = 5$ (a maximum minimal dominating set of P is a maximum minimal dominating set of $L_8(k)$).

The graph $L_9(k)$ consists of a Petersen graph of vertices $x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3, y_4, y_5$ (where the edges x_iy_i, $1 \le i \le 5$, form a perfect matching of P), two independent sets $A = \{a_1, a_2, \dots, a_k\}$ and $B = \{b_1, b_2, \dots, b_k\}$, the k edges a_ib_i of a perfect matching between A and B, and all the edges between A and a maximum independent set $\{x_1, x_3, y_4, y_5\}$ of P. The graph $L_9(k)$ is K_3-free. It satisfies $\alpha = k+4$ ($B \cup \{x_1, x_3, y_4, y_5\}$ is a maximum independent set) and $\Gamma \ge k+5$ ($B \cup \{x_1, x_2, x_3, x_4, x_5\}$ is a minimal dominating set).

The graph $L_{10}(k)$ consists of two disjoint graphs $K_3 \times K_2$, the first labeled $\{x_1, x_2, x_3, y_1, y_2, y_3\}$, and the second labeled $\{z_1, z_2, z_3, t_1, t_2, t_3\}$, where

$x_iy_i \in E$ and $z_it_i \in E$ for $1 \le i \le 3$; k triangles $u_1v_1w_1$, $w_1v_2w_2$, $w_2v_3w_3, \ldots, w_{k-1}v_kw_k$; and the four edges u_1x_1, u_1y_1, v_1z_1, v_1t_1. The graph $L_{10}(k)$ is C-free and K_4-free. It satisfies $\alpha = k+4$ since $\{v_1, v_2, \ldots, v_k, x_2, y_3, z_2, t_3\}$ is a maximum independent set, and $\Gamma \ge k+5$ since $\{v_2, v_3, \ldots, v_k, x_1, x_2, x_3, z_1, z_2, z_3\}$ is a minimal dominating set.

Theorem 7.2. *Let F be a connected graph and n_0 a given positive integer. The condition "G is F-free" implies $\alpha(G) = \Gamma(G)$ (resp. $\alpha(G) = IR(G)$) for any connected graph G of order at least n_0 if and only if F is a subgraph of P_4.*

Proof "only if" : Since $\alpha \ne \Gamma$ for the graphs $L_1(k)$ and $L_5(k)$, the graph F is a common induced subgraph of all the graphs of these two families. The only subgraphs common to all the $L_1(k)$ are paths, subdivisions of claws, or trees of maximum degree 3 with exactly two degree 3 vertices that are adjacent. But since $L_5(k)$ is C-free and P_5-free, the largest possible F is P_4.

"if" : By Theorem 2.4, if G is P_4-free, then $\alpha(G) = \Gamma(G) = IR(G)$. □

Theorem 7.3. *Let (X, Y) be a pair of connected graphs, neither of which is a subgraph of P_4 or a subgraph of each other and let n_0 be a given positive integer greater than 17. The condition "G is (X, Y)-free" implies $\alpha(G) = \Gamma(G)$ (resp. $\alpha(G) = IR(G)$) for any connected graph G of order at least n_0 if and only if (X, Y) is maximally one of the two pairs $(P_5, K_3 \times K_2)$ and $(Z_1, C_{1,2,2})$ (cf Figure 1).*

Proof "only if" : Suppose without loss of generality that X is an induced subgraph of an infinite number of graphs of the family L_1. The different possibilities for X are given in the previous proof. In particular, X is a tree of maximum degree at most 3.

If $X \simeq P_5$, then Y is an induced subgraph of $L_5(k)$, $L_6(k)$ and $L_7(k)$ which are P_5-free. The only maximal connected subgraph common to the graphs of these three families is $K_3 \times K_2$. So the only possible maximal pair (P_5, Y) is $(P_5, K_3 \times K_2)$.

If $X \simeq P_l$ with $l \ge 6$, then Y is still a subgraph of $L_5(k)$, $L_6(k)$, $L_7(k)$, and also of $L_4(k)$ and $L_8(k)$ which are P_6-free. Hence Y is a subgraph of $K_3 \times K_2$, $L_4(k)$ and $L_8(k)$. But $L_4(k)$ is K_3-free and $L_8(k)$ is C_4-free. Therefore the only possibility for Y is P_4, which is excluded.

If X is a tree of maximum degree 3, then Y is a subgraph of $L_2(k)$, $L_5(k)$, $L_6(k)$ and $L_{10}(k)$ which are C-free. Hence, Y is a subgraph of $K_3 \times K_2$, $L_2(k)$, $L_{10}(k)$, or of K_p, $L_2(k)$, $L_{10}(k)$ with $p \ge 4$. But $L_{10}(k)$ is K_4-free and $L_2(k)$ is C_4-free. Therefore Y is a subgraph of Z_1, different from a path, that is $Y \simeq K_3$ or Z_1. Now, since $L_3(k)$ and $L_9(k)$ are K_3-free, X is an induced subgraph of $L_3(k)$ and $L_9(k)$. No induced subtree of $L_3(k)$ contains two adjacent degree 3 vertices, so X is a subdivision of a claw.

Moreover, each induced subtree of $L_3(k)$ which is a subdivision of a claw has at least one branch of length 1. Therefore, $X \simeq C_{1,r,s}$ with r and $s \geq 1$. The only maximal subtree of $L_9(k)$ of this kind is $C_{1,2,2}$. This gives $(Z_1, C_{1,2,2})$ for the second possible maximal forbidden pair.
"if" : By Theorems 2.8 and 2.9, every $(P_5, K_3 \times K_2)$ or $(Z_1, C_{1,2,2})$-free graph of order at least 18 satifies $\alpha = \Gamma = IR$. □

8 Equality $\Gamma = IR$

Here we only study the families of one graph, the exclusion of which implies $\Gamma = IR$. First we describe two infinite classes of graphs for which $\Gamma \neq IR$.

Examples 8.1

The graph $L_{11}(k)$ consists of two disjoint cliques K_k, $k \geq 3$, with vertex sets $\{x_1, x_2, \cdots, x_k\}$ and $\{\xi_1, \xi_2, \cdots, \xi_k\}$, joined by a perfect matching $\{x_i\xi_i; 1 \leq i \leq k\}$, and two non-adjacent extra vertices, x joined to every vertex x_i, and ξ joined to every vertex ξ_i. This graph is P_5-free and C-free. Its only maximal induced subtree is P_4, and the only induced cycles have length 3 or 4. The graph $L_{11}(k)$ satisfies $IR = k$ and $\Gamma = 2$ ($\{x_1, x_2, \cdots, x_k\}$ is a maximum irredundant set and $\{x_1, \xi_1\}$ a maximum minimal dominating set).

Just as for $L_{11}(k)$, the vertex set of the graph $L_{12}(k)$, where k is a prime integer greater than 5, consists of two sets $A = \{x_1, x_2, \cdots, x_k\}$ and $B = \{\xi_1, \xi_2, \cdots, \xi_k\}$, and two non-adjacent extra vertices, x joined to every vertex x_i, and ξ joined to every vertex ξ_i. The set A induces the cycle $C = x_1x_2 \cdots x_kx_1$ and the set B induces the cycle $C' = \xi_1\xi_3\xi_5 \cdots \xi_k\xi_2\xi_4 \cdots \xi_{k-1}\xi_1$. Finally, the sets A and B are joined by the perfect matching $\{x_i\xi_i; 1 \leq i \leq k\}$. This graph is C_4-free and K_4-free.

Proposition 8.2. *The graph $L_{12}(k)$ satisfies $\Gamma < IR$.*

Proof : The set A is irredundant, so $IR \geq k$. We will prove that every minimal dominating set D has less than k vertices (recall that every minimal dominating set is irredundant).

If D contains x and ξ, then $|D| = 2$, since $\{x, \xi\}$ is a minimal dominating set. So we suppose that at least x or ξ is not in D, and we denote $|D \cap A| = a$, $|D \cap B| = b$.

If D contains x and not ξ, then $|D| = a + b + 1$. In this case, $b \neq 0$ since ξ must be dominated, $a \leq k - 3$ for otherwise x is redundant in D, and every D-private neighbor of every vertex of $D \cap A$ is in B. If $b = 1$, then $|D| \leq (k-3)+2 = k-1$. If $b > 1$, every D-private neighbor of every vertex of $D \cap B$ is also in B, and thus $a + 2b \leq k$. Hence $2(a + b) \leq 2k - 3$, and $|D| < k$. Similarly, by symmetry if D contains ξ and not x, then $|D| < k$.

Suppose now that $x \notin D$, $\xi \notin D$ and $|D| = k$. This implies $a \neq 0$ since B does not dominate x, and similarly $b \neq 0$. If $a = 1$, say $D \cap A = \{x_1\}$, and

$b = k-1$, then the $k-1$ vertices of $D \cap B$ have their D-private neighbors in A, which is impossible since these D-private neighbors cannot be x_1, x_2 nor x_k. So $a \geq 2$ and, by symmetry, $b \geq 2$. Let Y be the set of the non-isolated vertices of D, $Z = D - Y$ the set of the isolated vertices of D, Y' the set of the D-private neighbors of the vertices of Y, and $T = (A \cup B) - (D \cup Y')$. By the definition of Y', there is no edge between Z and Y', and $|Y'| \geq |Y|$. Moreover, $|Y \cup Z| = |Y' \cup T| = k$, and thus $|T| \leq |Z|$. The graph induced by $A \cup B$ is 3-regular, so the number $3|Z|$ of edges between Z and T is at most $3|T|$. Hence $|T| = |Z|$ and there is no edge between Y and T. By the connectedness of $G[A \cup B]$, $Z = \emptyset$, $|Y| = |Y'| = k$, and the edges between Y and Y' form a perfect matching of $Y \cup Y'$. Therefore, each of Y and Y' induces a 2-regular subgraph, respectively called Red Graph for Y and Blue Graph for Y'.

Each B-vertex has exactly two B-neighbors and one R-neighbor, and each R-vertex has exactly two R-neighbors and one B-neighbor. Since $a \neq 0$ and $b \neq 0$, the Red Graph uses at least one edge $x_i\xi_i$, say $x_1\xi_1$. Starting from x_1 and ξ_1, we color the vertices of G using the degree conditions on the red graph and the blue graph. Suppose first x_1, ξ_1, x_k, ξ_3 are R. We have to give the color B to x_2, ξ_2, x_3, x_4, ξ_4, ξ_{k-1}, x_{k-1}, x_{k-2}, ξ_{k-3}, x_{k-3}, and the color R to ξ_{k-2}, ξ_k, ξ_5, x_5, x_6, ξ_6, ... We can continue the coloring without any ambiguity. We find in this way the same pattern of length five for the coloration of $\{x_{k-1}, x_1, x_2, x_3, x_4, \xi_{k-1}, \xi_1, \xi_2, \xi_3, \xi_4\}$, repeated left and right, and thus k must be divisible by 5. This situation is impossible since we chose k prime greater than 5. Similarly, if we start with x_1, ξ_1, x_2, ξ_3 colored by R, we find the same pattern, and a simple exchange of the colors R and B leads to the same contradiction.

Therefore $|D| < k$ and $\Gamma(L_{12}(k)) < IR(L_{12}(k))$. □

In the following theorem, $2K_3 + e$ denotes the graph consisting of two vertex disjoint triangles joined by one edge.

Theorem 8.3. *Let F be a connected graph and n_0 a given positive integer. If the condition "G is F-free" implies $\Gamma(G) = IR(G)$ for any connected graph G of order at least n_0, then F is a subgraph of $2K_3 + e$.*

Proof : For any k, the graph F is an induced subgraph of $L_{11}(k)$ and $L_{12}(k)$. Hence, if F is a tree, it is contained in P_4. If not, the only possible cycles are disjoint triangles, and F is a subgraph of $2K_3 + e$.

By Theorem 2.4, we know that if G is P_4-free, then $\Gamma = IR$ (and even $\alpha = IR$). We think that P_4 may be the only possible graph F satisfying the conditions of Theorem 8.3. More precisely, does the triangle-free graph $L_{13}(k)$, similar to $L_{12}(k)$ with the only difference being that in $L_{13}(k)$, x is adjacent to x_1, x_3, x_5, ..., x_{k-2}, and ξ to ξ_1, ξ_5, ξ_9, ξ_{13},..., ξ_{2i+1},..., where the subscripts are taken modulo k, satisfy $\Gamma < IR$? More generally, the construction of any connected triangle-free arbitrarily large graph such that

$\Gamma < IR$, would be interesting.

Note: Recently, Cockayne and Mynhardt [4] succeeded to construct an infinite class $L_{14}(k)$ of triangle-free graphs for which $\Gamma < IR$ (moreover the difference $IR - \Gamma$ can be arbitrarily large). Hence the graph F of Theorem 8.3 also belongs to $L_{14}(k)$ and is triangle-free. Thanks to their result, we can thus complete the study of the property $\Gamma = IR$ in the case of one forbidden subgraph and get

Theorem 8.4. *Let F be a connected graph and n_0 a given positive integer. The condition "G is F-free" implies $\Gamma(G) = IR(G)$ for any connected graph G of order at least n_0 if and only if F is a subgraph of P_4.*

The following table summarizes the results of the paper. It needs two comments:

1. In the case corresponding to pairs of forbidden subgraphs and Property $ir = \gamma$, the result is only partial since we do not know if in a $(C_{1,2,3}, D)$-free graph, $ir = \gamma$.

2. In two cases, the references indicate that we used the corresponding results to complete our study.

Property $\mathcal{P}$	$i = \gamma$	$i = ir$	$\gamma = ir$	$i = \alpha$	$\alpha = \Gamma$ $\alpha = IR$	$\Gamma = IR$
One forbidden subgraph F	$K_{1,3}$	P_3	P_5 [P]	P_3	P_4	P_4 [CM]
Pairs of forbidden subgraphs (X,Y)	$(P_4, K_{3,3})$ or (H, C_4)	$(P_4, K_{3,3})$ or $(K_{1,3}, D)$	necessarily (P_6, W') or $(C_{1,2,3}, D)$	$\emptyset$	for $n \geq 18$ $(P_5, K_3 \times K_2)$ or $(C_{1,2,2}, Z_1)$	

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