Quasi-Orthogonal Latin Squares and Related Designs

David Bedford
Department of Mathematics
University of Keele
Keele, Staffordshire ST5 58G
England

ABSTRACT. We introduce a generalisation of the concept of a complete mapping of a group which we call a quasi-complete mapping and which leads us to a generalised form of orthogonality in latin squares. In particular, the existence of a quasi-complete mapping of a group is shown to be sufficient for the existence of a pair of latin squares such that if they are superimposed so as to form an array of unordered pairs each unordered pair of distinct elements occurs exactly twice. We call such a pair of latin squares quasi-orthogonal and prove that an abelian group possesses a quasi-complete mapping if and only if it is not of the form $Z_{4m+2} \oplus G, |G|$ odd. In developing the theory of quasi-complete mappings we show that the well known concept of a quasi-complete latin square arises quite naturally in this setting. We end the paper by giving a sufficient condition for the existence of a pair of quasi-orthogonal latin squares which are also quasi-row-complete.

1 Introduction and Basic Definitions

A *latin square* of order n is an $n \times n$ array defined on a symbol set S of size n such that each member of S occurs exactly once in each row and once in each column. A permutation θ of the elements of a finite group $(G, \cdot)$ is said to be a *complete mapping* if the mapping $\phi : g \mapsto g \cdot \theta(g)$ is again a permutation on G. A latin square of order n is called *row complete* if each of the $n(n-1)$ ordered pairs of distinct symbols occurs in adjacent positions in exactly one row of the latin square. Two latin squares are said to be *orthogonal* if, when they are superimposed so as to form an array of ordered pairs Λ, each of the n^2 possible ordered pairs occurs in Λ exactly once. In this paper we introduce a generalisation of orthogonality in latin squares which we call *quasi-orthogonality.*

Definition 1.1 *Two latin squares of order n, each defined on the same alphabet, are* quasi-orthogonal *if, when they are superimposed so as to form an array of unordered pairs* Λ, *each unordered pair of the form* $\{x, x\}$ *occurs in* Λ *exactly once and each unordered pair of distinct elements* $\{x, y\}$ *occurs in* Λ *exactly twice.*

In view of the non-existence of a pair of orthogonal latin squares of order 6, the following pair of quasi-orthogonal latin squares of order 6 is of interest.

1	2	3	4	5	6		1	3	4	2	6	5
2	1	6	5	4	3		2	4	1	3	5	6
3	5	1	2	6	4		3	2	5	6	4	1
4	6	2	1	3	5		4	1	6	5	2	3
5	3	4	6	2	1		5	6	3	4	1	2
6	4	5	3	1	2		6	5	2	1	3	4

It is well known that a group based latin square has an orthogonal mate if and only if the group possesses a complete mapping (Mann [11]). Similarly, a group based latin square has a quasi-orthogonal mate if and only if the group possesses a quasi-complete mapping (to be defined below). A group which has no complete mappings may nonetheless possess quasi-complete mappings. (The smallest group for which this is the case is the cyclic group of order 4, as we shall show.) There exist statistical experiments for which a pair of quasi-orthogonal latin squares may serve as well as an orthogonal pair so this fact is potentially useful.

An analogous situation arises as between row complete and quasi-row complete latin squares (see Freeman [5] and Bailey [3]). In this paper we show that, for group based latin squares, the analogy is no accident by developing the concept of a quasi-near complete mapping as a generalisation of a near-complete mapping (of which a sequencing is a particular case) and hence producing a unification of earlier results.

In the next section we will present a method for constructing sets of MQOLS by generalising the concept of a complete mapping of the elements of a group. Before we do this we define a concept which will be of use to us.

Definition 1.2 *Let* $(G, \cdot)$ *be a finite group of order n with identity element e. A list of elements* $a_1, a_2, \ldots, a_n$ *of G is a* quasi-ordering *of G if and only if:*

(i) the list contains exactly one occurrence of each element x of G such that $x^2 = e$;

(ii) for every element y of G such that $y^2 \neq e$, the list contains two occurrences of y and none of y^{-1}, or exactly one occurrence of each of y and y^{-1}, or exactly two occurrences of y^{-1} and none of y.

In this paper we will continue to denote the identity element of a group by e except where the group operation is $+$ in which case 0 will be used.

2 Quasi-Complete Mappings

Definition 2.1 *Let $(G,\cdot)$ be a finite group of order n, $G = \{g_1, g_2 \dots, g_n\}$. A mapping $\theta : G \to G$ is* quasi-complete *if the mapping ϕ defined by $\phi(x) = x\theta(x)$ is a permutation on G and θ is such that $\theta(g_1), \theta(g_2), \dots, \theta(g_n)$ is a quasi-ordering of G.*

Note that if, in the above definition, θ is a quasi-complete mapping of G in which θ is a permutation on G, then θ is a *complete mapping* as defined by Mann [11] and ϕ is an *orthomorphism* as defined by Johnson, Dulmage and Mendelsohn [10]. It therefore seems natural to call the mapping ϕ in Definition 2.1 a *quasi-orthomorphism* of G.

Example: a quasi-complete mapping θ of the elements of $(Z_4, +)$ together with its corresponding quasi-orthomorphism ϕ are

$$\theta = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 2 \end{pmatrix} \phi = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix} .$$

We now consider mappings which give rise to group based, quasi-orthogonal latin squares.

Definition 2.2 *Let $(G,\cdot)$ be a finite group and let p and q be permutations on G, then p and q are* quasi-orthogonal permutations *if $p(g_1)^{-1}q(g_1), p(g_2)^{-1}$ $q(g_2), \dots, p(g_n)^{-1}q(g_n)$ is a quasi-ordering of G.*

In order to demonstrate that two latin squares L_1, L_2 are quasi-orthogonal it will sometimes be convenient to superimpose L_1 and L_2 and to regard the resulting array as an array of ordered pairs Λ. It will then be the case that L_1 and L_2 are quasi-orthogonal if and only if each ordered pair of the form (a, a) occurs in Λ exactly once and for each pair of distinct elements a, b we have that (a, b) occurs twice in Λ with (b, a) absent, or (b, a) occurs twice in Λ with (a, b) absent, or each of (a, b) and (b, a) occur in Λ exactly once.

Theorem 2.3 *Let $(G,\cdot)$ be a finite group and let ϕ_1 and ϕ_2 be permutations on G. Let L be the Cayley table of $(G,\cdot)$ and L_{ϕ_1}, L_{ϕ_2} be obtained by permuting the columns of L according to ϕ_1 and ϕ_2 respectively. Then, if ϕ_1 and ϕ_2 are quasi-orthogonal, L_{ϕ_1} and L_{ϕ_2} are quasi-orthogonal.*

Proof: Let Λ be the array of ordered pairs obtained by superimposing L_{ϕ_1} and L_{ϕ_2}. Then the (i,j)th cell of Λ contains the ordered pair $(g_i\phi_1(g_j), g_i\phi_2(g_j))$. If we define the *difference* of an ordered pair (x,y) to be $x^{-1}y$, then the cells in the jth column of Λ contain all of the ordered pairs whose difference is $\phi_1(g_j)^{-1}\phi_2(g_j)$.

Now consider a particular ordered pair (a,b), where $a^{-1}b = d$, there are two possibilities:

(i) If $d^2 = e$, then, by the quasi-orthogonality of ϕ_1 and ϕ_2, there exists a unique element g_k in G such that $\phi_1(g_k)^{-1}\phi_2(g_k) = d$. Thus the kth column of Λ contains all ordered pairs whose difference is d, these include (a,b) and (b,a) (which are distinct if $d \neq e$).

(ii) If $d^2 \neq e$, then there exist exactly two distinct elements g_l, g_m of G such that $\phi_1(g_l)^{-1}\phi_2(g_l) = d$ or d^{-1} and $\phi_1(g_m)^{-1}\phi_2(g_m) = d$ or d^{-1}. If $\phi_1(g_l)^{-1}\phi_2(g_l) = d$, then the ordered pair (a,b) occurs once in the lth column of Λ. If $\phi_1(g_l)^{-1}\phi_2(g_l) = d^{-1}$, then the ordered pair (b,a) occurs once in the lth column of Λ. A similar argument may be applied to the mth column of Λ. Hence we must have one of the following four possibilities:

(a) $\phi_1(g_l)^{-1}\phi_2(g_l) = d; \phi_1(g_m)^{-1}\phi_2(g_m) = d$,

(b) $\phi_1(g_l)^{-1}\phi_2(g_l) = d; \phi_1(g_m)^{-1}\phi_2(g_m) = d^{-1}$,

(c) $\phi_1(g_l)^{-1}\phi_2(g_l) = d^{-1}; \phi_1(g_m)^{-1}\phi_2(g_m) = d$,

(d) $\phi_1(g_l)^{-1}\phi_2(g_l) = d^{-1}; \phi_1(g_m)^{-1}\phi_2(g_m) = d^{-1}$.

In cases (b) and (c) we have that each of the ordered pairs $(a,b), (b,a)$ occurs in Λ exactly once. In (a) we have that the ordered pair (a,b) occurs twice in Λ whilst (b,a) is absent, whereas in (d) we have that (b,a) occurs twice in Λ whilst (a,b) is absent.

Parts (i) and (ii) imply that L_{ϕ_1} and L_{ϕ_2} are quasi-orthogonal.

Corollary 2.4 *If $(G,\cdot)$ is a finite group which possesses a quasi-complete mapping θ, with corresponding quasi-orthomorphism ϕ, then L and L_ϕ are quasi-orthogonal.*

Proof: We need only to show that I, the identity permutation on G, and ϕ are quasi-orthogonal. This follows since $I(x)^{-1}\phi(x) = x^{-1}\phi(x) = \theta(x)$ and $\theta(g_1), \theta(g_2), \ldots, \theta(g_n)$ is a quasi-ordering of G by definition of θ.

The above theorem and corollary tell us that a set of m mutually quasi-orthogonal, quasi-orthomorphisms of the elements of a finite group $(G,\cdot)$ implies the existence of a set of $m+1$ MQOLS based on $(G,\cdot)$. For example, in $(Z_4,+)$ the following quasi-orthomorphisms are quasi-orthogonal.

$$\phi_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix} \phi_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 1 & 2 \end{pmatrix}$$

If, in the notation of Theorem 2.3, we superimpose L_{ϕ_1} and L_{ϕ_2} so as to obtain an array of ordered pairs Λ we obtain

$$\Lambda = \begin{matrix} 00 & 23 & 31 & 12 \\ 11 & 30 & 02 & 23 \\ 22 & 01 & 13 & 30 \\ 33 & 12 & 20 & 01 \end{matrix} .$$

In the above example $\{L, L_{\phi_1}, L_{\phi_2}\}$ is a set of MQOLS.

3 Which Groups Possess Quasi-Complete Mappings?

The following Lemmas provide necessary conditions for a group to possess a quasi-complete mapping. We will use these results to determine exactly which finite abelian groups possess quasi-complete mappings.

Lemma 3.1 *Let $(G,\cdot)$ be a finite group of order n with derived subgroup $(G',\cdot)$, and $x \in G$ be such that $\prod_{i=1}^{n} g_i \in xG'$. Let $(H,\cdot) = \langle g_1^2, g_2^2, \ldots, g_n^2\rangle$ and $a_1, a_2, \ldots, a_n$ be a quasi-ordering of G. Then $\prod_{i=1}^{n} a_i \in xHG'$.*

Proof: If $a_i = a_j, j > i$, define $h_j = a_j^2$ and $g_j = a_j^{-1}$. For all other elements a_k, define $h_k = e$ and $g_k = a_k$. We then obtain a list of elements $h_1g_1, h_2g_2, \ldots, h_ng_n$ in which $g_1, g_2, \ldots, g_n$ are all distinct, $h_i \in H$ and $h_ig_i = a_i, 1 \le i \le n$. Thus $\prod_{i=1}^{n} a_i = \prod_{i=1}^{n} h_ig_i$. Now H is normal in G since, if $g \in xHx^{-1}$, then $g = xg_i^2g_j^2 \ldots g_k^2x^{-1} = (xg_i^2x^{-1})(xg_j^2x^{-1})\cdots(xg_k^2x^{-1}) = (xg_ix^{-1})^2(xg_jx^{-1})^2\cdots(xg_kx^{-1})^2 \in H \Rightarrow xHx^{-1} \subseteq H$ Thus HG' is normal in G and $\prod_{i=1}^{n} h_ig_i \in (\prod_{i=1}^{n} h_i)(\prod_{i=1}^{n} g_i)G' \subseteq HxG' = xHG'$ as required.

Lemma 3.2 *Let $(G,\cdot)$ be a finite group of order n which possesses a quasi-complete mapping θ. Then there exists an indexing of the elements of G such that $\prod_{i=1}^{n}\theta(g_i) = e$.*

Proof: Let ϕ be the quasi-orthomorphism corresponding to θ written in cycle notation as

$$\phi = (g_{11}g_{12}\ldots g_{1k_1})\ldots(g_{s1}g_{s2}\ldots g_{sk_s}) .$$

Then

$$\prod_{i=1}^{s}\prod_{j=1}^{k_i}\theta(g_{ij}) = \prod_{i=1}^{s}(g_{i1}^{-1}g_{i2})(g_{i2}^{-1}g_{i3})\ldots(g_{ik_i}^{-1}g_{i1}) = e.$$

We note that Lemma 3.1 gives us no additional information about groups of odd order since for all such groups $(G,\cdot)$ we have that $H = G$. This is of little consequence however since it is already known that any group of odd order possesses at least one quasi-complete mapping, namely the identity mapping.

In [12], Paige proved that a finite abelian group possesses a complete mapping if and only if it is not of the form $Z_{2m} \oplus G$, where $|G|$ is odd. The following theorem tells us exactly when such a group possesses a quasi-complete mapping.

Theorem 3.3 *The finite group $Z_{2m} \oplus G$, where G is an abelian group of odd order, possesses a quasi-complete mapping if and only if m is even.*

Proof: Suppose that $Z_{2m} \oplus G$ possesses a quasi-complete mapping θ. The unique element of order 2 in $Z_{2m} \oplus G$ is $(m, 0)$. By Lemma 3.1, and the result due to Paige that the sum of all the distinct elements of an abelian group with a unique element of order 2 is that element of order 2, we have that, for any ordering $g_1, \ldots, g_n$ of the n distinct elements of $Z_{2m} \oplus G$, $\sum_{i=1}^{n} \theta(g_i) \in (m, 0) + H$, where $H = \langle 2g_1, 2g_2, \ldots, 2g_n \rangle = \{2g_1, 2g_2, \ldots, 2g_n\}$ since $Z_{2m} \oplus G$ is abelian. From Lemma 3.2 we have that $\sum_{i=1}^{n} \theta(g_i) = (0, 0)$. Together these imply that $(m, 0) \in H$, i.e. there exists $(x, y) \in Z_{2m} \oplus G$ such that $2(x, y) = (m, 0)$. Clearly we require $y = 0$ since $|G|$ is odd, and $2x = m$ has a solution in Z_{2m} if and only if m is even (in which case $x = \frac{m}{2}$).

Conversely, suppose m is even and define θ on $Z_{2m} \oplus G$ by $\theta(x, y) = (x, y)$ for $0 \leq x < m$; $\theta(x, y) = (x - m + 1, y)$ for $m \leq x < 2m$. Then θ is a quasi-complete mapping of $Z_{2m} \oplus G$.

Thus we have that an abelian group possesses a quasi-complete mapping if and only if it is not of the form $Z_{4m+2} \oplus G$, where $|G|$ is odd. As the next theorem demonstrates, Lemmas 3.1 and 3.2 can also be used to generate results for an infinite class of non-abelian groups.

Theorem 3.4 *The dihedral group of order $2n$ represented by $D_n = \langle a, b : a^n = e, b^2 = e, ab = ba^{-1} \rangle$ does not possess a quasi-complete mapping if n is odd.*

Proof: By Lemma 3.2, we have that a necessary condition for D_n to possess a quasi-complete mapping θ is that there exists a quasi-ordering of the elements of D_n whose total product is the identity. Now since any element of the form $a^i b$ has order 2 we must have that each of these elements occurs exactly once in any such quasi-ordering and so the total product is of the form $a^j b^n$, for some integer j, but this can only equal the identity if n is even.

If n is even, then D_n possesses a complete mapping. This is a consequence of a result by Hall and Paige [7] who proved that a necessary and sufficient condition for a finite soluble group to have a complete mapping is that its Sylow 2-subgroups be non-cyclic. As regards quasi-complete mappings of groups we have the following extension theorem.

Theorem 3.5 *Let $(G, \cdot)$ be a finite group and $(H, \cdot)$ a normal subgroup of $(G, \cdot)$. If there exists a complete mapping of H and a quasi-complete mapping of G/H, then there exists a quasi-complete mapping of G.*

Proof: Let $|G| = n$ and $(G/H, \cdot) \overset{\alpha}{\cong} (K, \cdot)$, where $K = \{k_1, \dots k_m\}$. Let $u_{k_i} \in G$ be such that $\alpha(Hu_{k_i}) = k_i$. Clearly each $g \in G$ has a unique representation in the form $g = u_k h$, (since the elements $u_{k_1}, \dots, u_{k_m}$ are all distinct and form a set of coset representatives of H in G). Let θ_1 be a quasi-complete mapping of K with corresponding quasi-orthomorphism ϕ_1 and θ_2 a complete mapping of H with corresponding orthomorphism ϕ_2. Define $\theta : G \to G$ by $\theta(u_k h) = \theta_2(h) u_{\theta_1(k)}$. We will show that θ is a quasi-complete mapping of G.

Firstly observe that

$$\{\theta(g_i) : g_i \in G\} = Hu_{\theta_1(k_1)} + \cdots + Hu_{\theta_1(k_m)} \qquad (\dagger)$$

since by assumption θ_2 is a permutation on H and so $\theta(u_k h) = \theta_2(h) u_{\theta_1(k)}$ exhausts the elements of $Hu_{\theta_1(k)}$ as h varies across the elements of H with k fixed.

If θ_1 is a permutation on K, then clearly $\{\theta(g_i) : g_i \in G\} = G$. We now consider what happens if $\theta_1(k_i) = \theta_1(k_j), i \neq j$. We then have that $Hu_{\theta_1(k_i)} = Hu_{\theta_1(k_j)}$ and so this coset occurs twice on the right hand side of ($\dagger$). In order to show that $\theta(g_1), \dots, \theta(g_n)$ is a quasi-ordering of G we are required to show that if $g \in Hu_{\theta_1(k_i)}$ and $g \in Hu_{\theta_1(k_j)}, i \neq j$, then $g^2 \neq e$ and $g^{-1} \notin \{\theta(g_i) : g_i \in G\}$.

Let $g \in Hu_{\theta_1(k_i)}$, so there exists $h \in H$ such that $g = hu_{\theta_1(k_i)}$. Now $g^2 = (hu_{\theta_1(k_i)})^2 \in Hu_{\theta_1(k_i)} Hu_{\theta_1(k_i)} = Hu_{\theta_1(k_i)^2}$ and since $\theta_1(k_i)^2 \neq e$ (by the assumptions that $\theta_1(k_i) = \theta_1(k_j), i \neq j$ and θ_1 is a quasi-complete mapping of K) we have that $g^2 \notin H \Rightarrow g^2 \neq e$.

Now consider g^{-1}.

$$g^{-1} = u_{\theta_1(k_i)}^{-1} h^{-1} \in u_{\theta_1(k_i)}^{-1} H = Hu_{\theta_1(k_i)}^{-1} = Hu_{\theta_1(k_i)^{-1}}$$

But $Hu_{\theta_1(k_i)^{-1}}$ cannot occur in the list of cosets ($\dagger$) since if $\theta_1(k_i) = \theta_1(k_j), i \neq j$, then $\nexists k \in K$ such that $\theta_1(k) = \theta_1(k_i)^{-1}$ since θ_1 is a quasi-complete mapping of K.

We therefore have that $\theta(g_1), \dots, \theta(g_n)$ is a quasi-ordering of G. It now only remains to show that the mapping ϕ defined by $\phi : g \mapsto g\theta(g)$ is a permutation on G.

Suppose $\phi(u_{k_i}h) = \phi(u_{k_j}h')$. We would then have that

$$\begin{array}{rcll} u_{k_i}h\theta_2(h)u_{\theta_1(k_i)} & = & u_{k_j}h'\theta_2(h')u_{\theta_1(k_j)} & \\ \Rightarrow u_{k_i}\phi_2(h)u_{\theta_1(k_i)} & = & u_{k_j}\phi_2(h')u_{\theta_1(k_j)} & (\ddagger) \\ \Rightarrow Hu_{k_i}u_{\theta_1(k_i)} & = & Hu_{k_j}u_{\theta_1(k_j)} & \\ \Rightarrow k_i\theta_1(k_i) & = & k_j\theta_1(k_j) & \\ \Rightarrow \phi_1(k_i) & = & \phi_1(k_j) & \\ \Rightarrow k_i & = & k_j & \text{since } \phi_1 \text{ is a permutation on } K \\ \Rightarrow \phi_2(h) & = & \phi_2(h') & \text{by } (\ddagger) \\ \Rightarrow h & = & h' & \text{since } \phi_2 \text{ is a permutation on } H \\ \Rightarrow u_{k_i}h & = & u_{k_j}h' & \text{as required.} \end{array}$$

4 Quasi-Near Complete Mappings and Quasi-Sequencings of Groups

In [5] Freeman defined a latin square A to be *quasi-row-complete* if the $n(n-1)$ pairs $\{a_{ij}, a_{i,j+1}\}$ contain each unordered pair of distinct elements exactly twice. Similarly A is *quasi-column-complete* if the $n(n-1)$ pairs $\{a_{ij}, a_{i+1,j}\}$ contain each unordered pair of distinct elements exactly twice. Finally, A is said to be *quasi-complete* if it is both quasi-row and quasi-column complete. Bailey [3] has shown that the existence of a quasi-complete latin square based on $(G, \cdot)$ is equivalent to the existence of a *quasi-sequencing* of G.

Definition 4.1 *Let $(G, \cdot)$ be a finite group of order n. If $a_1, a_2, \dots, a_n$ is a quasi-ordering of G such that the partial products $b_1 = a_1, b_2 = a_1a_2, \dots, b_n = a_1a_2\cdots a_n$ are all distinct, then $a_1, a_2, \dots, a_n$ is a* quasi-sequencing *of G.*

Bailey has called the list of partial products $b_1, b_2, \dots, b_n$, in the definition of a quasi-sequencing of G, a *terrace* of G. Note that in any quasi-sequencing of G it is necessarily the case that $a_1 = b_1 = e$.

Theorem 4.2 (Bailey [3]) *Let $(G, \cdot)$ be a finite group of order n and let $(a_1, a_2, \dots a_n)$ and $(b_1, b_2, \dots, b_n)$ be orderings of the distinct elements of G. Then the latin square whose (i, j)th entry is a_ib_j is quasi-row-complete if and only if $(b_1, b_2, \dots, b_n)$ is a terrace of G and quasi-column-complete if and only if $(a_1^{-1}, a_2^{-1}, \dots, a_n^{-1})$ is a terrace of G.*

If, in the notation of Definition 4.1, $a_1, a_2, \dots, a_n$ are all distinct, then the sequence is a *sequencing* of G as defined by Gordon [6]. In [9] Hsu and Keedwell generalised the concept of a complete mapping, in an entirely different way to that done in this paper, to that of a *near-complete mapping* and showed that a special type of near-complete mapping of G is equivalent to the existence of a sequencing of G. In an analogous way we generalise the idea of a quasi-complete mapping to that of a *quasi-near-complete mapping*, and show that a quasi-sequencing of G is a special type of quasi-near-complete mapping of G.

Definition 4.3 *If θ is a mapping from $G \setminus \{h\}$ to $G \setminus \{e\}$ with $e, \theta(g_2), \ldots, \theta(g_n)$ a quasi-ordering of G such that $\phi : x \mapsto x\theta(x)$ is a bijective mapping from $G \setminus \{h\}$ to $G \setminus \{k\}$, then θ is a* quasi-near-complete mapping *of G.*

If, in the above definition, θ is bijective, then θ is a *near-complete mapping* of G as defined by Hsu and Keedwell [9]. The following theorem is an exact analogue of a result in [9].

Theorem 4.4 *Let $(G, \cdot)$ be a finite group of order n. Then G possesses a quasi-sequencing $a_1, a_2, \ldots, a_n$ if and only if G possesses a quasi-near-complete mapping θ such that the bijection ϕ defined by $\phi(x) = x\theta(x) \forall x \in G$, can be expressed as a single non-cyclic sequence of distinct elements $[g_1, g_2, \ldots, g_n]$ where $\phi(g_i) = g_{i+1}$ for $1 \leq i < n$.*

Proof: Suppose that $(G, \cdot)$ possesses a quasi-sequencing $a_1 = e, a_2, \ldots, a_n$. Define $b_1 = a_1 = e, b_2 = a_1a_2, \ldots, b_n = a_1a_2 \cdots a_n$ and $\phi(b_i) = b_{i+1}, 1 \leq i < n$. Clearly ϕ is a bijection from $G \setminus \{b_n\}$ to $G \setminus \{e\}$ which can be represented by the non-cyclic sequence $[b_1, b_2, \ldots, b_n]$. Furthermore, if we define $\theta(b_i) = b_i^{-1}\phi(b_i) = b_i^{-1}b_{i+1} = a_{i+1}, 1 \leq i < n$, it is easily seen that θ is a mapping from $G \setminus \{b_n\}$ to $G \setminus \{e\}$ with $e, \theta(b_1), \ldots, \theta(b_{n-1})$ a quasi-ordering of G.

Conversely, suppose that $(G, \cdot)$ possesses a quasi-near-complete mapping θ such that $g_i\theta(g_i) = g_{i+1}, 1 \leq i < n$, in the non-cyclic sequence $[g_1, g_2, \ldots, g_n]$. If we define $a_{i+1} = \theta(g_i), 1 \leq i < n$ and $a_1 = e$, then $a_1, a_2, \ldots, a_n$ is easily seen to be a quasi-sequencing of G.

A group which possesses at least one sequencing is called *sequenceable*, we will similarly call a group which possesses a quasi-sequencing a *quasi-sequenceable* group.

Example using the group $(Z_5, +)$.

Quasi-sequencing:	0		4	2	2	4
Partial sums:		0	4	1	3	2

Quasi-complete mapping $\theta = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 2 & & 4 & 2 \end{pmatrix}$.

Corresponding quasi-orthomorphism $\phi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & & 2 & 1 \end{pmatrix}$.

Equivalent non-cyclic sequence for $\phi = [0, 4, 1, 3, 2]$.

5 Quasi-Orthogonal, Quasi-Row-Complete Latin Squares

We end this paper by giving a sufficient condition for the existence of quasi-orthogonal, quasi-row-complete latin squares. In [8] Heinrich gives the following sufficient conditions for the existence of a pair of orthogonal, row-complete latin squares.

Theorem 5.1 (Heinrich [8]) *If $(G, \cdot)$ is a sequenceable group of order n and if there are two orderings of the distinct elements of G, say $h_1, \ldots, h_n$ and $k_1, \ldots, k_n$, such that the list of elements $h_1k_1^{-1}, \ldots, h_nk_n^{-1}$ is again an ordering of the distinct elements of G, then we can construct a pair of orthogonal, row-complete latin squares based on $(G, \cdot)$.*

We refer the reader to Heinrich's original paper or to Dénes and Keedwell [4, p50], for a complete proof of this result but the method of construction is as follows. Let $a_1, a_2, \ldots, a_n$ be a sequencing of G with partial products $b_1 = a_1, b_2 = a_1a_2, \ldots, b_n = a_1a_2 \cdots a_n$. Then the latin squares H and K, whose (i,j)th entries are h_ib_j and k_ib_j respectively, are both row-complete and are also orthogonal. Heinrich also observes that one of the squares can be made complete, by permuting the rows of H and K, without affecting orthogonality.

As Heinrich points out, a group satisfying the conditions of Theorem 5.1 cannot be abelian since the product of the elements in a sequenceable abelian group is not the identity (since $a_1 = b_1 = e$) whereas clearly $\prod_{i=1}^{n} h_ik_i^{-1} = e$ since the k_i's are just the h_i's in a different order. In fact the conditions on $(G, \cdot)$ in Theorem 5.1 are equivalent to the requirement that $(G, \cdot)$ be a sequenceable group which also possesses a complete mapping. This follows since, if we define $\theta(h_i^{-1}) = h_ik_i^{-1}$, then θ is a permutation on G and the mapping $\phi : h_i^{-1} \mapsto h_i^{-1}\theta(h_i^{-1}) = k_i^{-1}$ is again a permutation on G. Conversely, if θ is a complete mapping of G, then $g_1^{-1}, g_2^{-1}, \ldots, g_n^{-1}$ and $\phi(g_1)^{-1}, \phi(g_2)^{-1}, \ldots \phi(g_n)^{-1}$ are two orderings of the distinct elements of G such that $g_1^{-1}[\phi(g_1)^{-1}]^{-1}, \ldots, g_n^{-1}[\phi(g_n)^{-1}]^{-1} = \theta(g_1), \ldots, \theta(g_n)$ is again an ordering of the distinct elements of G. We may therefore restate Theorem 5.1 as follows.

Theorem 5.2 *If $(G, \cdot)$ is a sequenceable group which possesses a complete mapping, then there exists a pair of orthogonal row-complete latin squares based on $(G, \cdot)$.*

This theorem generalises easily to give sufficient conditions for the existence of quasi-row-complete latin squares which are quasi-orthogonal.

Theorem 5.3 *If $(G, \cdot)$ is a quasi-sequenceable group which possesses a quasi-complete mapping θ with corresponding quasi-orthomorphism ϕ, then there exists a pair of quasi-orthogonal, quasi-row-complete latin squares based on $(G, \cdot)$.*

Proof: Let $a_1, a_2, \ldots, a_n$ be a quasi-sequencing of G with partial products $b_1 = a_1, b_2 = a_1a_2, \ldots, b_n = a_1a_2\cdots a_n$. Consider the latin squares H and K whose (i,j)th entries are given by $g_i^{-1}b_j$ and $\phi(g_i)^{-1}b_j$ respectively. Then, since $(b_1, b_2, \ldots, b_n)$ is a terrace of G, we have that H and K are each quasi-row-complete by Theorem 4.2.

Suppose H and K are superimposed so as to form an array of ordered pairs Λ. If we now define the *difference* of the ordered pair (a,b) to be ab^{-1} we have that the ijth cell of Λ contains the ordered pair $(g_i^{-1}b_j, \phi(g_i)^{-1}b_j)$ which has difference $\theta(g_i)$. Thus the ith row of Λ contains all the ordered pairs whose difference is $\theta(g_i)$ and the quasi-orthogonality of H and K follows from the fact that $\theta(g_1), \ldots, \theta(g_n)$ is a quasi-ordering of G in a manner analogous to that in the proof of Theorem 2.3.

We observe that if, in the notation of the above theorem, $(G, \cdot)$ is a sequenceable group, then H and K are both row-complete and, if θ is a complete mapping of G, then H and K are orthogonal.

Example using the group $(Z_8, +)$.

Quasi-sequencing	0		5		2		4		1		2		3		1	
Partial sums		0		5		7		3		4		6		1		2

$$\theta = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 2 & 3 & 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\phi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 2 & 4 & 6 & 5 & 7 & 1 & 3 \end{pmatrix}$$

$$\Lambda = \begin{matrix} 00 & 55 & 77 & 33 & 44 & 66 & 11 & 22 \\ 76 & 43 & 65 & 21 & 32 & 54 & 07 & 10 \\ 64 & 31 & 53 & 17 & 20 & 42 & 75 & 06 \\ 52 & 27 & 41 & 05 & 16 & 30 & 63 & 74 \\ 43 & 10 & 32 & 76 & 07 & 21 & 54 & 65 \\ 31 & 06 & 20 & 64 & 75 & 17 & 42 & 53 \\ 27 & 74 & 16 & 52 & 63 & 05 & 30 & 41 \\ 15 & 62 & 04 & 40 & 51 & 73 & 26 & 37 \end{matrix}$$

Note that we may permute the rows of the above array so as to make H quasi-complete without affecting the quasi-orthogonality of the latin squares or the quasi-row-completeness of K.

In [2] Anderson and Ihrig have shown that all groups of odd order are quasi-sequenceable. Since we know that any finite group of odd order possesses at least one complete mapping and that all such groups are quasi-sequenceable it follows that, if $(G, \cdot)$ is a finite abelian group of odd order, then there exists a pair of orthogonal quasi-row-complete latin squares based on $(G, \cdot)$.

References

[1] B.A. Anderson, Bailey's conjecture holds through 87 except possibly for 64, *Journal of Combinatorial Mathematics and Combinatorial Computing* **12** (1992), 187–195.

[2] B.A. Anderson and E.C. Ihrig, All groups of odd order have starter-translate 2-sequencings, *Australasian Journal of Combinatorics* **6** (1992), 135–146..

[3] R.A. Bailey, Quasi-complete latin squares: construction and randomization, *J. Royal Statist. Soc. B* **46** (1984), 323–334.

[4] J. Dénes and A. D. Keedwell, Latin Squares: New Developments in the Theory and Applications. (*Annals of Discrete Mathematics* No. 46, North Holland, 1991.)

[5] G.H. Freeman, Some two dimensional designs balanced for nearest neighbours, *J. Royal Statist. Soc. B* **41** (1979), 88–95.

[6] B. Gordon, Sequences in groups with distinct partial products. *Pacific J. Math.* **11** (1961), 1309–1313.

[7] M. Hall and L.J. Paige, Complete mappings of finite groups, *Pacific J. Math.* **5** (1955), 541–549.

[8] K. Heinrich, Pairwise orthogonal row-complete latin squares, Proc. Eighth S.E. Conf. on Combinatorics, Graph Theory and Computing, 1979, *Congressuss Numerantium* **23**, 505—510.

[9] D.F. Hsu and A.D. Keedwell, Generalised complete mappings, neofields, sequenceable groups and block designs I, *Pacific J. Math.* **111** (1984), 317–332.

[10] D.M. Johnson, A.L. Dulmage and N.S. Mendelsohn, Orthomorphisms of groups and orthogonal latin squares, I, *Canad. J. Math.* **13** (1961), 356–372.

[11] H.B. Mann, The construction of orthogonal latin squares, *Ann. Math. Statist.* **13** (1942), 418–423.

[12] L.J. Paige, A note on finite abelian groups, *Bull. Amer. Math. Soc.* **53** (1947), 590–593.