Characterization of polychrome paths and cycles

Katja Valentin
Mathematisches Institut
Arndtstr. 2
D-35392 Gießen
email: Katja.Valentin@math.uni-giessen.de

ABSTRACT. A polychrome labeling of a simple and connected graph $G = (V, E)$ by an abelian group A is a bijective map from V onto A such that the induced edge labeling $f^*(vw) = f(v) + f(w)$, $vw \in E$, is injective. The paper completes the characterization of polychrome paths and cycles begun in [3].

Let $G = (V, E)$ be a simple connected graph and A be an abelian group. A labeling f of G by A is a bijective map from V onto A. Let $f^* : E \to A$ be defined by $f^*(vw) = f(v) + f(w)$ for $vw \in E$. We say f is a *polychrome labeling* of G by A, if all edge labels are different.

In [3] the author tried to find all abelian groups of order n labeling the path P_n or the cycle C_n polychrome. We are now in the position to state for all such groups whether they label the path or the cycle polychrome or not. The key to the characterization is a result on harmonious groups due to Beals, Gallian, Headley and Jungreis [1]. Therefore we introduce now the notion of a harmonious group. Let H be a group of order n. Then H is *harmonious*, if the elements of H can be listed in the way $h_1, \ldots, h_n$ such that $H = \{h_1h_2, h_2h_3, \ldots, h_{n-1}h_n, h_nh_1\}$.

Remark. *Let A be an abelian group of order n. By definition, it is obvious that there is a polychrome labeling of C_n by A if and only if A is harmonious.*

The main results of this paper are Cor. 2 and Th. 6. In order to establish this we consider the cycle C_n. The first aim is a simple corollary of the next result.

Result 1 [1,Th. 6.6]. *Let H be a finite non-trivial abelian group. Then H is harmonious if and only if H has a non-cyclic or trivial sylow 2-subgroup and H is not an elementary abelian 2-group.*

By the remark we obtain

Corollary 2. *Let A be an abelian group of order n. Then there is a polychrome labeling of C_n by A if and only if A has a non-cyclic or trivial sylow 2-subgroup and A is not an elementary abelian 2-group.*

We shall see, the fact that a group is labeling the path polychrome depends on the structure of its sylow 2-group. Therefore we need a result due to Maamoun and Meyniel [2], which reads in our notation as follows.

Result 3. *If $d > 1$, then there is no polychrome labeling of P_{2^d} by $(\mathbf{Z}_2)^d$.*

Th. 6 will show that the elementary abelian 2-groups are the only groups, which do not label the path polychrome. We obtain this from Cor. 2 and the next theorem about groups having cyclic sylow 2-group.

Result 4 [3,Th. 2.5]. *Let A be an abelian group of order n with cyclic sylow 2-subgroup. Then there is a polychrome labeling of P_n by A.*

Lemma 5. *Let A be an abelian group and f be a polychrome labeling of C_n by A. Then there is a polychrome labeling of P_n by A, too.*

We are ready to prove the main theorem of this paper.

Theorem 6. *Let A be an abelian group of order n. Then there is a polychrome labeling of P_n by A if and only if A is not an elementary abelian 2-group or $A = \mathbf{Z}_2$.*

Proof: If A is an elementary 2-group different from $\mathbf{Z}_2$ the assertion follows from Res. 3. If A has cyclic sylow 2-subgroup, there is a polychrome labeling of C_n by Res. 4. In all other cases the assertion is true by Cor. 2 and Lem. 5. □

References

[1] R. Beals, J.A. Gallian, P. Headley and D. Jungreis, Harmonious groups, *J. Comb. Theory, Ser. A* **56**, No. 2 (1991), 223–238.

[2] M. Maamoun and H. Meyniel, On a problem of G. Hahn about coloured hamiltonian paths in K_{2n}, *Discrete Math.* **51** (1984), 213–214.

[3] K. Valentin, Polychrome labelings of trees and cycles, to appear in *Ars Combinatoria*.