On the Independence Number of Triangle Free Graphs with Maximum Degree Three

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ABSTRACT. In this paper we look at triangle free graphs with maximum degree three. By an inequality proved by K. Fraughnaugh* in 1990, the number of vertices v, edges e, and independence i of such a graph satisfy $e \geq 13v/2 - 14i$. We prove that there is a unique non-cubic, connected graph for which this inequality is sharp. For the cubic case, we describe a computer algorithm which established that two such extremal cubic graphs exist with $v = 14$, and none for $v = 28$ or 42. We give a complete list of cubic, and provide some new examples of non-cubic, triangle free graphs with $v \leq 36$ and independence ratio i/v less than 3/8.

1 Introduction

Let $\mathcal{G}$ be the set of connected triangle free graphs with maximum degree 3. In 1990, Fraughnaugh [7] established an inequality for the number of vertices v, edges e, and independence i for a graph G in $\mathcal{G}$, and also showed conditions for the extremal graphs in $\mathcal{G}$. Namely, she proved:

Theorem 1. (Fraughnaugh)

*formerly K. Jones

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(1) $e \geq 13v/2 - 14i$;   *and*

(2) *If* $e = 13v/2 - 14i$, *then every vertex has valence at least two. If there is a vertex with valence exactly 2, then* G *contains a four-cycle.*

We will refer to (1) as the FJ inequality, and graphs satisfying (2) as the edge-critical graphs. Our goal in this paper is to find all edge-critical graphs. We will prove

Theorem 2. *There are exactly 3 edge-critical graphs.*

These are the two edge-critical cubic graphs L *and* $P(7)$ *depicted in figure 1 and the unique non-cubic edge-critical graph* H *depicted in figure 2.*

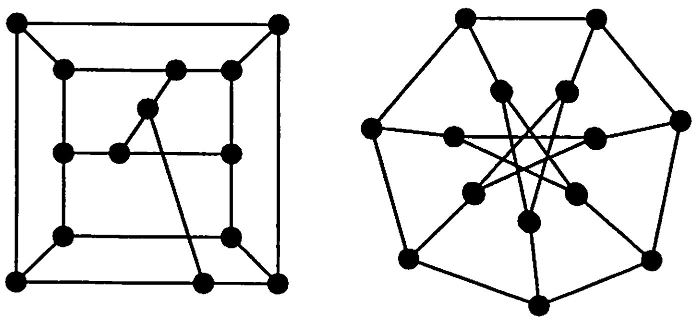

Figure 1. The graphs L and P(7)

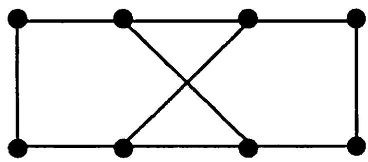

Figure 2. The H-Graph

For the non-cubic case we will in fact show the stronger result, that H is the unique edge-critical graph in $\mathcal{G}$ with girth 4. This result, together with Theorem 1 implies the non-cubic case in Theorem 2.

For the cubic case we have $e = 3v/2$, which implies that for an edge-critical graph, v must be divisible by 14. Two graphs on 14 vertices satisfying this condition are the graphs L (cf. [8]) and $P(7)$ (cf. [4]), see figure 1.

At the Southeastern Conference in Boca Raton in 1990 William Waller announced that there is no edge-critical cubic graph on 28 vertices. His proof was computer assisted and based on a large number of lemmas used to reduce the number of cases that had to be considered (personal communication).

With a computer search we confirmed that L and $P(7)$ are the only two edge-critical graphs on 14 vertices and that there are no such graphs on 28 vertices. We establish the result that there are no edge-critical graphs on 42 vertices.

After our work was finished, S. Locke and K. Fraughnaugh (cf. [5]) proved that every triangle free cubic graph with v vertices contains an independent set with at least $(11v-4)/30$ vertices. This implies that for $v > 14$ no edge-critical cubic graphs exist and therefore the first part of Theorem 2.

The FJ inequality implies that the independence ratio i/v of graphs in $\mathcal{G}$ is at least 5/14, which was first proved by Staton [13]. There are several conjectures about this independence ratio, let us mention

Conjecture 1: (Locke [8]) There are only finitely many cubic 3-connected graphs in $\mathcal{G}$ with an independence ratio less than 3/8; and

Conjecture 2: (Albertson, Bollobás, Tucker [1]) Every cubic planar graph in $\mathcal{G}$ has an independence ratio of at least 3/8.

Our computer search established these conjectures for graphs of order $v < 38$. In particular we find that, with the exception of the graphs L and the generalized Petersen graphs $P(7)$ and $P(11)$ (see figures 1 and 4), all such graphs with independence ratio less than 3/8 contain a bridge connected component isomorphic to the graphs F, $F1$, or $F2$ which are given in figures 5 and 6. We thereby state

Conjecture 3: There are exactly six 2-connected graphs in $\mathcal{G}$ with independence ratio less than 3/8, namely L, $P(7)$, $P(11)$, F, $F1$, and $F2$.

This conjecture implies Conjectures 1 and 2.

2 The underlying ideas of our methods

Though the theoretical proof of Theorem 2 and the computer search for the cubic case are different in nature and were first developed independently by the two authors, they both are based on similar methods that we will now describe.

Let $S = S_{v,e,i,g}$ be the set of all graphs in $\mathcal{G}$ that have v vertices, e edges, independence number at most i, and girth g. $S_{v,e,i,g+}$ denotes the sets where the girth is *at least* g.

To prove Theorem 2, we will assume that v, e, and i satisfy $e = 13v/2 - 14i$, and that $g = 4$. We will prove that $S = \{H\}$, if $v = 8$, $e = 10$, $i = 3$, and $g = 4$, and $S = \emptyset$ otherwise. For the cubic case, we will of course assume that $e = 3v/2$, but – in order to develop a more general program – we let v, i, and g be arbitrary ($g \geq 4$). We construct all members of $S_{v,e,i,g+}$ for all even $v \leq 42$ (except for $v = 38$) with $\lceil (5/14)v \rceil / v < 3/8$. In particular we give an independent proof for $S_{28,42,10,4+} = \emptyset$ and $S_{42,63,15,4+} = \emptyset$.

Our construction will be based on the following strategy:

Suppose v, e, i, and g are given. We start with a subgraph that must be contained in every graph in S. Then we extend this subgraph by adding edges and vertices in every possible way, so that the girth remains at least g, and the maximum degree remains at most 3. We also check each time if the current subgraph can possibly be extended to a member of S. To do this, we will need to estimate the independence number of graphs which can result from our subgraph.

Suppose that during the extension process we arrive at a subgraph $G_0 = (V_0, E_0)$ of a hypothetical graph $G \in S$. Let $V^\# \subseteq V_0$ be a set of saturated vertices of G_0, i.e. if $N_G(V^\#)$ denotes the neighborhood of $V^\#$ in G and $N_{G_0}(V^\#)$ denotes the neighborhood in G_0, then we require $N_G(V^\#) = N_{G_0}(V^\#)$. For $I^* \subseteq V^\#$ an independent set define $V^* := V^\# \cup N(I^*)$, $G^* = (V^*, E^*) := \langle V^* \rangle$ (the graph induced by the vertex set V^*), $V' := V - V^*$, and $G' = (V', E') := \langle V' \rangle$. Numbers v, e, v', e', etc. denote the cardinalities of the corresponding sets.

By the FJ inequality, $i' := \lceil 13v'/28 - e'/14 \rceil$ is a lower bound for the size of a maximal independent set I' of G'. Since $I^* \cup I'$ is an independent set in G for any I', $i^* + i'$ cannot exceed i. If there is a choice for I^* for which $i^* + i' > i$, we conclude that G_0 cannot be extended to a graph $G \in S$ with $V^\#$ saturated.

Thus our construction of S will repeatedly use the following routine.

- (Step 1) Fix the next subgraph $G_0 = (V_0, E_0)$ and the set of saturated vertices $V^\# \subseteq V_0$.
- (Step 2) Choose an independent set $I^* \subseteq V^\#$.
- (Step 3) Find G^* and G' as above.
- (Step 4) Find v^* and (a lower bound for) $\hat{e}$, the number of edges incident to at least one member of V^* ($v^* = v - v'$, $\hat{e} = e - e'$).
- (Step 5) Find i^* and $i' = \lceil 13(v - v*)/28 - (e - \hat{e})/14 \rceil$. $i^* + i' > i$ yields a contradiction. If a contradiction is not reached, we'll go back to (Step 2) (choose a different I^*) or (Step 1) (extend the graph further).

3 A Proposition

In this section G denotes an edge-critical graph of girth 4.

Proposition 1. *H is a subgraph of G.*

Our proof will follow the algorithmic approach described above: We will start with a 4-cycle as a subgraph of G, and then show that we must be able to extend this graph to H. We will achieve this through a string of lemmas, and the proof of each lemma will consist of the five steps described above. In each case, our choice of I^* will be such that $\langle I^* \cup N(I^*)\rangle = G_0$ (and hence we will choose $G^* := G_0$ in Step 3). Since G is edge-critical, thus $e = 13v/2 - 14i$, we have $i' = \lceil 13(v-v^*)/28 - (e-\hat{e})/14 \rceil = i - \lfloor 13v^*/28 - \hat{e}/14 \rfloor$. Hence we can replace Step 5 by

- (Step $\hat{5}$) Compute $\hat{\imath} = \lfloor 13v^*/28 - \hat{e}/14 \rfloor$. We get a contradiction if $i^* > \hat{\imath}$.

Lemma 1. *Each vertex in G has degree 2 or 3.*

Proof: Since G is connected with maximum degree 3, all vertices have degree $1, 2$ or 3. Suppose now that vertex w has a unique neighbor w_0.

- (Step 1) $G_0 := \langle w, w_0 \rangle$, $V^\# := \{w\}$;
- (Step 2) $I^* := \{w\}$;
- (Step 3) $G^* := G_0$;
- (Step 4) $v^* = 2, \hat{e} \geq 2$, since G is connected;
- (Step $\hat{5}$) $\hat{\imath} = 0$, a contradiction. □

Lemma 2. *Each degree 2 vertex in G has a neighbor of degree 2 and a neighbor of degree 3.*

Proof: Suppose that the two neighbors of vertex w are w_1 and w_2.

Case1: w_1 and w_2 both have degree 3.

- (Step 1) $G_0 := \langle w, w_1, w_2 \rangle$, $V^\# := \{w\}$;
- (Step 2) $I^* := \{w\}$;
- (Step 3) $G^* := G_0$;
- (Step 4) $v^* = 3, \hat{e} = 6$;
- (Step $\hat{5}$) $\hat{\imath} = 0$, a contradiction.

Case2: w_1 and w_2 both have degree 2. Let u_1 and u_2 be the other neighbors of w_1 and w_2, respectively.

Case2a: $u_1 = u_2 =: u$.

- (Step 1) $G_0 := \langle w, w_1, w_2, u \rangle$, $V^\# := \{w, w_1, w_2\}$;
- (Step 2) $I^* := \{w_1, w_2\}$;
- (Step 3) $G^* := G_0$;

- (Step 4) $v^* = 4, \hat{e} = 5$, since G is connected;
- (Step $\hat{5}$) $\hat{\imath} = 1$, a contradiction.

Case2b: u_1 and u_2 are distinct.

- (Step 1) $G_0 := \langle w, w_1, w_2, u_1, u_2 \rangle$, $V^\# := \{w, w_1, w_2\}$;
- (Step 2) $I^* := \{w_1, w_2\}$;
- (Step 3) $G^* := G_0$;
- (Step 4) $v^* = 5, 5 \leq \hat{e} \leq 8$;
- (Step $\hat{5}$) $\hat{\imath} = 1$, a contradiction. □

Lemma 3. *In a 4-cycle of G every vertex has degree 3.*

Proof: Suppose that the vertices a, b, c, d form a 4-cycle in G (in this order). By Lemma 2, in a possible counterexample to Lemma 3 (wlog) a and b have degree 2 and c and d have degree 3. In this case let c_1 be the third neighbor of c.

- (Step 1) $G_0 := \langle a, b, c, d, c_1 \rangle$, $V^\# := \{a, b, c\}$;
- (Step 2) $I^* := \{a, c\}$;
- (Step 3) $G^* := G_0$;
- (Step 4) $v^* = 5, 7 \leq \hat{e} \leq 8$;
- (Step $\hat{5}$) $\hat{\imath} = 1$, a contradiction. □

So assume that the vertices a, b, c, d form a 4-cycle in G (in this order) and these vertices have neighbors a_1, b_1, c_1, d_1 (different from a, b, c, d), respectively.

Lemma 4. *a_1, b_1, c_1, d_1 are all distinct.*

Proof: Since G is triangle free, by symmetry it is enough to prove that a_1 and c_1 are distinct. Suppose not.

- (Step 1) $G_0 := \langle a, b, c, d, a_1 = c_1 \rangle$, $V^\# := \{a, c\}$;
- (Step 2) $I^* := \{a, c\}$;
- (Step 3) $G^* := G_0$;
- (Step 4) $v^* = 5, \hat{e} = 9$, since a_1 must have degree 3 (lemma 2);
- (Step $\hat{5}$) $\hat{\imath} = 1$, a contradiction. □

To prove Proposition 1, we need to show that a_1 is adjacent to c_1 and b_1 is adjacent to d_1. Suppose that a_1 and c_1 are not adjacent.

Lemma 5. *a_1 or c_1 has degree 2.*

Proof: Assume $deg(a_1) = deg(c_1) = 3$.

- (Step 1) $G_0 := \langle a, b, c, d, a_1, c_1 \rangle$, $V^{\#} := \{a, c\}$;
- (Step 2) $I^* := \{a, c\}$;
- (Step 3) $G^* := G_0$;
- (Step 4) $v^* = 6, \hat{e} = 12$;
- (Step $\hat{5}$) $\hat{\imath} = 1$, a contradiction. □

So (wlog) assume that c_1 is of degree 2. Then by Lemma 3, c_1 cannot be in a 4-cycle, hence it cannot be connected to b_1 or to d_1. Therefore, c_1 has a neighbor c_2 which is distinct from all previous vertices. By Lemma 2, $deg(c_2) = 2$.

Lemma 6. *c_2 is not adjacent to a_1.*

Proof: Suppose it is. In this case $deg(a_1) = 3$ by Lemma 2.

- (Step 1) $G_0 := \langle a, b, c, d, a_1, c_1, c_2 \rangle$, $V^{\#} := \{a, c, c_1, c_2, \}$;
- (Step 2) $I^* := \{a, c, c_2\}$;
- (Step 3) $G^* := G_0$;
- (Step 4) $v^* = 7, \hat{e} = 11$;
- (Step $\hat{5}$) $\hat{\imath} = 2$, a contradiction.

Let c_3 be the other neighbor of c_2 (we may have $c_3 = b_1$ or $c_3 = d_1$, but c_3 is distinct from $a, b, c, d, a_1, c_1, c_2$). By Lemma 2, c_3 has degree 3. □

We can finish the proof of Proposition 1 now:

- (Step 1) $G_0 := \langle a, b, c, d, a_1, c_1, c_2, c_3 \rangle$, $V^{\#} := \{a, c, c_1, c_2\}$;
- (Step 2) $I^* := \{a, c, c_2\}$;
- (Step 3) $G^* := G_0$;
- (Step 4) $v^* = 8, 12 \leq \hat{e} \leq 14$ (The edges adjacent to a, c and c_2 give eight; since $deg(c_3) = 3$, two more edges are deleted at c_3; and at least one more at a_1, even if a_1 and c_3 are adjacent, since then a_1 has degree 3. As $b_1 \neq d_1$, c_3 can not be equal to both b_1 and d_1, so at least one further edge was deleted at b or d);

- (Step $\hat{5}$) $\hat{i} = 2$, a contradiction.

Thus a_1 is adjacent to c_1. Similarly, b_1 is adjacent to d_1, proving Proposition 1. □

4 Proof of Theorem 2

By Proposition 1, every edge-critical graph with girth 4 contains H as a subgraph. To prove Theorem 2, we will show that H cannot be a proper subgraph.

Suppose that H (with vertices denoted by $a, b, c, d, a_1, b_1, c_1$, and d_1 as in Section 3) is a proper subgraph of some $G \in S$. Choose G as small as possh this property. Since G is connected, at least one of the vertices a_1, b_1, c_1, d_1 has degree 3 in G. Wlog assume $deg(a_1) = 3$, then by Lemma 2, $deg(c_1) = 3$. We will try to get a contradiction as before.

- (Step 1) $G_0 := \langle a, b, c, d, a_1, c_1 \rangle$, $V^{\#} := \{a, c\}$;
- (Step 2) $I^* := \{a, c\}$;
- (Step 3) $G^* := G_0$;
- (Step 4) $v^* = 6, \hat{e} = 11$;
- (Step $\hat{5}$) $\hat{i} = 2$.

This is no contradiction yet, since i^* is not greater than $\hat{i}$. We can argue, however, as follows.

Lemma 7. *The connected components of $G' = G \setminus G^*$ are all isomorphic to H.*

Proof: We must have $i \geq i' + 2$, since $I^* \cup I'$ will be an independent set in G for any independent set I' of G'. Therefore

$$e' = e - 11 = 13v/2 - 14i - 11 \leq 13(v'+6)/2 - 14(i'+2) - 11 = 13v'/2 - 14i',$$

so by the FJ inequality we must have $e' = 13v'/2 - 14i'$ (and therefore $i = i' + 2$). Furthermore, no connected component of G' can be cubic (G wouldn't be connected). By our hypothesis on G, all connected components of G' must be isomorphic to H. □

Now define $I^{**} := \{b, d, c_1\}$. Though $N(I^{**})$ is not a subset of V^* (hence I^{**} is not independent from every independent set I' of G'), we will now find a maximum independent set I' of G' such that there will be no edges between I' and I^{**}. This will yield $i \geq i' + 3$, a contradiction.

Let $G' = kH$ for some natural number k. (Actually $k \leq 2$ can easily be seen.) This gives $i' = 3k$, and we need to find an independent set I' in G'

with $|I'| = 3k$ which is disjoint from $N(I^{**})$. Since $|N(I^{**}) \cap V'| = 3$, in each component H of G' we must have

– a degree 2 vertex which is not in $N(I^{**})$, and

– two degree 3 vertices that are not adjacent to this degree 2 vertex (or each other).

Take these three vertices in each component to form I'. □

5 The principle of the computer algorithm

Having completed the non-cubic case, we will now turn to edge-critical cubic graphs in $\mathcal{G}$. By the previous parts, such graphs must have order v divisible by 14 and girth g at least 5. We will describe a computer program that was used to confirm that L and $P(7)$ are the only such graphs for $v = 14$, and that there are no such graphs for v=28 or v=42.

Since programs are very sensitive to even small errors, it is necessary to be able to check the results independently in as many cases as possible. To be able to do this, we designed the program to work not only for the cases in question, but to be able to generate all sets $S = S_{v,e,i,g+}$ with v even, $e = (3/2)v$, and $g \geq 4$. For $v \leq 20$ we constructed all $S = S_{v,e,i,g+}$ for all possible values of g and i once with the described program and once filtering complete lists of cubic graphs with up to 20 vertices by an independent program computing the size of a maximal independent set. The results were the same.

Although the lists were generated by the computerprogram *minibaum* (see [3] or [2]), on which the following algorithm is based, thn be regarded as independent, as the numbers of graphs in the list were compared with other results (see [12] and [11]) and the graphs were tested for being non-isomorphic by using the (independent) computerprogram *nauty* (see [9] and [10]).

An important problem is of course to avoid the generation of isomorphic copies. This problem is solved efficiently in [2]. So the construction principle and all the routines necessary to avoid the generation of isomorphic copies are taken from the *minibaum* program described in that paper. Here we will not describe how these routines work, but refer the reader to [2].

Our computer algorithm, which we will now discuss, is based again on the 5-step routine described in section 2.

To construct all $G \in S = S_{v,e,i,g+}$ on the vertex set $\{1, \dots, v\}$, we will start with the initial graph G_0 containing only the edges $(1,2), (1,3)$, and $(1,4)$. (So $5, \dots, v$ are isolated vertices at the beginning.) During the construction there is always exactly one nontrivial connected component G_0. The vertices in G_0 are labelled $1 \dots v_0$. In the cubic case, vertices are saturated, iff they have valence 3 in G_0. The unsaturated vertex with the smallest label is called the border vertex and is denoted by $b(G_0)$.

- (Step 1) To construct the new G_0 out of the old G'_0, try to connect $b(G'_0)$ to other unsaturated vertices in every way compatible with the requirement that the vertices in G_0 have consecutive numbers. Of course only edges where the endpoints have distance at least $g-1$ in G'_0 may be inserted, in order to get only graphs with the required minimal girth.

  Define $V^{\#} := \{1, \ldots, b(G_0) - 1\}$ and proceed to Step 2 whenever $b(G_0)$ is not yet connected to a vertex with a number larger than $b(G_0)$.

During the construction process we always extend G_0 by connecting its first unsaturated vertex to other vertices, in such a way that vertices in V_0 are consecutively numbered (and that the maximum degree of G_0 remains 3 and its girth remains at least g). We proceed to (Step 2) whenever the first unsaturated vertex in G_0 is not yet connected to a vertex in $V - V^{\#}$. (This will first happen when $E_0 = \{(1,2),(1,3),(1,4)\}$, next (assuming $v \geq 6$) when $E_0 = \{(1,2),(1,3),(1,4),(2,5),(2,6)\}$, etc.)

- (Step2) Construct all possible independent sets $I^* \subseteq V^{\#}$.

Concerning the implementation it should be noted that after constructing the independent sets we store them. These sets can be used as parts in independent sets in all graphs containing G_0. This means that if e.g. 1000 graphs are evolving from G_0, then the computation of all the parts of the independent sets that contain vertices of $1, \ldots, b-1$ is carried out only once instead of 1000 times. This parallelization of tests in a generation process is similar to a technique used for the implementation of the isomorphism avoiding routines (see [2]).

- (Step 3) Find G^* and G'.

- (Step 4) Compute $v' = v - v^*$ and a bound for e' by $e' \leq e'' := \lfloor (3v' - D)/2 \rfloor$ with D the sum over all valencies of vertices in V'.

- (Step 5) Compute $i' = \lceil 13v'/28 - e''/14 \rceil$. As before, if $i^* + i' > i$, G_0 can not lead to a member of S. At this point, one can also use Theorem 2: If $(13v' - 2e'')/28$ is a natural number and $g \geq 5$ or v' is not a multiple of 8 or $e'' \neq 10v'/8$ then one can use $i' = 13(v' - 2e'')/28 + 1$.

As an example, suppose you want to construct $S_{14,21,5,4+}$. Figure 3 shows a stage where G_0 is spanned by $V_0 = \{1, \ldots, 8\}$, $b(G_0) = 5$. Vertices in I^* are marked by a circle (so $i^* = 3$) and members of $N(I^*)$ are surrounded by a *square. We now have $G^* = G_0$ (which is not always the case), $v' = 6$, and $e' \leq \lfloor(3*6-0)/2\rfloor = 9$, yielding $i' = 3$, a contradiction.*

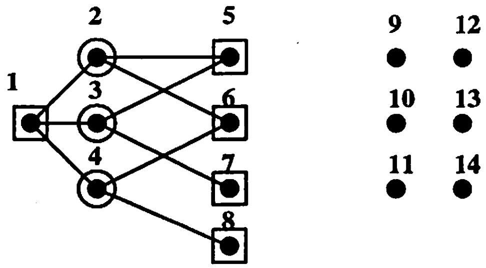

Figure 3. An example

A program based on this algorithm and run on a dec5000/240 took about 7.2 minutes to construct $S_{28,42,10,4+}$ (as a test) and about 5.2 minutes to construct $S_{28,42,10,5+}$.

For the larger cases, some improvements had to be made.

6 The function T

We can improve the algorithm described above if in (Step 4) we consider the degrees of individual vertices in G' rather than just the total valence D. Let us first introduce some notations. The valence vector $\mathcal{V}(G)$ of a not necessarily connected graph G with maximum degree 3 is defined as

$\mathcal{V}(G) = [x_0, x_1, x_2, x_3]$ with x_j the number of vertices of G with degree j.

Let $X = [x_j]_{0\leq j\leq 3}$ and $X' = [x'_j]_{0\leq j\leq 3}$ be vectors in $\mathbb{N}^4$. We say that X dominates X' (denoted by $X \succeq X'$) if we have

(i) $\sum_{j=0}^{3} x_j = \sum_{j=0}^{3} x'_j$; and (ii) $\sum_{j=m}^{3} x_j \geq \sum_{j=m}^{3} x'_j$ for $m \in \{1,2,3\}$.

If X and X' are valence vectors of graphs G and G', condition (i) just says that the graphs have the same number of vertices, while condition (ii) says that there is a bijection between the vertex sets of G and G' so that every vertex $w \in G$ is assigned a vertex $w' \in G'$ with $deg(w) \geq deg(w')$.

Now we define $T(X)$ to be the minimal independence number of all (not necessarily connected) triangle free graphs whose valence vector is dominated by X. It is easy to see that if X dominates X' then $T(X) \leq T(X')$. This is what makes the function T useful for us: We can give a lower bound on the size of a maximal independent set I' of any possible G' by computing the T value of a valence vector X which dominates the valence vector of any possible G'.

To calculate the values of the function T, the following lemma is helpful.

Lemma 8. *At least one of the following statements is true for any vector* $X = [x_j]_{0 \le j \le 3}$ *:*

a.) *There is a connected graph G with $\mathcal{V}(G) = X$ and $i(G) = T(X)$.*

b.) *$T(X) = T(X_1) + T(X_2)$ for some $X_1, X_2 \in \mathbb{N}^4$ with $X = X_1 + X_2$ and at least one element in each of X_1 and X_2 is not zero.*

c.) *There is some $m \in \{1,2,3\}$ so that $T(X) = T(X')$ with X' given by $x'_m = x_m - 1$, $x'_{m-1} = x_{m-1} + 1$, and $x'_j = x_j$ for $j \notin \{m-1$*

Proof: Let G be a graph with $\mathcal{V}(G)$ dominated by X and $i(G) = T(X)$. If G is not connected, say $G = G_1 \cup G_2$ with G_1 a connected component of G and G_2 nonempty, then clearly b.) is fulfilled with $X_1 = \mathcal{V}(G_1)$, $X_2 = \mathcal{V}(G_2)$ since in this case $T(X_1) \le i(G_1)$ and $T(X_2) \le i(G_2)$. A strict inequality in either case would give $T(X) \le T(X_1) + T(X_2) < i(G_1) + i(G_2) = i(G)$, a contradiction.

So suppose G is connected and let $X'' := \mathcal{V}(G)$. If $X'' = X$ then we have case a.), so suppose there is an index m with $x_m \ne x''_m$. Choose m as large as possible and define X' by $x'_m = x_m - 1$, $x'_{m-1} = x_{m-1} + 1$ and $x'_j = x_j$ for $j \notin \{m-1, m\}$.

Since $x''_m < x_m$, we have $X \succeq X' \succeq X''$ and therefore $i(G) = T(X) \le T(X') \le T(X'') \le i(G)$ with the last inequality given by the fact that G is one of the graphs evaluated for $T(X'')$. This gives c.). □

If we now had lists of all graphs in $\mathcal{G}$ of order up to v', then we could determine the minimal value of $i(G')$ for all possible G' with a given valence vector that has $\sum_{j=0}^{3} x_j$ at most v'. This data could then be used to determine the values of T: Starting with the zero vector and proceeding in an increasing order given by $\sum_{j=0}^{3} x_j$ and $\sum_{j=0}^{3} j * x_j$ inside the classes where the first sum is the same, we could easily determine the values of T by comparing the results we get by testing the 3 cases in the lemma. This means: Given a vector X, we have to compare all the values obtained in the following ways:

- Look for connected graphs with X as their valence vector (a.)) and determine their independence numbers.
- Split the vector as described in b.) and calculate the sum of the T-values of the two parts.
- Change the vector as described in c.) and look for the T-value of the modified vector.

Finally take the smallest of the values obtained this way as $T(X)$. The recursive order given above guarantees that all the values of T needed in this process are already known, since in b.) the first sum is decreased for both parts that have to be evaluated and in c.) the first sum stays the same, while the second is decreased. Note also that $T([x_0, x_1, x_2, x_3]) = x_0 + T([0, x_1, x_2, x_3])$..

We used the graph generator *multigraph* to generate such lists. This graph generator works similar to *minibaum* and has been tested by comparing the output e.g. with the well known generation program *MOLGEN* (see e.g. [6]). Of course this was a very time consuming task, so that the number up to which all the graphs were generated had to be a compromise between more information on G' (and hence faster generation of S) and a reasonable computing time for the lists used to compute T. We decided to generate these lists for $v' \leq 20$ and use only the information on the total valency D in case of graphs G' with more vertices. Since the results of the computations might be interesting for other applications as well, they are given in a table at the end of the text.

So we replace Steps 4 and 5 of our algorithm as follows:

- (Step $\bar{4}$) First calculate v'. If v' is larger than 20, determine an upper bound for e' by $e' \leq \lfloor (3v' - D)/2 \rfloor$, else determine a dominating valence vector X' by choosing for each vertex in V' the number of free valences as an upper bound for the degree in G'.
- (Step $\bar{5}$) Compute i' as in (step 5) if $v' > 20$ and by $i' = T(X')$ otherwise. Again, if $i^* + i' > i$ then G_0 can not lead to a member of S.

Using these data the generation of $S_{28,42,10,5+}$ took about 3 minutes and that of $S_{42,63,15,5+}$ about 170 hours. In the 42 vertex case the storage requirement had to be reduced to a value that could not guarantee optimal running times (see [2] for details).

It might also be of some interest that for both $v = 28$ and $v = 42$, we could always decide that the independent sets would become too large before completing the graphs. In the 28 vertex case the maximum number of edges inserted was 38 (out of 42) and in the 42 vertex case it was 54 (out of 63).

7 Summary and further results

Because of conjectures 1 and 2 about cubic graphs and other questions about 2-connected graphs with $8i < 3v$, all graphs in $\mathcal{G}$ with independence ratio smaller than 3/8 are interesting. In this section we will call these graphs 3/8-graphs and give complete lists of them for some vertex numbers and at least examples for others.

In order to determine for which vertex numbers such graphs can possibly exist, the FJ inequality can be used. Up to 42 vertices, the following numbers have to be checked: $v = 11, 14, 19, 22, 25, 27, 28, 30, 33, 35, 36, 38, 39, 41$, and 42. Note that for some numbers cubic and non-cubic 3/8-graphs might exist.

For the cubic case, v has to be even, so $v = 14, 22, 28, 30, 36, 38$, and 42. The running time of the program, based on the described algorithm, does not only depend on the number of vertices, but also on how close the ratio i/v is to 5/14. Because 38 is relatively large and 14/38 – the minimal possible independence ratio for this vertex number – is not close enough to 5/14, we could not do a complete search for $S_{38,57,14,4}$. So the program was used to construct $S_{14,21,5,4+}$, $S_{22,33,8,4+}$, $S_{28,42,10,4+}$, $S_{30,45,11,4+}$, $S_{36,54,13,4+}$, and $S_{42,63,15,4+}$.

The generated lists seem to relate Conjecture 1 and Conjecture 2 and motivated us to formulate Conjecture 3: Except the 3 graphs given in figure 1 and figure 4, which are the graphs L, $P(7)$, and $P(11)$ (a generalized Petersen graph on 22 vertices), they all contain one of the parts given in figure 5 and figure 6. These subgraphs are nonplanar and must be attached to the rest by a bridge. The F-graph in figure 5 was known before and used e.g. by Locke [8] to construct graphs with small independence number. The graphs in figure 6 did not appear in the literature up to now. So if one of these subgraphs had to be present in any large graph in $\mathcal{G}$ with independence ratio smaller than 3/8, both conjectures would be true.

Readers interested in the lists can easily construct them: Except for L, $P(7)$, and $P(11)$, the graphs can all be obtained by using the 4 building blocks in figure 2, figure 5, and figure 6. Here we will only give the numbers of graphs:

$|S_{14,21,5,4+}| = 2$, $|S_{22,33,8,4+}| = 2$, $|S_{28,42,10,4+}| = 0$, $|S_{30,45,11,4+}| = 3$, $|S_{36,54,13,4+}| = 0$, and $|S_{42,63,15,4+}| = 0$.

It is obvious that there are larger cubic 3/8-graphs that do not only consist of these building blocks (e.g. a 60 vertex graph built of a 5-cycle with 5 F-graphs attached). Conjecture 3 is motivated by the expectation that there is no large graph that does not contain one of these bridge connected components at all.

Now we will turn to the case of non-cubic graphs. First we will prove

Proposition 2. *F is the only graph in $\mathcal{G}$ with $v = 14k - 3$ and $i = 5k - 1$, $k \in \mathbb{N}$. (Implying $k = 1$.)*

Proof: By Theorem 1, we get $e \geq 21k - \frac{11}{2}$. Since G has maximum degree 3, this gives $e = 21k - 5$. Let x be the unique vertex of degree 2, and let y and z be its two neighbors.

- (Step 1) Define $G_0 := \langle x, y, z \rangle$, $V^{\#} := \{x\}$;

- (Step 2) Choose $I^* = \{x\}$;
- (Step 3) $G^* := G_0$, $G' := G - G^*$;
- (Step 4) $v^* = 3$, $\hat{e} = 6$ (both y and z have degree 3 in G);
- (Step 5) $i^* = 1$, $i' = 5k - 2$.

We have not arrived at a contradiction yet, but, just like in the proof of Theorem 2, we argue as follows.

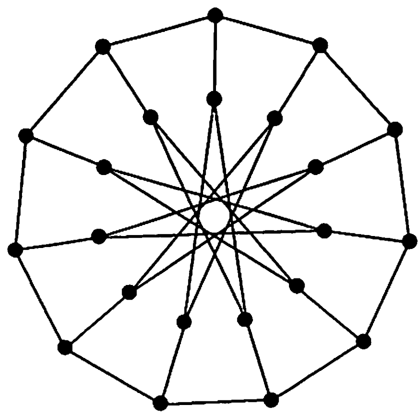

Figure 4. P(11): A generalized Petersen graph

Lemma 9. *G' is edge-critical.*

Proof: We have $v' = 14k - 6$, $e' = 21k - 11$, and $i' \leq i - 1 = 5k - 2$, hence by Theorem 1 we get $e' = 21k - 11 \leq 13(v'/2) - 14i' \leq 21k - 11$, proving our Lemma. □

By Theorem 2, G' (which has $m \leq 4$ connected components, none of which are cubic) must be the disjoint union of m copies of H, and therefore has $v' = 8m$ vertices. This gives $14k - 6 = 8m$, and the only solution of this Diophantine equation with $m \leq 4$ is $m = 1$, $k = 1$, giving $G' = H$. The symmetries of H then imply that $G = F$. □

Thus we have a unique 3/8-graph for $v = 11$, and none for $v = 25, 39$, etc. The case $v = 19$ was decided completely by the computer when doing the calculations for table 1: the two 3/8-graphs are $F1$ and $F2$ of figure 6. Examples for 3/8-graphs for 27, 30, 35, and 38 can be assembled from H, F, $F1$, and $F2$, e.g. $H + F1$ provides an example for $v = 27$. We don't know if these constructions provide all examples for 3/8-graphs for the numbers in question. The remaining numbers will be subject of further investigations.

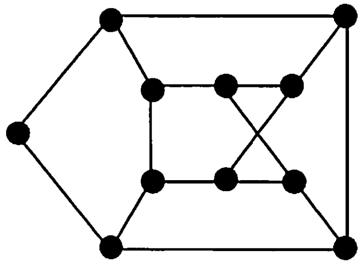

Figure 5. The F-graph

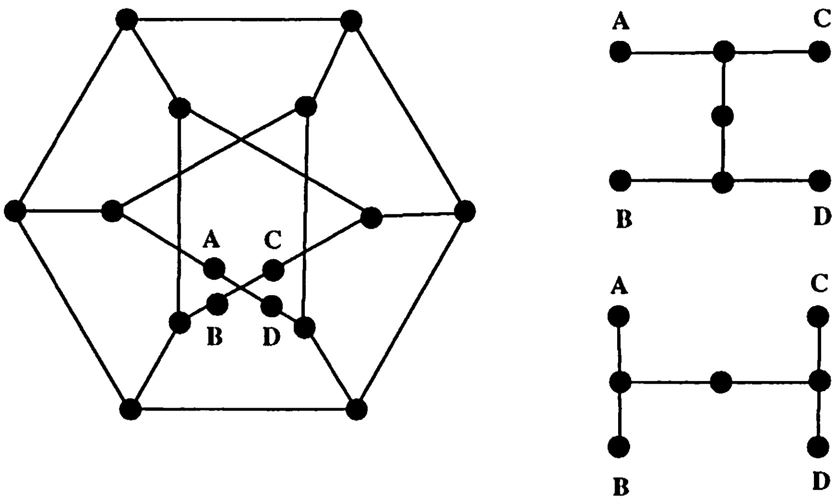

Figure 6. The graphs F1 and F2 as drawn by Tomaž Pisanski's **VEGA**-program. The graphs are obtained by gluing the graphs on the right into the graph on the left in the way described by the labels.

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Table 1.
xyz: A means that a connected triangle free graph with valence vector (x, y, z) has an independent set of size at least A