On strong domination in graphs

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ABSTRACT. Let $G = (V, E)$ be a graph. A vertex u *strongly dominates* a vertex v if $uv \in E$ and $deg\, u \geq deg\, v$. A set $S \subseteq V$ is a *strong dominating set* of G if every vertex in $V - S$ is strongly dominated by at least one vertex of S. The minimum cardinality among all strong dominating sets of G is called the *strong domination number* of G and is denoted by $\gamma_{st}(G)$. This parameter was introduced by Sampathkumar and Pushpa Latha in [4]. In this paper, we investigate sharp upper bounds on the strong domination number for a tree and a connected graph. We show that for any tree T of order $p \geq 2$ that is different from the tree obtained from a star $K_{1,3}$ by subdividing each edge once, $\gamma_{st}(T) \leq (4p-1)/7$ and this bound is sharp. For any connected graph G of order $p \geq 3$, it is shown that $\gamma_{st}(G) \leq 2(p-1)/3$ and this bound is sharp. We show that the decision problem corresponding to the computation of γ_{st} is NP-complete, even for bipartite or chordal graphs.

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1 Introduction

Let $G = (V, E)$ be a graph with vertex set V and edge set E, and let $v \in V$. The *neighborhood* of v, denoted by $N(v)$, is defined as the set of vertices adjacent to v, i.e., $N(v) = \{u \in V \mid uv \in E\}$. For $S \subseteq V$, the *neighborhood* of S, denoted by $N(S)$, is defined by $N(S) = \cup_{v \in S} N(v)$, and the *closed neighbourhood* $N[S]$ of S is the set $N[S] = N(S) \cup S$. For other graph theory terminology we follow [1].

A set $D \subseteq V$ is a *dominating set* of G if every vertex in $V - D$ is adjacent to at least one vertex of D. The minimum cardinality among all dominating sets of G is called the *domination number* of G and is denoted by $\gamma(G)$. The domination number has received considerable attention in the literature.

A vertex u *strongly dominates* a vertex v if $uv \in E$ and $deg\,u \geq deg\,v$. A set $S \subseteq V$ is a *strong dominating set* of G if every vertex in $V - S$ is strongly dominated by at least one vertex of S. The minimum cardinality among all strong dominating sets of G is called the *strong domination number* of G and is denoted by $\gamma_{st}(G)$. This parameter was introduced by Sampathkumar and Pushpa Latha in [4], who also introduced a similar parameter called the *weak domination number* of a graph which was studied further by Hattingh and Laskar [2].

We define the *strong neighborhood* $N_s(v)$ of v in G to be the set $N_s(v) = \{u \mid u \in N(v) \text{ and } deg\,u \geq deg\,v\}$. If S is a strong dominating set of G and $v \in S$, then the set of all vertices w of $V - S$ for which $N_s(w) \cap S = \{v\}$ is called the set of *private strong neighbors* of v and is denoted by $PN_s(v)$. We will need the following property of minimal strong dominating sets, first observed in [4].

Proposition 1 *Let S be a strong dominating set of a graph $G = (V, E)$. Then S is a minimal strong dominating set of G if and only if each $v \in S$ has at least one of the following two properties:*

P_1: *There exists a vertex $w \in V - S$ such that $w \in PN_s(v)$.*

P_2: $N_s(v) \cap S = \emptyset$. □

The paper is organized as follows. In Section 2, we investigate sharp upper bounds on the strong domination number for a tree and a connected graph. In Section 3, we show that the decision problem corresponding to the computation of γ_{st} is NP-complete, even for bipartite or chordal graphs.

2 Upper bounds on γ_{st}

In this section, we investigate upper bounds on the strong domination number of a connected graph. It is well-known (see Ore [3]) that for a connected graph G of order p, $\gamma(G) \leq p/2$. The following lemma shows, however, that the strong domination number of a connected graph of order p may exceed $p/2$.

Lemma 1 *Let $G = (V, E)$ is a connected graph of order p, and let W be the set of all vertices v of G satisfying $N_s(v) = \emptyset$; that is, $W = \{v \in V \mid deg\, v > deg\, u$ for all $u \in N(v)\}$. Then,*

$$\gamma_{st}(G) \leq \frac{p + |W|}{2}.$$

Proof. Among all minimum strong dominating sets of G, let S be chosen to maximize the sum of the degrees of the vertices in S. Let A be the set of all vertices of S that have property P_1; that is, A consists of all vertices v of S satisfying $PN_s(v) \neq \emptyset$. Now let $A' = \cup_{v \in A} PN_s(v)$. We note that $|A'| \geq |A|$. Further, we define $B = \{v \in S - A \mid N_s(v) = \emptyset\}$ and $C = S - (A \cup B)$. By Proposition 1, each vertex of C has property P_2. Let $C' = V - (S \cup A')$. We show that $|C'| \geq |C|$.

We show first that each vertex of C is strongly dominated by some vertex of C'. Let $v \in C$. Then there must exist a vertex $w \in V - S$ such that w strongly dominates v. If $w \in A'$, then $w \in PN_s(a)$ for some vertex a in A and so $deg\, w > deg\, v$. We now consider the set $S' = (S - \{v\}) \cup \{w\}$. Since v does not have property P_1, every vertex of $V - S$ that is strongly dominated by v is also strongly dominated by some vertex of $S - \{v\}$. Hence S' is a strong dominating set of G. Since $\gamma_{st}(G) = |S|$, S' is a minimum such set. However, the sum of the degrees of the vertices in S' exceeds that of S. This contradicts our choice of S. Hence $w \notin A'$; so $w \in C'$.

We now show that $|C'| \geq |C|$. Since each vertex of C is strongly dominated by some vertex of C', the set $(S - C) \cup C'$ is a strong dominating set of G. However, since S is a minimum strong dominating set of G, it follows that $|C| \leq |C'|$. Hence $|A| + |C| \leq |A'| + |C'| = |V - S| = p - \gamma_{st}(G)$. Thus, since $B \subseteq W$, we have $\gamma_{st}(G) = |A| + |B| + |C| \leq |W| + p - \gamma_{st}(G)$. Hence, $\gamma_{st}(G) \leq (p + |W|)/2$, as asserted. □

An immediate corollary now follows.

Corollary 1 *Let $G = (V, E)$ is a connected graph of order p. If $N_s(v) \neq \emptyset$ for all $v \in V$, then $\gamma_{st}(G) \leq p/2$.* □

Using Lemma 1, we may establish a sharp upper bound on the strong domination number of a tree. Let T^* be the tree obtained from a star $K_{1,3}$ by subdividing each edge once. (The tree T^* is shown in Figure 1. The darkened vertices form a minimum strong dominating set of T^*.) Then T^* is a tree of order $p = 7$ with $\gamma_{st}(T^*) = 4 = 4p/7$.

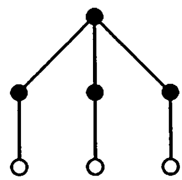

Figure 1: A tree T^* of order p with $\gamma_{st}(T^*) = 4p/7$.

Theorem 1 *For any tree T of order $p \geq 2$ that is different from the tree T^* of Figure 1,*

$$\gamma_{st}(T) \leq \frac{4p-1}{7},$$

and this bound is sharp.

Proof. We proceed by induction on the number m of vertices in the tree whose strong neighborhoods are empty. The base case when $m = 0$ follows from Corollary 1. So, assume that for all trees T' of order $p' \geq 2$ different from the tree T^* and with less than $m \geq 1$ vertices whose strong neighborhoods are empty that $\gamma_{st}(T') \leq (4p'-1)/7$. Let $T = (V, E)$ be a tree of order p with m vertices whose strong neighborhoods are empty. Let W be the set of all vertices v of T satisfying $N_s(v) = \emptyset$, so $|W| = m \geq 1$. Then W is an independent set. Since each vertex in $N(W)$ is strongly dominated by some vertex of W, $V - N(W)$ is a strong dominating set of T. Hence, we have the following lemma.

Lemma 2 $\gamma_{st}(T) \leq p - |N(W)|$.

If W has a vertex of degree 2, then T is a star $K_{1,2}$ and $\gamma_{st}(T) = 1 < (4p-1)/7$. So we may assume in what follows that each vertex of W has degree at least 3. Let S be the set of vertices of W of degree exactly 3, and let $R = W - S$. Then each vertex of R has degree at least 4.

Lemma 3 $|N(R)| \geq 3|R| + 1$.

Proof. Let $H = \langle R \cup N(R) \rangle$ be the subgraph induced by $R \cup N(R)$. Then H is a forest with $|E(H)| \geq 4|R|$. Therefore $|V(H)| \geq 4|R| + 1$, and hence $|N(R)| \geq 3|R| + 1$. □

From Lemmas 1 and 2, it is easy to obtain

Lemma 4 *If $|N(W)| \geq 3|W| + 1$, then $\gamma_{st}(T) \leq (4p-1)/7$.*

Lemma 5 *If each vertex of W has degree at least 4, then $\gamma_{st}(T) \leq (4p-1)/7$.*

Proof. If each vertex of W has degree at least 4, then $S = \emptyset$ and $W = R$. The result now follows from Lemmas 3 and 4. $\square$

Lemma 6 *If S contains a vertex u at distance 2 from some other vertex w of W, then $\gamma_{st}(T) \leq (4p-2)/7$.*

Proof. Let u, v, w be the u-w path in T. Then $deg\, v = 2$. Let T' be the tree obtained from $T - \{u, v\}$ by joining w with an edge to each of the two neighbors of u in T different from v. Then the degree of w in T' is one more than its degree in T, while the degrees of the remaining vertices of T' are equal to their degrees in T. It follows that $W - \{u\}$ is the set of vertices of T' whose strong neighborhoods are empty. Thus, T' is a tree of order $p' = p - 2 \geq 5$ with $m - 1$ vertices whose strong neighborhoods are empty. Furthermore, since T' contains a vertex, namely w, of degree at least 4, T' is different from the tree T^* of Figure 1. Hence, by induction, $\gamma_{st}(T') \leq (4p'-1)/7 = (4p-9)/7$. Let D' be a minimum strong dominating set of T'. Then $D' \cup \{u\}$ is a strong dominating set of T of cardinality at most $(4p-2)/7$. $\square$

In what follows we may assume that $S \neq \emptyset$ and that every vertex of S is at distance at least 3 from every other vertex of W, for otherwise $\gamma_{st}(T) \leq (4p-1)/7$ by Lemmas 5 and 6. Hence $|N(S)| = 3|S|$ and $N(S) \cap N(R) = \emptyset$. Thus $|N(W)| = |N(S)| + |N(R)|$.

Lemma 7 *If $R \neq \emptyset$, then $\gamma_{st}(T) \leq (4p-1)/7$.*

Proof. By Lemma 3, $|N(R)| \geq 3|R| + 1$. Hence, $|N(W)| = |N(S)| + |N(R)| \geq 3|S| + 3|R| + 1 = 3|W| + 1$. Thus, by Lemma 4, $\gamma_{st}(T) \leq (4p-1)/7$. $\square$

Lemma 8 *If $W = S$, then $\gamma_{st}(T) \leq 4p/7$.*

Proof. Since $W = S$, $|N(W)| = 3|W|$. Therefore, by Lemmas 1 and 2 we obtain the desired result. $\square$

In what follows we may assume that $W = S$, for otherwise $\gamma_{st}(T) \leq (4p-1)/7$ by Lemma 7. Thus, by Lemma 8, we know that $\gamma_{st}(T) \leq 4p/7$. It remains for us to show that if $\gamma_{st}(T) = 4p/7$, then T must be the tree T^* of Figure 1. Suppose, then, that $\gamma_{st}(T) = 4p/7$. Then all the inequalities in Lemmas 1 and 2 must be equalities. Hence $\gamma_{st}(T) = (p + |W|)/2$ and $\gamma_{st}(T) = p - 3|W|$. Thus, $p = 7|W|$ and $V - N(W)$ is a minimum strong dominating set of T. Let $X = V - N[W]$. Then $|X| = 3|W|$. However, since each vertex of $N(W)$ has degree at most 2, each vertex of $N(W)$ is adjacent to at most one vertex of X. Consequently, each vertex of $N(W)$ has degree exactly two and is adjacent to a vertex of W and to a vertex of X, while each vertex of X is adjacent to a unique vertex of $N(W)$. Furthermore, we note that X is an independent set, for if $x, y \in X$ with $xy \in E(T)$, then $W \cup (X - \{x\})$ or $W \cup (X - \{y\})$ would be a strong dominating set of T, contradicting the fact that $W \cup X$ is a minimum strong dominating set of T. It follows, therefore, that if $\gamma_{st}(T) = 4p/7$, then T must be the tree T^* of Figure 1.

That the upper bound in the statement of the theorem is sharp, may be seen as follows. Let F_1 be the tree obtained from a star $K_{1,4}$ by subdividing each edge once, and, for $k \geq 2$, let $F_2, \ldots, F_k$, be $k-1$ disjoint copies of the tree T^* shown in Figure 1. For $i = 1, 2, \ldots, k$, let v_i denote the central vertex of F_i, and let w_i be a vertex adjacent to v_i in F_i. Let $W = \{v_1, v_2, \ldots, v_k\}$. For $k \geq 2$, let T_k be the tree obtained from the disjoint union $\cup_{i=1}^{k} F_i$ of $F_1, F_2, \ldots, F_k$ by the addition of the edges $w_i v_{i+1}$ for $i = 1, \ldots, k-1$. (The tree T_4 is shown in Figure 2. The darkened vertices form a minimum strong dominating set of T_4.) Then T_k is a tree of order $p = 7k + 2$ with $\gamma_{st}(T) = |W| + |N(W)| = 4k + 1 = (4p-1)/7$. □

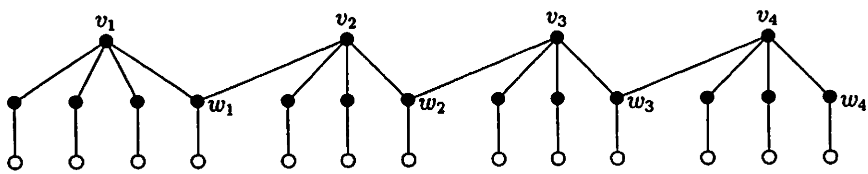

Figure 2: The tree T_4 of order p with $\gamma_{st}(T) = (4p-1)/7$.

Using Lemma 1, we may also establish a sharp upper bound on the strong domination number of a connected graph.

Theorem 2 *For any connected graph G of order $p \geq 3$,*

$$\gamma_{st}(G) \leq \frac{2}{3}(p-1),$$

and this bound is sharp.

Proof. If $p = 3$, then $\gamma_{st}(T) = 1 < 2(p-1)/3$. So we may assume in what follows that $p \geq 4$. Let W be the set of all vertices v of G satisfying $N_s(v) = \emptyset$. Then W is an independent set, and $V - N(W)$ is a strong dominating set of G, so $\gamma_{st}(G) \leq p - |N(W)|$. Since $p \geq 4$, each vertex of W has degree at least 3. Let $W = \{w_1, \ldots, w_k\}$, so $|W| = k$. If $|N(W)| \geq k+2$, then $\gamma_{st}(G) \leq p - |N(W)| \leq p - k - 2$. Thus, $|W| = k \leq p - \gamma_{st}(G) - 2$. Hence, by Lemma 1, $\gamma_{st}(G) \leq (p + |W|)/2 \leq (2p - \gamma_{st}(G) - 2)/2$; or, equivalently, $\gamma_{st}(G) \leq 2(p-1)/3$. Hence in what follows we may assume that $|N(W)| \leq k+1$. Before proceeding further, we prove the following claim.

Claim 1 *W can be matched to a subset of $N(W)$.*

Proof. Let H be the subgraph of G with vertex set $W \cup N(W)$ and edge set all edges of G incident with vertices of W. Since W is an independent set, H is bipartite. We show that $|N(S)| \geq |S|$ for every nonempty subset S of W. We proceed by induction on the cardinality $|S|$ of the sets S. The base case when $|S| = 1$ is trivial since each vertex of W has degree at least 3. So, assume that $|N(S)| \geq |S|$ for every subset S of W with $1 \leq |S| < t \leq |W|$. Let T be a subset of W with $|T| = t$. Let $T' \subset T$ with $|T'| = t - 1$. Without loss of generality, we may assume $T' = \{w_1, \ldots, w_{t-1}\}$. Let H' be the graph obtained from H be deleting the vertices in $W - T'$. Then H' is a bipartite graph with partite sets T' and $N(W)$. By induction, $|N(S)| \geq |S|$ for every nonempty subset S of T'. Hence it follows from a well known theorem attributed to König and Hall that T' can be matched to a subset of $N(W)$ in H'. Let $M = \{w_1v_1, \ldots, w_{t-1}v_{t-1}\}$ denote such a matching of T' to a subset $\{v_1, \ldots, v_{t-1}\}$ of $N(W)$. Then $deg\, v_i \leq deg\, w_i - 1$ for each $i = 1, \ldots, t-1$. A simple counting argument on the edges joining T' and $N(T')$ shows that $|N(T')| \geq t$. Thus $|N(T)| \geq |N(T')| \geq t = |T|$. Hence, by the principle of mathematical induction, $|N(S)| \geq |S|$ for every nonempty subset S of W. The desired result now follows. □

By Claim 1, there exists a matching $M = \{w_1v_1, \ldots, w_kv_k\}$ of W to a subset $\{v_1, \ldots, v_k\}$ of $N(W)$. By our definition of W, we know that $deg\, v_i \leq deg\, w_i - 1$ for each $i = 1, \ldots, k$. A simple counting argument on the edges joining W and $N(W)$ shows that $|N(W)| \geq k + 1$. Thus $|N(W)| = k + 1$.

Claim 2 $G \cong K_{k,k+1}$.

Proof. Let v_{k+1} denote the vertex of $N(W)$ that is not incident with an edge of M. Let q_W denote the number of edges joining W and $N(W)$. Then

$$\begin{aligned}\sum_{i=1}^{k} deg\, w_i = q_W \leq \sum_{i=1}^{k+1} deg\, v_i &\leq \sum_{i=1}^{k} (deg\, w_i - 1) + deg\, v_{k+1} \\ &= \sum_{i=1}^{k} deg\, w_i - k + deg\, v_{k+1},\end{aligned}$$

so $deg\, v_{k+1} \geq k$. Without loss of generality, we may assume that v_{k+1} is adjacent to w_1. Hence $k + 1 \geq deg\, w_1 > deg\, v_{k+1} \geq k$. Consequently, $deg\, w_1 = k+1$ and $deg\, v_{k+1} = k$. We show next that $deg\, w_i = k+1$ for every $i = 2, \ldots, k$. Let H be the subgraph of G with vertex set $W \cup N(W)$ and edge set all edges of G incident with vertices of W, and let $H' = H - v_{k+1}$.

Then H' is a bipartite graph with partite sets W and $N(W) - \{v_{k+1}\}$. Then

$$\sum_{i=1}^{k}(deg\, w_i - 1) \leq q(H') \leq \sum_{i=1}^{k} deg\, v_i \leq \sum_{i=1}^{k}(deg\, w_i - 1).$$

Hence we must have equality throughout. In particular, $q(H') = \sum_{i=1}^{k} deg\, v_i$, and so every vertex v_i is adjacent in G only to vertices of W. Furthermore, $deg\, v_i = deg\, w_i - 1$ for every $i = 1, 2, \ldots, k$. Consequently, $deg\, v_1 = k$ and v_1 is adjacent to every vertex of W. Hence for every $i = 2, \ldots, k$, $k + 1 \geq deg\, w_i > deg\, v_1 = k$ and therefore $deg\, w_i = k + 1$. Thus every vertex of W is adjacent to every vertex of $N(W)$. It follows that $G \cong K_{k,k+1}$. □

By Claim 2, $G \cong K_{k,k+1}$ and so $\gamma_{st}(G) = |W| = k < 2(p - 1)/3$. This establishes that $2(p - 1)/3$ is an upper bound on $\gamma_{st}(G)$.

That this upper bound is sharp, may be seen by taking a complete bipartite graph $K_{k,k+2}$ and adding an adjacent end-vertex to each vertex of the partite set of cardinality $k + 2$, i.e., for each vertex v in the partite set of cardinality $k+2$ we add a new vertex v' and the edge vv'. Let G denote the resulting graph. Then G is a connected graph of order $p = 3k + 4$ with $\gamma_{st}(G) = 2(k + 1) = 2(p - 1)/3$. □

3 Complexity and algorithmic results

In this section we show that the decision problem

STRONG DOMINATING SET (SDS)

INSTANCE: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

QUESTION: Is there a strong dominating set of cardinality at most k?

is NP-complete, even when restricted to bipartite and chordal graphs, by describing polynomial transformations from the following well-known NP-complete problem:

EXACT COVER BY 3-SETS (X3C)

INSTANCE: A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.

QUESTION: Does C contain an exact cover for X, that is, a subcollection $C' \subseteq C$ such that every element of X occurs in exactly one member of C'.

Theorem 3 *SDS is NP-complete, even for bipartite graphs.*

Proof. It is clear that SDS is in NP. To show that SDS is an NP-complete problem, we will establish a polynomial transformation from **X3C**. Let $X = \{x_1, \ldots, x_{3q}\}$ and $C = \{C_1, \ldots, C_m\}$ be an arbitrary instance of **X3C**.

We will construct a bipartite graph G and a positive integer k such that this instance of **X3C** will have an exact three cover if and only if G has a strong dominating set of cardinality at most k.

The graph G is constructed as follows. Corresponding to each variable $x_i \in X$, we associate the single vertex named x_i. Corresponding to each set C_j, we associate the graph F_j which is obtained from the disjoint union of $K_{2,m}$ and $K_{1,m+3}$ by joining a vertex c_j of degree m in the $K_{2,m}$ with the central vertex w_j of the $K_{1,m+3}$. Let w'_j be the vertex of degree m at distance 2 from c_j in F_j. The construction of the bipartite graph G is completed by adding the edges $\{x_i c_j \,|\, x_i \in C_j\}$. It is easy to see that the construction of the graph G can be accomplished in polynomial time. Let $X = \{x_1, \ldots, x_{3q}\}$, $C = \{c_1, \ldots, c_m\}$, and set $k = 2m + q$. We show that C has an exact 3-cover if and only if G has a strong dominating set of cardinality at most k.

Suppose C' is an exact 3-cover for X. Then $|C'| = q$. If $m \geq 2$, let $S = \cup_{j=1}^{m}\{w_j, w'_j\} \cup \{c_j \,|\, C_j \in C'\}$ and if $m = 1$, let S consists of c_1 and its two neighbors in F_1. Then S is a strong dominating set of cardinality $k = 2m + q$. Suppose, conversely, that S is a strong dominating set of G of cardinality at most k. Note that $|(V(F_j) - \{c_j\}) \cap S| \geq 2$ for $j = 1, \ldots, m$. Let $S' = S \cap (X \cup C)$. Then $|S'| \leq k - \sum_{j=1}^{m} |(V(F_j) - \{c_j\}) \cap S| \leq k - 2m = q$. We show now that $S' \subseteq C$. Suppose $|S \cap X| = x$. Then $|S \cap C| \leq |S'| - |S \cap X| \leq q - x$, so that $|N[S \cap C] \cap X| \leq 3(q - x)$. It then follows that $|X - (S \cap X) - (N[S \cap C] \cap X)| \geq 3q - x - (3q - 3x) = 2x$. If $x > 0$, then $x_i \notin N[S]$ for some $i = 1, \ldots, 3q$, which contradicts the fact that S is also a dominating set of G. This implies that $S' \subseteq C$ and S' strongly dominates X. Let $C' = \{C_j \,|\, c_j \in S\}$. Then $|C'| = |S'| \leq q$ and, since S' strongly dominates X, C' must be a cover for X. However every cover of X has cardinality at least q. Consequently, $|C'| = q$ and C' is an exact 3-cover for X. □

Theorem 4 *SDS is NP-complete, even for chordal graphs.*

Proof. It is clear that SDS is in NP. To show that SDS is an NP-complete problem, we will establish a polynomial transformation from **X3C**. Let $X = \{x_1, \ldots, x_{3q}\}$ and $C = \{C_1, \ldots, C_m\}$ be an arbitrary instance of **X3C**.

We will construct a chordal graph G and a positive integer k such that this instance of **X3C** will have an exact three cover if and only if G has a strong dominating set of cardinality at most k.

The graph G is constructed as follows. Corresponding to each variable $x_i \in X$ associate the single vertex x_i. Corresponding to each set C_j associate the single vertex c_j. The construction of the chordal graph G is completed by adding the edges $\{x_i c_j \,|\, x_i \in C_j\}$ and edges so that the c_j's induce a clique; that is, $\langle\{c_1,\ldots,c_m\}\rangle \cong K_m$. It is easy to see that the construction of the graph G can be accomplished in polynomial time. Let $X = \{x_1,\ldots,x_{3q}\}$, $C = \{c_1,\ldots,c_m\}$, and set $k = q$. We show that C has an exact 3-cover if and only if G has a strong dominating set of cardinality at most k.

Suppose C' is an exact 3-cover for X. Then $|C'| = q$, and $\{c_j \mid C_j \in C'\}$ is a strong dominating set of cardinality $k = q$. Suppose, conversely, that S is a strong dominating set of G of cardinality at most $k = q$. We show that $S \subseteq C$. Suppose $|S \cap X| = x$. Then $|S \cap C| \leq |S| - |S \cap X| \leq q - x$, so that $|N[S\cap C]\cap X| \leq 3(q-x)$. It then follows that $|X-(S\cap X)-(N[S\cap C]\cap X)| \geq 3q - x - (3q - 3x) = 2x$. If $x > 0$, then $x_i \notin N[S]$ for some $i = 1,\ldots,3q$, which contradicts the fact that S is also a dominating set of G. This implies that $S \subseteq C$. Let $C' = \{C_j \mid c_j \in S\}$. Then $|C'| = |S| \leq q$ and, since S is a strongly dominating set of G, C' must be a cover for X. However every cover of X has cardinality at least q. Consequently, $|C'| = q$ and C' is an exact 3-cover for X. □

A linear algorithm for computing $\gamma_{st}(T)$ for a tree T is readily obtained by constructing a dynamic style algorithm using the methodology of Wimer (see [5]). We omit the details of the algorithm since a similar algorithm is presented in [2] and can easily be adapted to compute the value of $\gamma_{st}(T)$ for any tree T.

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