Efficient and Excess Domination in Graphs

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Abstract

Each vertex of a graph $G = (V, E)$ dominates every vertex in its closed neighborhood. Set $S \subset V$ is a *dominating set* if each vertex in V is dominated by at least one vertex of S and is an *efficient dominating set* if each vertex in V is dominated by exactly one vertex of S. The *domination excess* $de(G)$ is the smallest number of times that the vertices of G are dominated more than once by a minimum dominating set. We study graphs having efficient dominating sets. In particular, we characterize such coronas and caterpillars as well as the graphs G for which both G and $\overline{G}$ have efficient dominating sets. Then we investigate bounds on the domination excess in graphs which do not have efficient dominating sets and show that for any tree T of order n, $de(T) \leq 2n/3 - 2$.

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1 Introduction

In general, we follow the notation and terminology of [15]. Let $G = (V, E)$ be a graph with $|V| = n$ and $|E| = m$. For a generic invariant $\mu(G)$, let $\mu = \mu(G)$ and $\overline{\mu} = \mu(\overline{G})$. Each vertex $v \in V$ *dominates* every vertex in its closed neighborhood $N[v]$. Set $S \subset V$ is a *dominating set* if each vertex in V is dominated by some vertex of S. In an *independent dominating set*, no two vertices of S are adjacent. The *domination number* $\gamma(G)$ and *independent domination number* $i(G)$ are the smallest orders of a dominating set and an independent dominating set, respectively. If $S \subset V$ is a minimum cardinality dominating set, then we say that S is a $\gamma(G)$-set, or briefly a γ-set. For comprehensive works on domination, see [22, 23].

Much attention [1, 18, 19, 28, 29, 30] has been given to graphs satisfying $i = \gamma$. When $i = \gamma$ there exists a minimum dominating set S in which each vertex of S is dominated exactly once. We are concerned with an even stricter constraint requiring that each vertex of V be dominated exactly once. Set S is an *efficient dominating set* if each vertex in V is dominated exactly once by S. In other words, S is a dominating set and the distance $d(u, v)$ between any pair of vertices u and v in S is at least three (that is, S is a packing that dominates). For brevity we call a graph G *efficient* if G has an efficient dominating set. This concept was first considered by Biggs [4] who looked at it in terms of perfect 1-error correcting codes. Not every graph is efficient. Bange, Barkauskas, and Slater [2] introduced a parameter that measures how close one can come to dominating every vertex given that no vertex can be dominated more than once. Specifically, the *efficient domination number* of G, denoted $F(G)$, equals the maximum number of vertices one can dominate with each vertex dominated at most once. Let $deg(s)$ denote the degree of vertex s. Then $|N[s]| = 1 + deg(s)$ and

$$F(G) = \max \{\sum_{s \in S}(1 + deg(s)) : u, v \in S \text{ implies } d(u, v) \geq 3\}$$

is the maximum amount of domination one can do with a packing. Hence when $F(G) = n$, G is efficient. Bange et al [2] showed that when $F(G) = n$ then any packing S that is an efficient dominating set has $|S| = \gamma(G)$. They further showed that deciding if $F(G) = n$ is an NP-complete problem, and they presented a linear time algorithm to determine $F(T)$ for a tree T. Later, together with Host [3], they studied efficient domination in grid graphs. Grinstead and Slater [13] presented a linear time "template" algorithm for several parameters, including F, for series-parallel graphs. Efficient domination was further investigated by Livingston and Stout [26] who called it "perfect domination". They determined the existence of efficient dominating sets in several families of graphs popular in network design including trees, meshes, and hypercubes. Wieren, Livingston, and

Stout [31] gave an algorithm to construct efficient dominating sets for cube-connected cycles in the cases for which they exist and to prove non-existence otherwise.

Obviously, every efficient dominating set is also an independent dominating set. However, not all independent dominating sets are efficient. For instance, the cycle C_4 has $\gamma = i = 2$ and two vertices are dominated twice by any minimum dominating set. We define the *domination excess* $de(G)$ to be the minimum taken over all minimum dominating sets $S = \{v_1, v_2, \dots, v_\gamma\}$ of $\sum_{i=1}^{\gamma} |N[v_i]| - n$. That is, the domination excess is the smallest number of times that the vertices of G are dominated more than once by any minimum dominating set. Then $de(G) = 0$ if and only if G is efficient. For example, the cycles C_4, C_5, C_6 have $de = 2, 1, 0$, respectively, and the join $P_4 + P_4$ has $de = 4$ achieved by a γ-set formed from any two vertices of degree five. Hence the domination excess number is another measure of how close a given graph is to having an efficient dominating set.

In Grinstead and Slater [12, 13] the *influence* of a vertex set S is defined by $I(S) = \sum_{s \in S}(1 + deg(s))$. Thus $F(G)$ equals the maximum influence of a packing. Rather than having each vertex dominated at most once, if we require that each vertex be dominated at least once then the redundance $R(G)$ defined in [12, 13] and also studied by Johnson and Slater [24] is the minimum influence of a dominating set. Namely, $R(G) = \min\ \{\sum_{s \in S}(1 + deg(s)) : S \text{ is a dominating set}\}$. Note that $F(G) \le n \le R(G)$, and $F(G) = n$ if and only if $R(G) = n$. As shown in some following examples, the packing that achieves $F(G)$ might not be a maximum cardinality packing, and the dominating set that achieves $R(G)$ might not be a minimum cardinality dominating set. Smart and Slater [27] have the following families of parameters under study. For $\gamma(G) \le k \le n$, let $R(G; k)$ be the minimum influence of a dominating set of order at most k. Let $\rho(G)$ denote the maximum order of a packing and for $1 \le k \le \rho(G)$, let $F(G; k)$ denote the maximum influence of a packing of order at least k. We observe that the domination excess satisfies $de(G) = R(G; \gamma) - n$.

In Section 2, we study efficient graphs, i.e., graphs G having $de(G) = 0$. Then in Section 3, we turn our attention to graphs which are not efficient and investigate bounds on the domination excess of these graphs.

2 Graphs With Efficient Dominating Sets

Of course, graph G is efficient if and only if each of its components is. Hence we need to consider only connected graphs. Many interesting families of graphs are efficient, for example, the complete graph K_n, the path P_n, and for $n \equiv 0 \ (mod\ 3)$, the cycle C_n. We begin with two fundamental observations from [2].

Observation 1 *Graph G is efficient if and only if some subcollection of the closed neighborhoods* $\{N[v_1], N[v_2], \ldots, N[v_n]\}$ *partitions* V.

Observation 1 characterizes efficient graphs. On the other hand, Bange, Barkauskas, and Slater [2] show that there is no forbidden subgraph characterization for such graphs. The next observation from [2] establishes that any efficient dominating set is a minimum dominating set.

Observation 2 *If G is efficient, then all efficient dominating sets of G have the same cardinality* γ.

We now make additional observations which also follow directly from the definition of an efficient dominating set. We first mention that if $\gamma = 1$, then G is efficient. We write *diam*(G) for the diameter of G. A comprehensive survey of distance in graphs is given in [6].

Observation 3 *If G is efficient, then* $diam(G) \geq 3$ *or* $\gamma(G) = 1$.

Our first theorem establishes an upper bound on the size of a graph having an efficient dominating set.

Theorem 1 *If G is efficient, then*

$$m \leq n - \gamma + \binom{n-\gamma}{2}$$

and this bound is sharp.

Proof. If G has an efficient dominating set S, then there is no edge with both its vertices in S. Further, each vertex in $V - S$ has exactly one neighbor in S. This accounts for $n - \gamma$ edges. The maximum number of edges in the induced subgraph $< V - S >$ is $\binom{n-\gamma}{2}$. Complete graphs achieve the upper bound. □

Recall that δ and Δ are the minimum and maximum degrees of G, respectively.

Theorem 2 *If G is efficient, then*

$$\frac{n}{\Delta + 1} \leq \gamma \leq \frac{n}{\delta + 1}.$$

Proof. The lower bound is well known [25]. Let $S = \{v_1, v_2, \ldots, v_\gamma\}$ be an efficient dominating set for G. Then S partitions V into $N[v_1] \cup N[v_2] \cup \ldots \cup N[v_\gamma]$. Since $deg(v_i) \geq \delta$ and $N(v_i) \cap S = \emptyset$ for all $v_i \in S$, we have $|V - S| \geq \delta\gamma$. Hence $n - \gamma \geq \delta\gamma$ so $n \geq (\delta + 1)\gamma$. □

Corollary 2.1 *If an efficient graph G is r-regular, then $\gamma = n/(r+1)$.*

Obviously, not all regular graphs are efficient, e.g., $K_{r,r}$ has $de = 2$ for $r \geq 2$. We note that an r-regular graph G is efficient if and only if $(r+1)|n$ and $\gamma = n/(r+1)$. Moreover, if a regular graph is efficient, then all its minimum dominating sets are efficient. A theorem due to Livingston and Stout [26] characterizes the efficient hypercubes.

Theorem 3 ([26]) *The hypercube Q_h is efficient if and only if $(h+1)|2^h$.*

We mentioned above that the cycle C_n is efficient if and only if $n \equiv 0\,(mod\,3)$. We now consider other families of efficient graphs. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *corona* $G \circ H$, introduced by Frucht and Harary [9], is the graph formed by one copy of G and n copies H_1 to H_n of H in which each $v_i \in V(G)$ is joined to every vertex in the copy H_i.

Theorem 4 *The corona $G \circ H$ is efficient if and only if G is $\overline{K}_n$ or $\gamma(H) =$ 1.*

Proof. Obviously, $V(G)$ dominates $G \circ H$ so $\gamma(G \circ H) \leq n$. Further, each H_i must be dominated by either $v_i \in V(G)$ or a dominating set of H_i. Hence $\gamma(G \circ H) = n$.

If G is $\overline{K}_n$, then $V(G)$ is an efficient dominating set for $G \circ H$. If $\gamma(H) = 1$, then the union of the dominating sets of the copies of H is an efficient dominating set for $G \circ H$.

Conversely, let the corona $G \circ H$ have an efficient dominating set S. If $S = V(G)$, then $V(G)$ is an independent set and G is $\overline{K}_n$. Thus assume that at least one vertex in S is in H_i. Then $v_i \notin S$ since S is an efficient dominating set. Further if another vertex of H_i is in S, then S would excessively dominate v_i. Thus $\gamma(H) = 1$. □

A *caterpillar*, introduced by Harary and Schwenk [21], is a tree T which contains a path $u_1, u_2, \ldots, u_k$, called its *spine*, such that each vertex of T is either on the spine or adjacent to a vertex on the spine. A sequence of non-negative integers $(t_1, t_2, \ldots, t_k)$ where t_i is the number of endvertices adjacent to u_i for $k \geq 2$ is associated with T. Both this sequence and its reverse sequence define T. The *code of the caterpillar* is the larger of these two sequences, as in Hage and Harary [14]. Although Bange, Barkauskas, and Slater [2] gave a recursive characterization of efficient trees, we give an alternative simple characterization for caterpillars in terms of their codes.

Theorem 5 *A nontrivial caterpillar G is efficient if and only if its code can be obtained recursively from codes $T_1, T_2, \ldots, T_k = T$ for $T_1 = (1)$ or $T_1 = (r, 0)$ for $r \geq 1$ and T_{i+1} formed by concatenating one of the following two sequences to code T_i:*

$s_1 = (0,t,0) \quad for\ t \geq 0$

$s_2 = (1)$

If the final code $T = (t_1, t_2, \ldots, t_k)$ *has* $t_k = 0$, *then* $T = (t_1, t_2, \ldots, t_{k-1}+1)$.

Proof. It is an easy exercise to see that a code formed by concatenating the above sequences yields an efficient caterpillar. We construct a caterpillar G with an efficient dominating set S. Note that if $t_i \geq 2$, then u_i is in any minimum dominating set. Also, for $t_i = 1$, either u_i or its neighboring endvertex must be in any minimum dominating set. By definition $t_1 \geq 1$. Hence either $u_1 \in S$ or $t_1 = 1$ and u_1 is dominated by the endvertex adjacent to it. If $u_1 \in S$, then u_2 is dominated by u_1 and $u_2 \notin S$ implying that $t_2 = 0$. Thus the initial code T_1 is either (1) or $(t_1, 0)$ for $t_1 \geq 1$. In either case, the last vertex on the spine of T_1, say u_i, is dominated by S but is not in S. If the final code T is not yet constructed, then the next vertex u_{i+1} is not in S since S is an efficient dominating set. Hence $t_{i+1} = 0$ or 1.
Case 1. $t_{i+1} = 0$. Then $u_{i+2} \in S$ to dominate u_{i+1} implying that $t_{i+2} \geq 0$. Furthermore, if the spine is longer than $i+2$, then u_{i+3} is dominated by u_{i+2} and $u_{i+3} \notin S$. Thus $t_{i+3} = 0$. Here we have concatenated the sequence s_1 to the code.
Case 2. $t_{i+1} = 1$. Since $u_{i+1} \notin S$, its adjacent endvertex must be in S. Here we have concatenated the sequence s_2 to the code.

Either the caterpillar is constructed or in both cases, the last vertex on the spine of G is not in S but is dominated by S. We have the same situation as before and proceed recursively. □

We now consider the graphs for which both G and $\overline{G}$ are efficient, i.e., graphs for which $de = \overline{de} = 0$. Obviously, the trivial graph K_1 has this property. We use the following result from [5] and [17].

Theorem 6 ([5, 17]) *If* $\overline{\gamma} \geq 3$, *then* $diam(G) \leq 2$.

If graph G has no isolates, then $\overline{\gamma} \geq 2$. Three corollaries follow directly from Theorem 6 and the definition of efficient domination.

Corollary 6.1 *If an efficient graph G has no isolated vertices, then* $\overline{\gamma} = 2$.

A pair of adjacent vertices which dominates G is called a *dominating edge.*

Corollary 6.2 *If G is efficient and* $\gamma \geq 2$, *then $\overline{G}$ has a dominating edge.*

Corollary 6.3 *If neither G nor $\overline{G}$ has isolates and both are efficient, then* $diam(G) = diam(\overline{G}) = 3$ *and* $\gamma = \overline{\gamma} = 2$.

The converse to Corollary 6.3 is not true as can be seen by the self-complementary graph G obtained from the path $P_4 = x, y, z, w$ and a self-complementary graph H by joining each of y and z to every vertex H. A graph constructed in this manner has $diam = 3$ and $\gamma = 2$, but is not efficient.

Next we give a characterization of the graphs with no isolates for which $de = \overline{de} = 0$. The basic forms of these graphs are displayed schematically in Figure 1.

Theorem 7 *Let G and $\overline{G}$ have no isolates. Then both G and $\overline{G}$ are efficient if and only if G has an induced $P_4 = < u, x, y, v >$ and each vertex in $V - \{u, x, y, v\}$ has exactly one neighbor in each of $\{u, v\}$ and $\{x, y\}$.*

Proof. Let both G and $\overline{G}$ be efficient with no isolates. From Corollary 6.3, $\gamma = \overline{\gamma} = 2$ and $diam(G) = diam(\overline{G}) = 3$. Let $\{u, v\}$ be an efficient dominating set for G. Since $diam(G) = 3$ and the distance between u and v is at least three, $d(u, v) = 3$. Further two vertices in an efficient dominating set for $\overline{G}$, say x and y, are the vertices of a dominating edge in G. As u and v do not have a common neighbor, neither u nor v can be a vertex of a dominating edge of G. Since $N[u] \cap N[v] = \emptyset$, without loss of generality, the dominating edge xy must have $x \in N(u)$ and $y \in N(v)$. Hence G has an induced $P_4 = < u, x, y, v >$. Since $\{u, v\}$ is an efficient dominating set for G, $\{N[u], N[v]\}$ partitions V. Moreover, since $\{x, y\}$ is an efficient dominating set for $\overline{G}$, $\{N[x], N[y]\}$ partitions V in $\overline{G}$ implying that $\{N(x), N(y)\}$ partitions V in G. Thus, these are the graphs described in the theorem for which $V - \{u, x, y, v\}$ induces an arbitrary subgraph. Observe that if G is a graph described in the theorem, then so is $\overline{G}$. Conversely, it is easy to see that such a graph is efficient. □

We note that no cycle has this property and the only such tree is P_4.

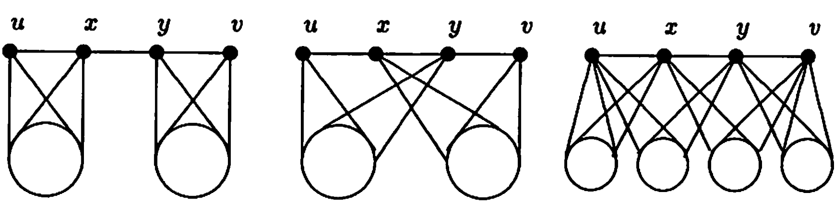

Figure 1: Schemas of the graphs G having $de(G) = de(\overline{G}) = 0$.

3 Domination Excess

Since almost all graphs have diameter two [6, 20], it follows that almost no graphs are efficient, that is, almost all graphs G have $de(G) \geq 1$. First, we give a lower bound on the domination excess of graphs with diameter two.

Theorem 8 *If $diam(G) = 2$, then*

$$de \geq \lfloor \frac{\gamma(\gamma-1)}{2} \rfloor.$$

Proof. Let S be a minimum dominating set of G and $diam(G) = 2$. For each pair of vertices u, v in S, either u is adjacent to v or u and v have a common neighbor. Thus for each of the $\binom{\gamma}{2}$ pairs of vertices in S, at least one vertex is excessively dominated. □

If $de = 1$, then G has a dominating set S and vertices $u, v \in S$ such that $S - \{u, v\}$ is a packing and $d(u, v) = 2$. For example, cycles with $n \equiv 1 \,(mod\, 3)$ have $de = 1$. Obviously, if $de \leq 1$, then $\gamma = i$. But graphs with $\gamma = i$ may not have $de \leq 1$, for instance, the complete bipartite graph $K_{2,t}$ for $t \geq 2$ has $\gamma = i = 2$ and $de = 2$. Observe that if $\gamma = i$, then $de \leq (n - \gamma)(\gamma - 1)$.

In general de can become much larger than n. For example, the corona $K_p \circ \overline{K}_2$ has $n = 3p$, $\gamma = p$, and $de = p(p-1)$. We conclude this paper by deriving an upper bound on $R(T, \gamma(T))$ for a tree T, and hence an upper bound on domination excess in trees. Goddard, Oellermann, Slater, and Swart [11] gave upper and lower bounds, respectively, for $R(T)$ and efficiency $F(T)$ for a tree T.

Theorem 9 ([11]) *If T is a nontrivial tree of order n, then $R(T) \leq 3n/2-1$ and $F(T) \geq \sqrt{8(n+2)}-4$. For a caterpillar C with k vertices on its spine, $F(C) \geq (n + 2k + 2)/3$.*

For efficiency, the tree T in Figure 2 on $n = 2k^2+4k$ vertices achieves the lower bound of $F(T) = 4k = \sqrt{8(2k^2+4k+2)}-4 = \sqrt{8(n+2)}-4$, but if one must use a maximum packing then $F(T, \rho(T)) = F(T, k+1) = 2k+2$. For the caterpillar $C_{k,k}$ with code (k, k) and $k \geq 2$ we have $F(C_{(k,k)}) = k+2$, whereas $F(C_{(k,k)}, \rho(C_{(k,k)}))$ is constant for all k, namely $F(C_{(k,k)}, 2) = 4$.

For redundance, the upper bound is achieved by the caterpillar of Figure 3 with code $(2, 3, 3, ..., 3, 3, 2)$ for which $n = 4k-2$ and $R(C_{(2,3,3,...,3,3,2)}) = 6k-4 = 3(4k-2)/2-1$. For this caterpillar we also have $R(C_{(2,3,3,...,3,3,2)}, \gamma) = 6k-4$. However, there are trees with $R(T, \gamma(T)) \geq 3n/2-1$. For example, let $T^{\#}$ be any tree on k vertices and $T = T^{\#} \circ \overline{K}_2$. For T we have $n = 3k$ and $\gamma(T) = k$. The unique $\gamma(T)$-set is $V(T^{\#})$, and $R(T, k) = 5k-2 = 5n/3-2$. Note that if $T^{\#}$ is the path P_k then we have a caterpillar C on $n = 3k$ vertices with $R(C, \gamma(C)) = 5n/3-2$.

Theorem 10 *If T is a tree of order $n \geq 3$, then $R(T, \gamma(T)) \leq 5n/3-2$.*

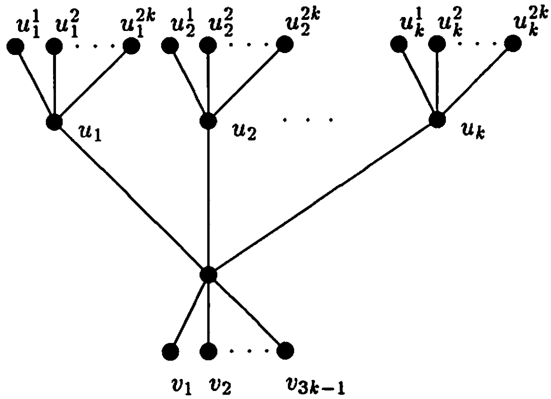

Figure 2: Tree T with $n = 2k^2 + 4k$ and $F(T) = \sqrt{8(n-2)} - 4$.

Proof. As noted, the upper bound is achieved, for example, by $T^{\#} \circ \overline{K}_2$ for any tree $T^{\#}$. We will show that $5n/3-2$ is an upper bound on $R(T, \gamma(T))$ by induction on n. First, if T is a star $K_{1,n-1}$ then $\gamma(T) = 1$ and $R(T, 1) = n$. Actually, if we assume $n \geq 4$ and vertex x is adjacent to three endvertices, w_1, w_2, w_3, then x is in every $\gamma(T)$-set and every $\gamma(T - w_3)$-set. It follows inductively that $R(T, \gamma(T)) = R(T-w_3, \gamma(T-w_3))+1 \leq 5(n-1)/3-2+1 = 5n/3 - 2 - 5/3 + 1 < 5n/3 - 2$, and the result holds for T.

If T is not a star, let v be a vertex in $V(T)$ of eccentricity $e(v) \geq 3$ and root T at v. Let $x \in V(T)$ satisfy $d(v, x) = e(v) - 2$. Then each descendant of x is either an endvertex or all of its neighbors except for x are endvertices. As noted, we can assume no vertex has three or more endvertices in its neighborhood. Assume $N(x) = \{x', u_1, u_2, ..., u_i, v_1, v_2, ..., v_j, w_1, ..., w_k\}$ where x is a descendent of x', $deg(u_t) = 3$ for $1 \leq t \leq i$, $deg(v_t) = 2$ for $1 \leq t \leq j$ and $deg(w_t) = 1$ for $1 \leq t \leq k$ (and $k \in \{0, 1, 2\}$). Also, because $d(v, x) = e(v)-2$ we have $i+j \geq 1$. Let $U_1 = \{u_1, ..., u_i, u'_1, ..., u'_i, u''_1, ..., u''_i\}$ and $U_2 = \{v_1, ..., v_j, v'_1, ..., v'_j\}$ as in Figure 4.

Case 1. Assume $k = 2$. Let $T^* = T-U_1-U_2$, then $\gamma(T^*) = \gamma(T)-i-j$ and x is in every $\gamma(T)$-set and in every $\gamma(T^*)$-set. It follows that $R(T, \gamma(T)) = R(T^*, \gamma(T^*)) + 5i + 3j \leq 5(n - 3i - 2j)/3 - 2 + 5i + 3j = 5n/3 - 2 - j/3 \leq 5n/3 - 2$, and the theorem holds for T by induction.

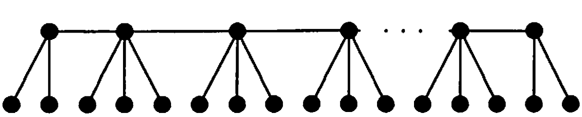

Figure 3: $R(C_{(2,3,3,3,...3,2)}) = 3n/2 - 1$.

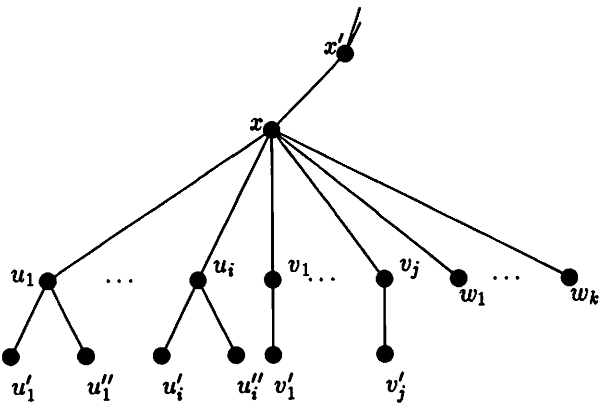

Figure 4: Vertex x of height 2.

Case 2. Assume $i \geq 2$. Let $T^* = T - \{u_i, u_i', u_i''\}$, then $\gamma(T^*) = \gamma(T) - 1$. Let S^* be a $\gamma(T^*)$-set with $R(T^*, \gamma(T^*)) = I(S^*)$. Let $S = S^* \cup \{u_i\}$. If $x \notin S^*$ then $I(S) = I(S^*) + 4$, and if $x \in S^*$ then $I(S) = I(S^*) + 5$. Thus $R(T, \gamma(T)) \leq I(S) \leq R(T^*, \gamma(T^*)) + 5 \leq 5(n-3)/3 - 2 + 5 = 5n/3 - 2$ by induction.

Case 3. Assume $j \geq 2$. Let $T^* = T - \{v_j, v_j'\}$. Then $\gamma(T^*) = \gamma(T) - 1$. Let S^* be a $\gamma(T^*)$-set with $R(T^*, \gamma(T^*)) = I(S^*)$ and let $S = S^* \cup \{v_j'\}$. If $x \notin S^*$ then $I(S) = I(S^*) + 2$, and if $x \in S^*$ then $I(S) = I(S^*) + 3$. Inductively $R(T, \gamma(T)) \leq I(S) \leq R(T^*, \gamma(T^*)) + 3 \leq 5(n-2)/3 - 2 + 3 = 5n/3 - 7/3 < 5n/3 - 2$.

Case 4. Assume $i = j = 1$. This case is similar to Case 3.

Case 5. The remaining situations have $i + j = 1$ and $0 \leq k \leq 1$.

Case 5a. Assume $i = 1$ and $j = k = 0$, and let $T^* = T - \{x, u_1, u_1', u_1''\}$, so that $\gamma(T^*) = \gamma(T) - 1$. As above, let S^* be an $R(T^*, \gamma(T^*))$-set and $S = S^* \cup \{u_1\}$. It follows inductively that $R(T, \gamma(T)) \leq I(S) \leq R(T^*, \gamma(T^*)) + 5 \leq 5(n-4)/3 - 2 + 5 < 5n/3 - 2$.

Case 5b. Assume $i = k = 1$ and $j = 0$, and let $T^* = T - \{u_1, u_1', u_1''\}$, so that $\gamma(T^*) = \gamma(T) - 1$. Let S^* be an $R(T^*, \gamma(T^*))$-set and $S = S^* \cup \{u_1\}$. It follows inductively that $R(T, \gamma(T)) \leq I(S) \leq R(T^*, \gamma(T^*)) + 5 \leq 5(n-3)/3 - 2 + 5 = 5n/3 - 2$.

Case 5c. Assume $j = k = 1$ and $i = 0$, and let $T^* = T - \{v_1, v_1'\}$. Let S^* be an $R(T^*, \gamma(T^*))$- set and $S = S^* \cup \{v_1'\}$. A similar argument shows that $R(T, \gamma(T)) \leq I(S) \leq R(T^*, \gamma(T^*)) + 3 \leq 5(n-2)/3 - 2 + 3 < 5n/3 - 2$.

Case 5d. Finally, assume $j = 1$ and $i = k = 0$, and let $T^* = T - \{x, v_1, v_1'\}$, so that $\gamma(T^*) = \gamma(T) - 1$. Let S^* be an $R(T^*, \gamma(T^*))$-set and $S = S^* \cup \{v_1\}$. We have that $R(T, \gamma(T)) \leq R(T^*, \gamma(T^*)) + 4 \leq 5(n-3)/3 - 2 + 4 < 5n/3 - 2$ inductively. □

Corollary 10.1 *If T is a nontrivial tree of order n, then $de(T) \leq 2n/3-2$.*

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