Undecidable Generalized Colouring Problems

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Abstract

Let H be a graph. An H-colouring of a graph G is an edge-preserving mapping of the vertices of G to the vertices of H. We consider the Extendable H-colouring Problem, that is, the problem of deciding whether a partial H-colouring of some finite subset of the vertices of G can be extended to an H-colouring of G. We show that, for a class of finitely described infinite graphs, Extendable H-colouring is undecidable for all finite non-bipartite graphs H, and also for some finite bipartite graphs H. Similar results are established when H is a finite reflexive graph.

1 Introduction

Let $N \in \mathbf{Z}^+$, and G be a graph with vertex set $V(G) = \{v_{xy}^{(k)} : x, y \in \mathbf{Z}, k = 1, 2, \ldots, N\}$, and suppose that whenever $v_{xy}^{(k)} v_{x'y'}^{(l)}$ is an edge of G, then $|x - x'| \leq 1$ and $|y - y'| \leq 1$. The graph G is called *doubly periodic* (or DP) if the adjacency of $v_{xy}^{(k)}$ and $v_{x'y'}^{(l)}$ depends only on $|x - x'|$, $|y - y'|$, k and l. Doubly periodic graphs are clearly finitary objects. The subgraphs induced by $\{v_{xy}^{(k)} : k = 1, 2, \ldots, N\}$ for fixed x and y are called *cells*. If C_{xy} and $C_{x'y'}$ are distinct cells, and $|x - x'| \leq 1$ and $|y - y'| \leq 1$, then we say C_{xy} and $C_{x'y'}$ are *neighbouring cells*.

In [5], S. Burr studied the following problem concerning extending partial n-colourings of infinite graphs.

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Extendable n-colouring ($n \geq 3$ fixed)
INSTANCE: A graph G and an n-colouring c of some finite subset of $V(G)$.
QUESTION: Can c be extended to an n-colouring of G?

Burr proved that for any fixed integer $n \geq 3$ Extendable n-colouring is undecidable over the class of doubly periodic infinite graphs. We establish a generalization of this result.

Let G and H be graphs. A *homomorphism* of G to H is a function $f : V(G) \to V(H)$ such that $f(x)f(y)$ is an edge of H whenever xy is an edge of G. That is, a homomorphism is a mapping of the vertices of G to the vertices of H that preserves edges. Often we write $G \to H$ when there is a homomorphism of G to H. Since an n-colouring of a graph G is a homomorphism of G to K_n, the term *H-colouring* has been employed to describe a homomorphism of G to H. If the vertices of H are regarded as colours, then an H-colouring of G is an assignment of these colours to the vertices of G so that adjacent vertices in G are assigned adjacent colours in H.

In the case when $H = K_n$, the following problem is Extendable n-colouring.

Extendable H-colouring (H fixed)
INSTANCE: A graph G and an H-colouring c of (the subgraph induced by) some finite subset of $V(G)$.
QUESTION: Can c be extended to an H-colouring of G?

We consider here only the case where H is a fixed finite graph.

Suppose H is a subgraph of G. A *retraction* of G to H is a homomorphism r of G to H such that $r(h) = h$ for all vertices h of H. If there exists a retraction of G to H, then H is called a *retract* of G. The H-retract problem is formally defined below.

H-retract
INSTANCE: A graph G for which H is a labelled subgraph.
QUESTION: Is there a retraction of G to H?

Let X and Y be disjoint graphs, and $A = (a_1, a_2, \ldots, a_k)$ and $B = (b_1, b_2, \ldots, b_k)$ be finite sequences of vertices of X and Y, respectively (repetitions allowed). We write $X_A \cdot Y_B$ to denote the graph formed from $X \cup Y$, by identifying a_i and b_i for $i = 1, 2, \ldots, k$.

We now reformulate Extendable H-colouring as a problem involving retraction of a graph to a labelled copy of H. Let H be a fixed finite graph with vertex set $V(H) = \{h_1, h_2, \ldots, h_n\}$. Suppose an instance of Extendable H-colouring (a graph G and an H colouring $c : U \rightarrow \{h_1, h_2, \ldots, h_n\}$, for some finite subset $U = \{u_1, u_2, \ldots, u_t\}$ of $V(G)$) is given. Let $A = (u_1, u_2, \ldots, u_t)$, $B = (c(u_1), c(u_2), \ldots, c(u_t))$, and $G' = G_A \cdot H_B$. Any extension of c to an H-colouring of G defines a retraction of G' to this copy of H and, conversely, any retraction of G' to this copy of H yields an extension of c to an H-colouring of G.

Let G be a graph. For a subset X of $V(G)$, we denote by $G[X]$ the subgraph of G induced by X.

A graph G is called *almost doubly periodic* (or ADP) if there exists a doubly periodic infinite graph F with cells $F_{xy} = F[\{v_{xy}^{(1)}, v_{xy}^{(2)}, \ldots, v_{xy}^{(N)}\}], (x, y \in \mathbf{Z})$, such that G can be obtained from F by selecting a proper subset $T \subset \{1, 2, \ldots, N\}$ and, for each $t \in T$, identifying all vertices of F belonging to the set $\{v_{xy}^{(t)} : x, y \in \mathbf{Z}\}$. See Figure 1 for a schematic representation of such a graph. We will say that G *arises from F and T*.

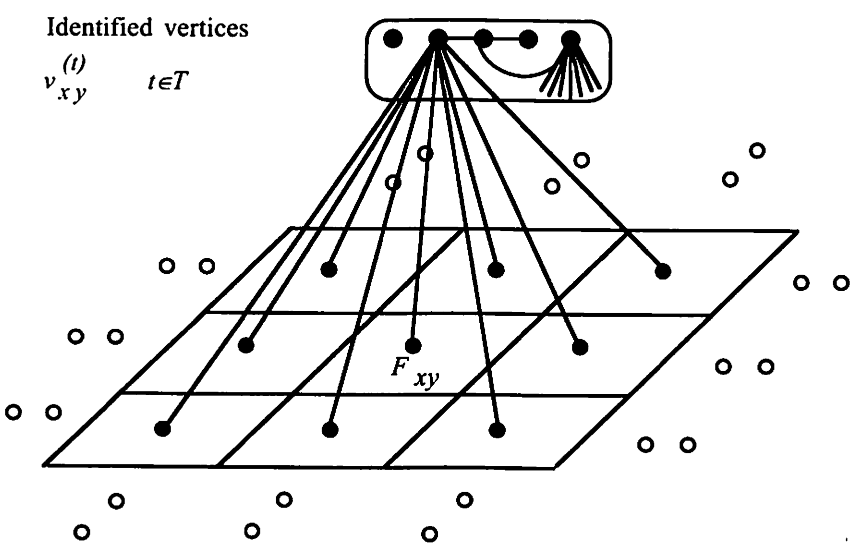


Figure 1. An ADP graph.

A graph G' is called *nearly almost doubly periodic* (or NADP) if there exists a nonempty almost doubly periodic graph G and a (possibly empty) finite graph S such that $G' = G_A \cdot S_B$ for some finite sequences A and B. If G is doubly periodic we call G' a *nearly doubly periodic* (or NDP) graph.

It follows from the earlier remarks on equivalence of H-retract and Extendable H-colouring that the usefulness of NADP graphs lies in modelling finite partial H-colourings of ADP graphs. In view of the above construction and definitions, Burr's result asserts that for every integer $n \geq 3$, K_n-retract is undecidable over the class of NDP graphs. We establish undecidability of Extendable H-colouring, over the class of ADP graphs, for any finite non-bipartite graph H and for many finite bipartite graphs H. This is accomplished by proving undecidability of H-retract over the class of NADP graphs.

2 Non-Bipartite Colour Graphs

It is shown in [8] that H-colouring is NP-complete whenever H is non-bipartite, and is polynomial if H is bipartite. The complexity of H-colouring for infinite graphs H is examined in [3, 11]. In this section we generalize the results in [5, 8] to show that Extendable H-colouring of ADP graphs is undecidable whenever H is non-bipartite. We first discuss a special class of graphs which simplifies our considerations on the structure of H.

If there is no retraction from a graph H to a proper subgraph, H is called *a core*. It is known that every finite graph H contains a unique (up to isomorphism) subgraph C which both is a core and the image of some retraction $r : H \rightarrow C$ [6, 13]. This subgraph C is called *the core* of H [8, 9]. Much less is known about cores in infinite graphs; these are investigated in [4]. If H is a finite graph with core H', then $G \rightarrow H$ if and only $G \rightarrow H'$. This follows since $H \rightarrow H'$ (the retraction), and $H' \rightarrow H$ (the inclusion). Therefore, it suffices to consider H-colouring problems when H is a core. While, in general, this comment applies neither to Extendable H-colouring problems nor to problems involving retractions, we do have the following.

LEMMA 2.1 Suppose H is a retract of the finite graph H'. If H-retract is undecidable, then so is H'-retract.

Proof. We show that if H'-retract is decidable, then so is H-retract. Let an instance of H-retract, a graph G for which H is a labelled subgraph, be given. Let $V(H) = \{v_1, v_2, \ldots, v_n\}$ and $A = (v_1, v_2, \ldots, v_n)$. Let $G' =$

$G_A \cdot H'_A$. We claim that there is a retraction of G to H if and only if there is a retracton of G' to H'. A retraction of G to H can be extended to a retraction of G' to H' by mapping all vertices of $H' - H$ to themselves. On the other hand, suppose r_1 is a retraction of G' to H'. Let r_2 be a retraction of H' to H. Then $r_2 \circ r_1$ is a retraction of G to H. This completes the proof. □

COROLLARY 2.2 Suppose H is a retract of the finite graph H'. If Extendable-H-colouring is undecidable, then so is Extendable-H'-colouring.

For non-bipartite graphs H, the converse is a consequense of the following theorem.

THEOREM 2.3 For any finite non-bipartite graph H, Extendable H-colouring of ADP graphs is undecidable.

Proof. By Corollary 2.2, it suffices to prove the result when H is a core.

We transform the problem to one involving retraction (and ultimately H-colouring, since H is a core) of a NADP graph to a labelled copy of H. Let Y be an ADP graph, and let $c : U' \to V(H)$ be an H-colouing of the subgraph induced by some finite subset $U' = \{u_1, u_2, \ldots, u_k\}$ of $V(Y)$. Let Let $U = (u_1, u_2, \ldots, u_k)$, and $L = (c(u_1), c(u_2), \ldots, c(u_k))$. Define the NADP graph G by $G = Y_U \cdot H_L$. The problem is equivalent to deciding if H is a retract of G. Since H is a core this, in turn, is equivalent to deciding if G is H-colourable.

Suppose, by way of contradiction to the main theorem, that there exists a finite non-bipartite graph H such that H-colouring is decidable for all NADP graphs G. Among all such counterexamples, let H be one with the smallest possible number of vertices and, among all counterexamples with this number of vertices, one with the largest possible number of edges. By Burr's theorem, H is not complete. The exact same argument as in [8] to deny the existence of a smallest counterexample - repeatedly applying the three transformations described below - works here; all we need to do is show that each transformed instance is still a NADP graph.

In order that we may define new cells in a transformed doubly periodic infinite graph, we classify the bordering cells of a given cell into two types, as follows.

Let C_{xy} be an arbitrary cell in a doubly periodic infinite graph G. Then the cells $C_{x'y'}$, with $(x', y') \in \{(x, y+1), (x+1, y+1), (x+1, y), (x+1, y-1)\}$, are called *A-cells* (with respect to C_{xy}); the cells $C_{x'y'}$, with $(x', y') \in \{(x-1, y+1), (x-1, y), (x-1, y-1), (x, y-1)\}$, are called *B-cells* (with respect to C_{xy}.)

An important observation is that if $C_{x'y'}$ is an A-cell with respect to C_{xy}, then C_{xy} is a B-cell with respect to $C_{x'y'}$.

A. *The Indicator Construction*

Let I be a fixed finite graph, and let i and j be distinct vertices of I such that some automorphism of I maps i to j and j to i. The *indicator construction* with respect to the *indicator* (I, i, j) transforms a graph H into the graph $H*$ defined to have the same vertex set as H and to have as the edge set all pairs hh' for which there is a homomorphism of I to H taking i to h and j to h'. Because of the assumption on I, the edges of $H*$ are undirected.

Let H be a fixed finite graph, (I, i, j) be an indicator, and $H*$ be the result of applying the indicator construction with respect to (I, i, j) to H. In [8] it is proved that a graph G is $H*$-colourable if and only if the graph $*G$ obtained from X by replacing each edge $uu' \in E(X)$ with a disjoint copy of I, and identifying u with i and u' with j is H-colourable. The same argument works when G is infinite. Further, we have:

LEMMA 2.4 If G is a NADP graph, then so is $*G$.

Proof. Let G be NADP with cells C_{xy}, (x,$y \in \mathbf{Z}$). As required in the definition of NADP graphs, let $G = Y_A \cdot S_B$, where Y is an ADP graph, and S is a finite graph.

We note that $*G = (*Y_A) \cdot (*S_B)$ for the same sequences A and B as above. Thus, it suffices to show that $*Y$ is ADP. Suppose Y arises from the DP graph F and set T. Consider $*F$, and let $*F_{xy}$ be the family of graphs obtained from the cells F_{xy} (of F) upon replacement of each edge $vv' \in E(F_{xy})$ with a copy of I, and identifying v with i and v' with j. Now, the only edges in $*F$ which are not wholly contained in some $*F_{xy}$ are the edges belonging to copies of I which replaced edges between neighbouring cells of F. Because of the symmetry of I we can assume, without loss of generality, that when a copy of I replaces an edge uv, with $u \in V(F_{xy})$ and v in an A-cell, that i is identified with u and j is identified with v. By the definition of A- and B-cells, it follows that if I replaces an edge st joining

F_{xy} to a B-cell, with $s \in V(F_{xy})$ and t in a B-cell, that j is identified with s and i is identified with t.

Consider the subgraphs $D_{xy} = *F\big[V(*F_{xy}) \bigcup (\cup V(I-i)\big]$, where the union is over all copies of $I-i$ joining $*F_{xy}$ to a B-cell. These will be the cells in $*F$. It follows from the fact that F is doubly periodic and the construction of $*F$ that the subgraphs D_{xy} are all isomorphic. Since the only edges between cells D_{xy} are edges contained in a copy of I which replaced edges joining neighbouring cells, it follows that vertices in D_{xy} are adjacent in $*F$ only to vertices in the same cell, or in neighbouring cells. If an edge uu' joins two neighbouring cells D_{xy} and $D_{x'y'}$, then (say) u is identified with i in a copy of I and $u' \in N_I(i)$. The definition of D_{xy} guarantees that adjacency of two vertices neighbouring cells depends on neither x nor y, but only on the relative position of the neighbouring cells. Hence $*F$ is doubly periodic with cells D_{xy}. Now, every cell D_{xy} contains the vertices $\{v_{xy}^{(t)} : t \in T\}$. Identifying all vertices in $\{v_{xy}^{(t)} : x, y \in \mathbf{Z}\}$ for each $t \in T$ yields $*Y$. Therefore, $*Y$ is ADP. □

B. *The Sub-indicator Construction*

Let J be a fixed finite graph, with specified (distinct) vertices j and $k_1, k_2, \ldots, k_t$. The ***sub-indicator construction*** with respect to the ***sub-indicator*** $(J, j, k_1, k_2, \ldots, k_t)$ transforms a finite core H, with specified (distinct) vertices $h_1, h_2 \ldots, h_t$, into its subgraph $H^{\sim}$ induced by the vertex set $V^{\sim}$ defined as follows: Let $A = (k_1, k_2, \ldots, k_t)$, $B = (h_1, h_2 \ldots, h_t)$, and $W = J_A \cdot H_B$. A vertex v of H belongs to $V^{\sim}$ if and only if there is a retraction of W to H in which j maps to v.

Let $(J, j, k_1, k_2, \ldots, k_t)$ be a fixed sub-indicator, and let H be a finite non-bipartite core with specified (distinct) vertices $h_1, h_2, \ldots, h_t$. In [8] it is proved that a graph G is $H^{\sim}$-colourable if and only if the graph ${}^{\sim}G$ obtained from G, H, and $|V(G)|$ copies of J by identifying each vertex v of G with j in the vth copy of J, and identifying, for all copies of J, the vertices $k_1, k_2, \ldots, k_t$ with $h_1, \ldots, h_t$, respectively, is H-colourable. The same argument works when G is infinite. Further, we have:

LEMMA 2.5 If G is NADP, then so is ${}^{\sim}G$.

Proof. Suppose $A = (a_1, a_2, \ldots, a_k)$, $B = (b_1, b_2, \ldots, b_k)$, and $G = Y_A \cdot S_B$. Supose $V(H) = \{h_1, h_2, \ldots, h_p\}$, and set $A' = \{a_1, a_2, \ldots, a_k, h_1, h_2, \ldots, h_p)$ and $B' = \{b_1, b_2, \ldots, b_k, h_1, h_2, \ldots, h_p)$. Then, it is not hard to check that ${}^{\sim}G = ({}^{\sim}Y_{A'}) \cdot ({}^{\sim}S_{B'})$. Thus, it suffices to show that ${}^{\sim}Y$ is ADP. Suppose

Y arises from the DP graph F and set X. Let $M = (k_1, k_2, \ldots, k_t)$, $P = (h_1, h_2 \ldots, h_t)$, and $J' = J_M \cdot H_P$. Let C_{xy} be the graph obtained from F_{xy} and $|V(F_{xy})|$ copies of J' by identifying each vertex v of F_{xy} with vertex j in the vth copy of J'. Let C be the doubly periodic graph with cells C_{xy} and the same edges between neighbouring cells as in F. Then $\tilde{}Y$ is obtained from C by identifying corresponding vertices of H belonging to copies of J' over all cells C_{xy}. Thus $\tilde{}Y$ is ADP. □

C. *The Edge-sub-indicator Construction*

Let J be a fixed graph with a specified edge jj', and specified vertices $k_1, k_2, \ldots, k_t$, such that some automorphism of J fixes $k_1, k_2, \ldots, k_t$, while exchanging the vertices j and j'. The *edge-sub-indicator construction* with respect to the *edge sub-indicator* $(J, jj', k_1, k_2, \ldots, k_t)$ transforms a finite core H, with specified vertices $k_1, \ldots, k_t$, into its subgraph $H\hat{}$ induced by the edges hh' of H which are images of the edge jj' under retractions of W (defined as in B above) to H. Note that because of our assumption on J, the edges of $H\hat{}$ are undirected.

Let $(J, jj', k_1, k_2, \ldots, k_t)$ be a fixed edge-sub-indicator. Let H be a finite non-bipartite core with specified vertices $h_1, h_2, \ldots, h_t$. In [8] it is proved that a graph G is $H\hat{}$-colourable if and only if the graph $\hat{}G$ obtained from G, H, and $|E(G)|$ copies of J by identifying each edge uv of G with jj' in the uvth copy of J, and identifying, for all copies of J, the vertices $k_1, k_2, \ldots, k_t$ with $h_1, \ldots, h_t$, respectively, is H-colourable. The same argument works when G is infinite. Further, we have:

LEMMA 2.6 If G is NADP, then so is $\hat{}G$.

Proof. Suppose $G = S_A \cdot Y_B$. Then, using the notation of the previous proof, $\hat{}G = (\hat{}S_{A'}) \cdot (\hat{}Y_{B'})$. Thus, it suffices to show that $\hat{}Y$ is ADP. Suppose Y arises from the DP graph F and set T.

For fixed x and y, let $E_A = \{u_1 u_1', \ldots, u_r u_r'\}$ be the set of all edges joining F_{xy} to an A-cell.

Let $M = (k_1, k_2, \ldots, k_t)$, $P = (h_1, h_2 \ldots, h_t)$, and $J' = J_M \cdot H_P$. Let T_{xy} be the graph obtained from F_{xy} and $|E(F_{xy})| + |E_A|$ copies of J' by performing the following three steps: (i) identify each edge uv of F_{xy} with jj' in the uvth copy of J'; (ii) identify the vertex u, where uv joins F_{xy} to an A-cell and $u \in V(F_{xy})$, with vertex j in the uvth copy of J', and (iii) identifying corresponding vertices in all copies of H in J'. Let T be the doubly periodic graph with cells T_{xy} obtained by identifying each vertex j'

with the corresponding vertex v in (ii) above. Then $\tilde{}Y$ is obtained from T by identifying corresponding vertices belonging to all copies of H (in J') over all cells T_{xy}. Thus $\tilde{}Y$ is ADP. □

To recap: Suppose G is some NADP instance of H-colouring, where H is non-bipartite. Then the graphs $*G$, $\tilde{}G$, and $\hat{}G$ are also NADP. Further, the same argument as in [8] (or see [11]) establishes that for graphs G and H,

(1) there exists a homomorphism $G \rightarrow H^*$ if and only if there exists a homomorphism $*G \rightarrow H$,
(2) there exists a homomorphism $G \rightarrow H^{\tilde{}}$ if and only if there exists a homomorphism $\tilde{}G \rightarrow H$,
(3) there exists a homomorphism $G \rightarrow H^{\hat{}}$ if and only if there exists a homomorphism $\hat{}G \rightarrow H$.

One then follows the exact same steps an in Hell and Nešetřil's proof, repeatedly applying these transformations and establishes the non-existence of a minimum counterexample. □

3 Bipartite Colour Graphs

We now show that for some bipartite graphs H, Extendable H-colouring is undecidable over the class of ADP graphs. In particular, we show that for $n \geq 3$, Extendable C_{2n}-colouring is undecidable. Undecidability of a larger class of Extendable H-colouring problems then follows from Corollary 2.2. When the input graphs are finite, the same constructions yield NP-completeness.

THEOREM 3.1 Extendable C_{2n}-colouring of ADP graphs is undecidable for $n \geq 3$.

Proof. Once again, we model an extendable H-colouring problem for ADP graphs by an H-retraction problem for NADP graphs. First we construct a gadget - the graph R_{2n} eventually described below - that will be used to replace edges. It is based on the graph X_{2n}, which we describe first.

The graph $X_{2n}, (n \geq 3)$ is constructed from the $2n$-cycles $C_{2n}^{(k)} = v_0^{(k)}, v_1^{(k)}, \ldots, v_{2n-1}^{(k)}, v_0^{(k)}$ for $k = 1, 2, \ldots, n$, and the vertex b, by adding the edges $v_l^{(k)} v_l^{(k+1)}$ for $k = 1, 2, \ldots, n-1$ and $l = 0, 1, \ldots, 2n-1$ (in addition to those on the $2n$-cycles), and also a path of length $n-2$ from $c = v_n^{(n+1)}$ to b (this requires the addition of $n-3$ new vertices, excluding b). Let $a = v_0^{(n+1)}$. See Figure 2 for the graph X_6.

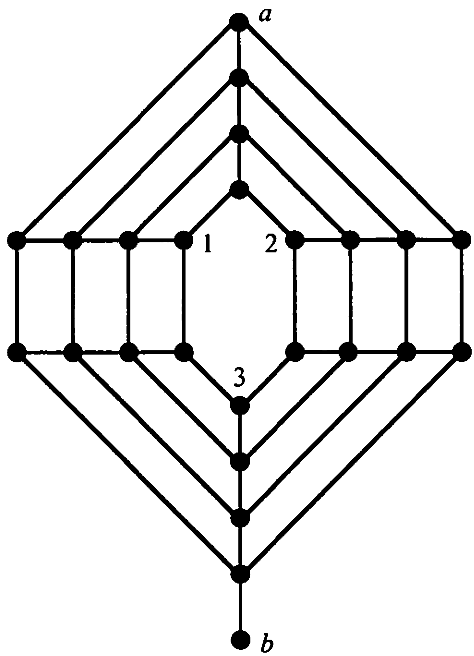


Figure 2. The graph X_6.

Let $S = \{v_0^{(1)}, v_2^{(1)}, \ldots, v_{2n-2}^{(1)}\}$, and define the *inner cycle* of X_{2n} to be $v_0^{(1)}, v_1^{(1)}, \ldots, v_{2n-1}^{(1)}, v_0^{(1)}$.

We claim: *(i) Any retraction of X_{2n} to the inner cycle maps a and b to distinct vertices in S*, and *(ii) If u and v are distinct elements of S, then there is a retaction of X_{2n} to the inner cycle mapping a to u and b to v. That is, a pre-colouring of a and b by distinct elements of S can be extended to a retraction of X_{2n} to the inner cycle.* Suppose r is a retraction of X_{2n} to its inner cycle. Since X_{2n} is connected and bipartite, for every vertex x, the image $r(x)$ belongs to the same partite set as x. Thus, $r(a), r(b) \in S$. Further, it is easily verified that the cycle $C_{2n}^{(2)}$ must map under r to $C_{2n}^{(1)}$ by either a left or a right rotation. Applying the same argument repeatedly, it follows that a and c must map to antipodal vertices of $C_{2n}^{(1)}$. Further, any such pair of vertices, where a is in S, is possible. (For example, any set of rotations that includes exactly two left rotations will map a to $v_{n-4}^{(1)}$.) Since the path from c to b is less than half the length of the inner cycle $C_{2n}^{(1)}$, it is not possible that $r(b) = r(a)$. However, it is clearly possible for b to map to any other element of S.

Let F and F' be two disjoint copies of X_{2n}. For notational simplicity we use the same vertex names as for X_{2n} (but the reader is reminded that F and F' actually have disjoint vertex sets). Let $P = (v_0^{(1)}, v_1^{(1)}, \ldots, v_{2n-1}^{(1)}, a, b)$ and $Q = (v_{n+1}^{(1)}, v_{n+2}^{(1)}, \ldots, v_n^{(1)}, b, a)$, where subscripts in Q are modulo $2n$. Let $R_{2n} = F_P \cdot F'_Q$. We adopt the convention that the identified vertices retain the labels from list P. This should not cause confusion since there is an automorphism of R_{2n} that exchanges the two vertices resulting from identifying a in F (resp. F') and b in F' (resp. F). Note that R_{2n} also has properties (i) and (ii) from above.

Let G be an NADP graph. Construct the graph G' by replacing every edge uu' of G by a copy of R_{2n}, identifying u with a and u' with b, and then identifying corresponding vertices belonging to the $2n$-cycles $C_{2n}^{(1)}$ in each copy of R_{2n}. Note that G' is bipartite.

We claim that the graph G' is NADP. To see this we describe an alternate method of consrtucting G'. Suppose $G = (G_2)_A \cdot W_B$, where G_2 is ADP. Then G_2 arises from a DP graph G_1 and set T_1. Following the steps in the proof of Lemma 2.4, with $I = R_{2n}$, we see that $*G_1$ is DP. Let C_1 denote the set of indices of vertices belonging to inner $2n$-cycles (in any one cell of $*G_1$), and set $T_1' = T_1 \cup C_1$. Identifying, for all $t \in T_1'$, the vertices in each cell of $*G_1$ indexed by t yields an ADP graph H_1. Let W' be the graph obtained by replacing every edge uu' of W by a copy of R_{2n}, identifying u with a and u' with b, and then identifying corresponding vertices belonging to the $2n$-cycles $C_{2n}^{(1)}$ in each copy of R_{2n}. Note that $V(G_2) \subseteq V(H_1)$ and $V(W) \subset V(W')$. Let A' and B' denote, respectively, the lists obtained from A and B by adding the vertices of *the* inner $2n$-cycles (made from identifying vertices, as described above), in the same order, on the end of the list. Then $G' = (H_1)_{A'} \cdot W'_{B'}$, and so G' is NADP.

Since the vertices of G are identified with a and b in copies of R_{2n}, (i) and (ii) above assert that a retraction of G' to C_{2n} models an n-colouring of G, the colours $1, 2, \ldots, n$ being associated with the vertices $v_0^{(1)}, v_2^{(1)}, \ldots, v_{2n-2}^{(1)}$, respectively. Conversely, for any n-colouring of G, (ii) implies that there exists a retraction of G' to $C_{2n}^{(1)}$ which maps each vertex of G (in G') to the vertex of $C_{2n}^{(1)}$ corresponding to its colour. Hence, $G \to K_n$ if and only if there is a retraction of $^!G$ to $C_{2n}^{(1)}$.

Now, suppose G is NADP with some finite subset of its vertices being pre-K_n-coloured with the colours $1, \ldots, n$. Construct $^!G$, maintaining the same pre-K_n-colouring, and also put colours $1, \ldots, n$ on the vertices of the identified $2n$-cycle $C_{2n}^{(1)}$ as shown in Figure 2. As before, the pre-K_n-colouring of G is extendable if and only if the constructed C_{2n}-pre-colouring of $^!G$ is extendable. Since Extendable n-colouring ($n \geq 3$) is undecidable [5], so is Extendable C_{2n}-colouring. This proves Theorem 3.1. □

We are now in a position to define an entire family of bipartite graphs H for which extendable H-colouring is undecidable.

COROLLARY 3.2 If H is a finite bipartite graph that admits a retraction to a cycle of length at least six, then extendable H-colouring is undecidable.

It remains to charactarize the complexity of extendable H-colouring when H is a forest or when the only cycle to which H retracts is C_4. In a personal communication, Bruce Bauslaugh of the University of Calgary reports that he has proved that Extendable K_2-colouring is decidable for doubly periodic graphs.

4 Reflexive Colour Graphs

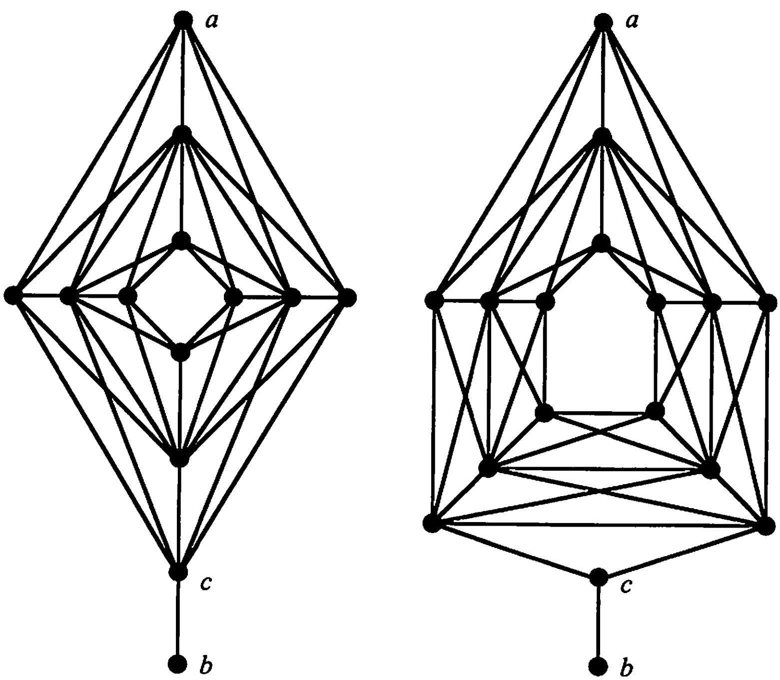
We now turn our attention to those graphs H which are *reflexive*; that is, graphs which have a loop at every vertex. Homomorphisms to reflexive graphs are examined in [10]. If H is reflexive and $G \to H$, then adjacent vertices of G can map to the same vertex of H. In this section, C_n denotes the reflexive cycle on n vertices. It will be shown that, for the class of NADP graphs, extendable C_n-colouring is undecidable for $n \geq 4$. We will use a similar construction as in the previous section.

THEOREM 4.1 Extendable C_n-colouring of ADP graphs is undecidable for $n \geq 4$.

Proof. As before, consider the C_n-retract problem for NADP graphs. Since the proof is very similar to the one given in the previous section, we simply describe the graphs that replace edges, and omit the remaining details. Construct the graph X'_n as follows.

Case 1: n is even.

Suppose $n = 2m, m \in \mathbf{Z}$. Consider $m+1$ n-cycles $C_n^{(k)} = v_0^{(k)}, v_1^{(k)}, \ldots, v_{n-1}^{(k)}, v_0^{(k)}$ for $k = 1, 2, \ldots, m+1$, and the vertex b, joined with edges $v_l^{(k)} v_l^{(k+1)}$, $v_l^{(k)} v_{l-1}^{(k+1)}$, and $v_l^{(k)} v_{l+1}^{(k+1)}$ for $k = 1, 2, \ldots, m$ and $l = 0, 1, \ldots, n-1$, where the subscripts are taken modulo n. Let $a = v_0^{(m+1)}$ and $c = v_m^{(m+1)}$, and add a path of length $m-1$ from c to b. Finally, place a loop at every vertex, thus forming a reflexive graph. The resulting graph is X'_n. See Figure 3a for the graph X'_4. (Loops are omitted in the figures.)



(a) (b)

Figure 3. The graphs X'_4 and X'_5.

Case 2: n is odd.
Suppose $n = 2m+1$, $m \in \mathbf{Z}$. Consider m n-cycles $C_n^{(k)} = v_0^{(k)}, v_1^{(k)}, \ldots, v_{n-1}^{(k)}, v_0^{(k)}$ for $k = 1, 2, \ldots, m$, and the vertex b, joined with edges $v_l^{(k)} v_l^{(k+1)}$, $v_l^{(k)} v_{l-1}^{(k+1)}$, and $v_l^{(k)} v_{l+1}^{(k+1)}$ for $k = 1, 2, \ldots, m-1$ and $l = 0, 1, \ldots, n-1$, where the subscripts are taken modulo n. Let $a = v_0^{(m)}$ and join the vertex c to $v_m^{(m)}$ and $v_{m+1}^{(m)}$; add a path of length $m-1$ from c to b. Again, place a loop at every vertex, thus forming a reflexive graph. The resulting graph is X_n'. See Figure 3b for the graph X_5'.

The graphs which replace edges are made, as in the proof of the previous theorem, using two copies of X_n'. □

COROLLARY 4.2 If H is a reflexive graph of which, for some $n \geq 4$, C_n is a retract, then extendable H-colouring of NADP graphs is undecidable.

We are now left with examining the decidability of Extendable H-colouring when H contains no cycle (other than loops) or when the only cycle which is a retract of H is C_3. We show that a subclass of these remaining graphs H are such that Extendable H-colouring of NDP graphs is decidable. Our proof uses of the following theorem of Hell.

THEOREM 4.3 [7] Let F be an infinite graph and H a finite labelled subgraph of H. If H is a retract of every finite subgraph of F which contains H, then H is a retract of F.

If G is a graph with vertices u and v, let $d_G(u,v)$ be the length of a shortest path from u to v, whenever such a path exists and define $d_G(u,v)$ to be infinite otherwise. We say that a subgraph H of G is *isometric* if $d_G(u,v) = d_H(u,v)$ for all $u, v \in V(H)$. See [12] for further discussion on isometric subgraphs. In [10] and [1], absolute retracts are defined. For our purposes, we call a finite graph H an *absolute retract* if it is a retract of every graph of which it is an isometric subgraph. For example, each reflexive complete graph is an absolute retract. Several other, inequivalent, definitions have also been used [2, 10]. Using Theorem 4.3, we can establish the following:

COROLLARY 4.4 If a graph H is an absolute retract, then extendable H-colouring of DP graphs is decidable.

Proof. Once again, the interplay between retractions and extending partial colourings allows us to formulate the result in terms of retractions to NDP graphs. Let F be an DP graph, $B' = \{b_1, b_2, \dots, b_k\} \subseteq V(F)$ and suppose $c : b' \to V(H)$ is a pre-H-colouring. Let $A = (c(b_1), c(b_2), \dots, c(b_k))$ and $B = (b_1, b_2, \dots, b_k)$. Consider the NDP graph $G = H_A \cdot F_B$. Suppose F has cells $F_{xy} = F[\{v_{xy}^{(1)}, v_{x,y}^{(2)}, \dots, v_{xy}^{(N)}\}]$ $(x, y \in \mathbf{Z})$. Let D be the finite subgraph of G induced by $V(H)$, and all vertices at distance at most $diam(H)$ from some vertex of H

Since all vertices of G have finite degree, it is easy to find D. Note that if H is an isometric subgraph of D, then it is also an isometric subgraph in any graph containing D as an induced subgraph. Thus, we need only establish whether H is an isometric subgraph of D, then the results follows from Theorem 4.3. This is easy to do, since D is a finite graph. □

The above argument fails for ADP graphs, since these may have vertices of infinite degree.

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