Generalized Hooked, Extended, and Near-Skolem Sequences

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ABSTRACT. In this paper, we introduce generalized hooked, extended, and near-Skolem sequences and determine necessary conditions for their existence, the minimum number of hooks and their permissible locations. We also produce computational results for small orders in each case.

1 Introduction

While studying Steiner Triple Systems, Skolem (1957) asked whether it was possible to partition the set $\{1, 2, \ldots, 2n\}$ into n pairs (a_r, b_r), such that $b_r - a_r = r$, where $r = 1, 2, \ldots, n$ [26]. He showed that such a partition exists if and only if $n \equiv 0$ or 1 (mod 4). Nickerson [11] was first to write the system in the form of a sequence. For example, if $n = 5$ the sequence $3, 5, 2, 3, 2, 4, 5, 1, 1, 4$ is equivalent to the partition of the numbers $1, \ldots 10$ into the pairs $(8, 9), (3, 5), (1, 4), (6, 10), (2, 7)$; this sequence is now known as a *Skolem sequence* of order 5. When $n \equiv 2, 3$ (mod 4), Skolem considered distributing the numbers $1, 2, \ldots, 2n-1, 2n+1$, into n disjoint

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pairs $(a_r, b_r), r = 1, 2, \ldots n$, such that $b_r - a_r = r$. O'Keefe [13] showed that such a partition exists; the solution written in the form of Nickerson's notation requires leaving a space (or zero) for the missing integer, called a *hook*. For example, $3, 1, 1, 3, 2, 0, 2$, is a *hood Skolem sequence* of order 3, and is equivalent to distributing the numbers $1, 2, 3, 4, 5, 7$, into the pairs $(2, 3), (5, 7), (1, 4)$.

Rosa (1966) [15] modified the notion of Skolem sequences by adding an extra hook in the middle. Another modification of Skolem sequences was introduced by Abrham and Kotzig [1]: in an *extended Skolem sequence* the zero (or hook) may occur anywhere in the sequence. They showed that such sequences exist for all n. However, the existence problem for extended Skolem sequences with a prescribed position of the zero (subject to a parity condition) remains open.

Stanton and Goulden (1981) [27] introduced, in effect, *Near-Skolem sequences* for the purpose of constructing cyclic Steiner triple systems. They asked for a set of $n - 1$ pairs $P(1, n)/m$ with each of the integers of $\{1, 2, \ldots, 2n-2\}$ appearing exactly once and each of the integers of $\{1, \ldots, m-1, m+1, \ldots, n\}$ occurring as a difference exactly once. For example, the pairs $(3, 4), (6, 8), (1, 5), (2, 7)$ form a $P(1, 5)/3$, or the 3-near-Skolem sequence of order $5 : 4, 5, 1, 1, 4, 2, 5, 2$.

In [23, 24], it was shown that the necessary and sufficient conditions for the existence of near- and hooked near-Skolem sequences are:

For the near-Skolem sequences:

(1) if $m \equiv 0, 1 \pmod 4$, then m must be odd;

(2) if $n \equiv 2, 3 \pmod 4$, then m must be even.

Langford (1957) [7] noticed that his son, while playing with coloured blocks, placed them in one pile so that between the red pair there was one block, between the blue two, and between the yellow three. He expressed the case of three colours ($n = 3$) as $3, 1, 2, 1, 3, 2$. This is now known as a special case of a *Langford sequence*. Adding one to each term and appending a pair of 1's results in a Skolem sequence of order $n + 1$. Thus an (n, d)-*Langford sequence* $(L_1, L_2, \ldots, L_{2n-2d})$ is a sequence in which each of the integers $k \in \{d, d+1, \ldots, d+n-1\}$ is repeated exactly twice and whenever $L_i = L_j = k$ then $j - i = k$. The works of Priday, Davies, Bermond, Brouwer, Germa, and Simpson [13, 4, 3, 25], proved that the necessary and sufficient conditions for the existence of Langford sequences and hooked Langford sequences are:

For Langford sequences:

(1) $n \geq 2d - 1$;

(2) if $n \equiv 0, 1 \pmod 4$, then d must be odd;

(3) if $n \equiv 2,3 \pmod 4$, then d must be even.

For hooked Langford sequences:

(1) $n(n-2d+1)+2 \geq 0$;

(2) if $n \equiv 2,3 \pmod 4$, then d must be odd;

(3) if $n \equiv 1,2 \pmod 4$, then d must be even.

Gillespie and Utz (1966) [6] were first to introduce the notion of a *generalized perfect s-Langford* sequence of order n, as a sequence of sn terms in which each of the integers $1,2,\ldots,n$ occurs exactly s-times, and between any two consecutive occurrences of the integer i, there are i entries, $1 \leq i \leq n$. They showed that there is no generalized perfect 3-sequence for $n = 2,3,4,5,6$. Levine (1968) [8, 9] showed that if $s = 3$ then a necessary condition for the existence of a perfect 3-sequence is that n must be $\equiv -1, 0$ or $1 \pmod 9$. He conjectured that such a condition is sufficient for the existence of the perfect 3-Langford sequence with $n > 8$. Levine also generalized this necessary condition for the case of perfect s-sequence, $s = pt$ where p is a prime, to be $n \equiv -1,0,1,\ldots,$ or $p-2 \pmod{p^2}$. Roselle and Thomasson [18] generalized this necessary condition to $s = p^e t$ where p is a prime and e is any positive integer, to be $n \equiv -1,0,1,\ldots,$ or $p-2$ (mod p^{e+1}). Roselle [17] also introduced the *generalized Skolem (s,n) sequence* which is a generalized s-Langford that starts from the integer 0 rather than 1. For example,

$$0,0,0,1,9,1,6,1,8,2,5,7,2,6,9,2,5,8,4,7,6,3,5,4,9,3,8,7,4,3$$

is a Skolem $(3,9)$-sequence.

Henceforth we will use the Nickerson-Simpson notation for writing sequences. Thus the above sequence is written as

$$1,1,1,2,10,2,7,2,9,3,6,8,3,7,10,3,6,9,5,8,7,4,6,5,10,4,9,8,5,4,$$

and is a Skolem $(3,10)$-sequence.

Motivated by the analogy with the case of $s = 2$, in this paper we introduce the generalized hooked, extended and near-Skolem sequences ($s > 2$) and determine necessary conditions for their existence, the minimum number of hooks and their permissible locations (when $s = p^e t$ where p is a prime and e is any positive integer). We also produce computational results for small orders in each case.

2 Necessary Conditions For The Existence of Generalized Sequences

In this section, we generalize the theorem of Roselle and Thomasson [18] and deduce from it several corollaries.

Formally, a *Skolem sequence* of order n is a sequence $S = (s_1, s_2, \dots, s_{2n})$ of $2n$ integers satisfying the following conditions:

1. For every $k \in \{1, 2, \dots, n\}$ there exist exactly two elements s_i, s_j, in S, such that $s_i = s_j = k$.
2. If $s_i = s_j = k$ then $j - i = k$.

   A *hooked Skolem sequence* of order n is a sequence $HS = (s_1, s_2, \dots, s_{2n+1})$ of $2n + 1$ integers satisfying conditions 1 and 2 and
3. $s_{2n} = 0$.

   An *extended Skolem sequence* of order n is a sequence $ES = (s_1, s_2, \dots, s_{2n+1})$ of $2n + 1$ integers satisfying conditions 1 and 2 and 3. There is exactly one $s_i = 0$ $(1 \leq i \leq 2n + 1$

The necessary condition for the location of the subscript i of the 0 element in the extended Skolem sequence was determined by Abrham and Kotzig [1] to be $\equiv \frac{1}{2}(n+1)(3n+2) \pmod 2$.

A *near-Skolem sequence* of order n and *defect* $m, n \geq m$, is a sequence $NS = (s_1, s_2, \dots, s_{2n-2})$ of $2n - 2$ integers satisfying the following conditions:

1. For every $k \in \{1, 2, \dots, m-1, m+1, \dots, n\}$ there are exactly two elements s_i, s_j, in NS, such that $s_i = s_j = k$.
2. If $s_i = s_j = k$ with $i < j$, then $j - i = k$.

   A *hooked near-Skolem sequence* of order n and *defect* m is a sequence $HNS = (s_1, s_2, \dots, s_{2n-1})$ of $2n - 1$ integers $s_i \in \{1, 2, \dots, m-1, m+1, \dots, n\}$ satisfying conditions 1 and 2 and the condition:
3. $s_{2n-2} = 0$.

For example, there exist near-Skolem sequences of order 7 and defects 6, 4, and 2:

7,1,1,2,5,2,4,7,3,5,4,3
1,1,6,3,7,5,3,2,6,2,5,7
1,1,5,6,7,3,4,5,3,6,4,7

and there exist hooked near-Skolem sequences of order 7 and defects 7, 5, 3, and 1:

1,1,3,4,5,3,6,4,2,5,2,0,6
2,3,2,6,3,7,4,1,1,6,4,0,7
2,5,2,4,6,7,5,4,1,1,6,0,7
2,5,2,6,4,7,5,3,4,6,3,0,7.

A *generalized Skolem sequence* of order n and multiplicity s is a sequence $GS = (a_1^i, a_2^i, \ldots, a_{ns}^i), i \in \{1,..s\}$, of ns integers from $\{1, 2, \ldots, n\}$ satisfying the following conditions:

1. For every $k \in \{1, 2, \ldots, n\}$ and every $i \in \{1,..s\}$ there exist exactly s elements in GS, $\{a_{j1}^i, a_{j2}^i, \ldots, a_{js}^i\}$ such that $a_{j1}^i = a_{j2}^i = \ldots = a_{js}^i = k$.
2. If $a_{ju}^i = a_{j(u+1)}^i$ then $j_{(u+1)} - j_u = k, (1 \leq u \leq s-1)$.

A *generalized extended Skolem sequence* of order n and multiplicity s is a sequence $GES = (a_1^i, a_2^i, \ldots, a_{ns+d}^i), i \in \{1,..s\}$, of ns integers from $\{1, 2, \ldots, n\}, d \geq 1$, satisfying the following conditions:

1. For every $k \in \{1, 2, \ldots, n\}$ and every $i \in \{1,..s\}$ there exist exactly s elements in GES, $\{a_{j1}^i, a_{j2}^i, \ldots, a_{js}^i\}$ such that $a_{ji}^i = a_{j2}^i = \ldots = a_{js}^i = k$.
2. If $a_{ju}^i = a_{j(u+1)}^i$ then $j_{(u+1)} - j_u = k, (1 \leq u \leq s-1)$.
3. There are exactly d zeros in the sequence (where d is the minimum number of zeros that can exist in the sequence).

For example, the sequence $1, 1, 1, 6, 2, 0, 2, 5, 2, 6, 4, 0, 5, 3, 4, 6, 3, 5, 4, 3$, is a generalized extended Skolem sequence of order 6, $s = 3$ and $d = 2$.

A *generalized near-Skolem sequence* of order n, multiplicity s, and defect m is a sequence $GNS = (a_1^i, a_2^i, \ldots, a_{(n-1)s}^i), i \in \{1,..s\}$, of $(n-1)s$ integers from $\{1, 2, .., m-1, m+1.., n\}$ satisfying the following conditions:

1. For every $k \in \{1, 2, ..m-1, m+1, .., n\}$ and every $i \in \{1,..s\}$ there exist exactly s elements in $GNS, \{a_{j1}^i, a_{j2}^i, \ldots, a_{js}^i\}$ such that $a_{j1}^i = a_{j2}^i = \ldots = a_{js}^i = k$.
2. If $a_{ju}^i = a_{j(u+1)}^i$ then $j_{(u+1)} - j_u = k, (1 \leq u \leq s-1)$.

The *generalized extended near-Skolem sequence* is defined in a manner similar to that of the generalized extended Skolem sequence. For example, the sequence $7, 8, 9, 5, 1, 1, 1, 7, 5, 8, 6, 9, 3, 5, 7, 3, 6, 8, 3, 2, 9, 2, 6, 2$ is a generalized near-Skolem sequence of order 9, multiplicity 3, and defect 4 (the smallest that can be found), and the sequence $4, 1, 1, 1, 4, 5, 6, 2, 4, 2, 5, 2, 6, 0, 0, 5, 0, 0, 6$ is a generalized extended near-Skolem sequence with $n = 6, m = 3$ and $s = 3$.

The following theorem extends Theorem 1 in [18].

Theorem 1 *Let $s = p^e t$, where p is the smallest prime factor in s and e, t are any positive integers.*

(i) *If a generalized Skolem sequence of order n and multiplicity s exists, then n must satisfy one of the congruences*

$$n \equiv 0, 1, \ldots, p-1 (mod\, p^{e+1}). \tag{2.1}$$

(ii) *A necessary condition for a generalized extended Skolem sequence of order n to exist is that n satisfy one of the congruences*

$$n \equiv kp, kp+1, \ldots, (k+1)p-1 (mod\, p^{e+1}) \text{ where } 1 \le k \le p-1.$$

In case (ii), a lower bound of number of zeros is given by $(p-k)(s-1)$.

Proof: First we consider the case where $e = 1$. Let $A = (a_1, a_2, \ldots, a_{ns+d})$ be the generalized sequence.

(i) In this case A is a generalized Skolem sequence, i.e. a perfect sequence, and hence $d = 0$. With $s = pt$, arrange the terms of A in an $nt \times p$ array $B = (b_{i,j})$ according to the rule $b_{i,j} = a_{(i-1)p+j}$ where $1 \le j \le p$ and $1 \le i \le nt$. We note that, if $1 \le b \le n$ and $b \not\equiv 0 \pmod p$, then b appears exactly t times in every column in the array B. Hence, every column of B contains the same number of elements $b \not\equiv 0 \pmod p$. On the other hand, if $1 \le b \le n$ and $b \equiv 0 \pmod p$, then b appears all $s = pt$ times in the same column in the array. Denoting by $[n/p]$, the largest integer $\le n/p$, this means that the number $[n/p]$ of elements $\equiv 0 \pmod p$ must be a multiple of p in the case of a perfect sequence, which implies n satisfies one of the congruency classes in (2.1).

(ii) If the number $[n/p]$ is not a multiple of p, i.e., if $[n/p]$ is congruent $(\text{mod}\, p)$ to $k, 1 \le k \le p-1$, then the array B must have $p-k$ columns each containing s zeros. Since the zeros in B translate into hooks in the sequence A, we see that the special case when these columns occupy the positions $k+1, \ldots, p$ and the entries $b_{nt,k+1}, b_{nt,k+2}, \ldots, b_{nt,p}$ are all zeros, then these last $p-k$ zeros lie outside the sequence A. Hence, we obtain at least $(p-k)(s-1)$ hooks, $1 \le k \le p-1$ (see Fig. 1). This completes the proof for the case $e = 1$.

For the case when $e > 1$ we consider the $ntp^{e-1} \times p$ array $B = (b_{i,j})$ as in (i), i.e., we arrange the terms of A according to the rule $b_{i,j} = a_{(i-1)p+j}$ where $1 \le j \le p$ and $1 \le i \le ntp^{e-1}$.

If $1 \le b \le n$ and $b \not\equiv 0 \pmod p$ then the number $(i-1)p+j$ will force b to be distributed equally among the residue classes of p, i.e. b will appear in every column exactly tp^{e-1} times. If $b \equiv 0 \pmod p$ then b will appear all

$p^e t$ times in a single column. Thus, to have a perfect sequence, $[n/p]$ must be a multiple of p. Note that this process depends on whether $(b, p) = 1$ and not on whether $(b, p^e) = 1$.

In case $[n/p]$ is not a multiple of p, the array B will have $p - k$ columns each containing $p^e t$ zeros. Consequently the lower bound is attained when the entries $b_{ntp(e-1),k+1}, b_{ntp(e-1),k+2}, \cdots, b_{ntp(e-1),p}$ are all zeros. These zeros will all lie outside the sequence; hence, we attain the lower bound $(p - k)(s - 1)$ of zeros (see the example below for $s = 4$). This completes the proof. □

$$\begin{pmatrix}
b_{1,1} & b_{1,2} & \ldots & b_{1,k} & 0 & \ldots & 0 & \ldots & b_{1,p} \\
b_{2,1} & b-2,2 & \ldots & b_{2,k} & b_{2,k+1} & \ldots & 0 & \ldots & b_{2,p} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
b_{nt,1} & b_{nt,2} & \ldots & b_{nt,k} & * & * & \ldots & \ldots & *
\end{pmatrix}$$

Figure 1

(Zeros are scattered between columns $k + 1$ and p and the last row all $b_{nt,k+1} \cdots, b_{nt,p}$ are zeros.)

The following example shows the smallest n for $s = 4$ where the lower bound on the number of zeros is attained and is the minimum: $A = (1, 1, 1, 1, 5, 11, 2, 0, 2, 5, 2, 10, 2, 0, 5, 9, 11, 6, 4, 5, 8, 10, 4, 4, 10, 7, 9, 3, 6, 8, 3, 11, 7, 3, 10, 9, 3, 8, 0, 7)$. The corresponding array B is

$$\begin{pmatrix}
1 & 1 & 5 & 2 & 2 & 2 & 2 & 5 & 11 & 4 & 8 & 4 & 9 & 4 & 8 & 4 & 7 & 3 & 8 & 11 & 3 & 9 & 8 & 7 \\
1 & 1 & 11 & 0 & 5 & 10 & 0 & 9 & 6 & 5 & 10 & 6 & 7 & 11 & 6 & 10 & 9 & 6 & 3 & 7 & 10 & 3 & 0 & *
\end{pmatrix}^T$$

We observe that the *generalized hooked Skolem sequence* is a special case of the generalized extended Skolem sequence. In the generalized hooked Skolem sequence, the minimum number of zeros that can exist in the extended sequence will occupy the bottom right corner of the array B as in Fig 2.

For example, the sequence $1, 1, 1, 2, 6, 2, 3, 2, 5, 3, 6, 4, 3, 5, 0, 4, 6, 0, 5, 4$ is a generalized hooked Skolem sequence of order 6 with $s = 3$ and $d = 2$ and the corresponding array B is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 6 & 2 \\ 3 & 2 & 5 \\ 3 & 6 & 4 \\ 3 & 5 & 0 \\ 4 & 6 & 0 \\ 5 & 4 & * \end{pmatrix}$$

(Note that the last zero lies outside the sequence represented by an aserick.) d extended sequence with the same parameters given above, has the following array B:

$$\begin{pmatrix} 1 & 1 & 1 \\ 6 & 2 & 0 \\ 2 & 5 & 2 \\ 6 & 4 & 0 \\ 5 & 3 & 4 \\ 6 & 3 & 5 \\ 4 & 3 & * \end{pmatrix}$$

In fact, the argument used in the proof of Theorem 1 will also determine the locations of the zeros in the generalized hooked or extended Skolem sequence from the corresponding locations of the zeros in the array B. In the following two corollaries s, p, t, and e are the same as in the statement of Theorem 1.

Corollary 2 *If a generalized extended Skolem sequence $A = (a_1, a_2, \ldots, a_{ns+d})$ of order n and multiplicity $s = tp^e$ exists with the number of zeros $(p-k)(s-1)$, then for the congruences*

$$n \equiv kp, kp+1, \ldots, (k+1)p-1 (mod\ p^{e+1}), 1 \leq k \leq p-1,$$

these zeros can only occur in the locations $a_i, i \equiv k+1, ..p-1, 0$ (mod p).

If the sequence A is hooked, the minimum number of zeros will occupy the right most possible places in the sequence, i.e., in the bottom right corner in the array (see Fig. 2).

$$\begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,k} & b_{1,k+1} & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,k} & b_{2,k+1} & \cdots & b_{2,p} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{nt-s-1,1} & \cdots & \cdots & b_{nt-s-1,k} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{nt,1} & b_{nt,2} & \cdots & b_{nt,k} & * & \cdots & * \end{pmatrix}.$$

Figure 2

(All zeros are in the right lower corner and the last line of zeros lie outside the sequence (indicated by astericks).)

Similar results can be obtained for the generalized near-, hooked near-, and generalized near-Skolem sequences.

Corollary 3 *Let $s = p^e t$. Then*

(i) *If a generalized near-Skolem sequence of order n, defect m and multiplicity s exists, then n and m must satisfy the congruences*

$$\begin{aligned} n &\equiv 0, 1, \ldots, p-1 (mod\, p^{e+1}) \\ m &\not\equiv 0 (mod\, p). \end{aligned}$$

If $m \equiv 0$ $(mod\, p)$ then the sequence is hooked and a lower bound for the number of hooks is $(p-k)(s-1)$.

(ii) *If a generalized hooked or extended near-Skolem sequence of order n and defect m exists, then n and m satisfy the congruences*

$$\begin{aligned} n &\equiv kp, kp+1, \ldots, (k+1)p - 1 (mod\, p^{e+1}), 1 \leq k \leq p-1, \\ m &\not\equiv 0 (mod\, p), \end{aligned}$$

in which case a lower bound for the number of zeros is $(p-k)(s-1)$. If $m \equiv 0$ $(mod\, p)$, then this lower bound on the number of zeros is $(p-k+1)(s-1)$.

Proof: We apply the same matrix argument to the generalized extended near- and near-Skolem sequences. If $m \not\equiv 0$ (mod p), then every column in the matrix will be missing m exactly t times. Thus, every column will maintain equal length.

If $m \equiv 0$ (mod p) then one more column will have at least $(s-1)$ zeros, and this will increase the number of zeros by $s-1$.

It is also an easy corollary to determine the possible locations of zeros for the generalized extended and hooked near-Skolem sequences.

3 Computational Results

In this section, we exhibit computational results for some small orders of the generalized extended, hooked, extended near-, hooked near- and near-Skolem sequences. These results were obtained by an exhaustive computer search, and with a few small exceptions where the sequences degenerate the results are in agreement with the necessary conditions proved. Note that the sequence and its reverse are considered the same.

To obtain the computational results which follow we generalized Marsh's algorithm [10] for constructing Skolem sequences by an exhaustive search.

1. Let M be the set $\{1, 2, \ldots, sn\}$;

2. Select $m_i \in M$;

3. For $i = 1$ to n, do

   Compute $m_i + i, m_i + 2i, \ldots, m_i + si$.
   If (all $m_i + ji, 1 \leq j \leq s$, are in M),
   Then
   $M := M - \{m_i + ji | 1 \leq j \leq s\}$.
   Else,
   No sequence exists;

4. Repeat 2 and 3 until M is exhausted.

3.1 Generalized Hooked Skolem Sequences $s = 3$ and $n \leq 8$

Lemma 4 *There does not exist a generalized hooked Skolem sequence for $n \leq 5$ and $s = 3$.*

Lemma 5 *For $n = 6, 7,$ and 8, $s = 3$ there exist exactly two generalized hooked Skolem sequences.*

n	Sequences
6	1 1 1 2 6 2 3 2 5 3 6 4 3 5 0 4 6 0 5 4
	4 1 1 1 4 5 6 2 4 2 5 2 6 3 0 5 3 0 6 3
7	1 1 1 6 2 7 2 3 2 6 3 5 7 3 4 6 5 0 4 7 0 5 4
	4 1 1 1 4 5 6 7 4 2 5 2 6 2 7 5 3 0 6 3 0 7 3
8	4 1 1 1 4 2 8 2 4 2 7 5 3 6 8 3 5 7 3 6 0 5 8 0 7 6
	5 7 1 1 1 5 8 2 7 2 5 2 6 3 8 7 3 4 6 3 0 4 8 0 6 4

Table 1

Generalized Extended Skolem Sequences $1 \leq n \leq 8, s = 3$

We note that unlike the generalized hooked sequences the generalized extended exist for all small orders.

Lemma 6 *For $n = 3, 4, 5, 6, 7, 8$, there are exactly $2, 7, 18, 9, 17, 38$, respectively, generalized extended Skolem sequences with $s = 3$.*

Here we give an example for each case. The full details will be supplied in a technical report which can be obtained by writing to us.

n	Sequence
3	1 1 1 2 0 2 3 2 0 3 0 0 3
4	1 1 1 2 0 2 4 2 0 3 4 0 3 0 4 3
5	1 1 1 2 0 2 4 2 5 3 4 0 3 5 4 3 0 0 5
6	1 1 1 2 6 2 3 2 5 3 6 4 3 5 0 4 6 0 5 4
7	1 1 1 2 7 2 4 2 0 6 4 7 5 3 4 6 3 5 7 3 0 6 5
8	1 1 1 2 4 2 7 2 4 8 6 0 4 7 5 3 6 8 3 5 7 3 6 0 5 8

Table 2

3.2 Generalized Near-Skolem Sequences $s = 3$ and $n \leq 8$

Lemma 7 *The numbers of generalized near-Skolem or extended near-Skolem for orders $n, 2 \leq n \leq 9$, and all possible defects m are:*

n	m	Number of Sequences	n	m	Number of Sequences
2	1	1	7	4	5
3	1	1	7	5	4
3	2	1	7	6	78
4	1	6	8	1	5
4	2	3	8	2	5
4	3	2	8	3	133
5	1	6	8	4	10
5	2	7	8	5	13
5	3	1	8	6	129
5	4	1	8	7	15
6	1	3	9	1	4
6	2	3	9	2	5
6	3	6	9	3	21
6	4	5	9	4	1
6	5	5	9	5	3
7	1	4	9	6	19
7	2	5	9	7	6
7	3	29	9	8	32

Table 3

n	m	Sequence
2	1	2 0 2 0 2
3	1	2 0 2 3 2 0 3 0 0 3
3	2	1 1 1 3 0 0 3 0 0 3
4	1	2 0 2 4 2 0 3 4 0 3 0 4 3
4	2	1 1 1 3 4 0 3 0 4 3 0 0 4
4	3	1 1 1 2 4 2 0 2 4 0 0 0 4
5	1	2 0 2 4 2 5 3 4 0 3 5 4 3 0 0 5
5	2	1 1 1 3 4 5 3 0 4 3 5 0 4 0 0 5
5	3	5 1 1 1 4 5 0 2 4 2 5 2 4
5	4	5 1 1 1 0 5 3 0 0 3 5 2 3 2 0 2
6	1	2 6 2 3 2 5 3 6 4 3 5 0 4 6 0 5 4
6	2	1 1 1 5 6 0 3 4 5 3 6 4 3 5 0 4 6
6	3	1 1 1 6 2 0 2 5 2 6 4 0 5 0 4 6 0 5 4
6	4	1 1 1 6 2 0 2 5 2 6 3 0 5 3 0 6 3 5
6	5	1 1 1 2 6 2 0 2 3 4 6 3 0 4 3 0 6 4
7	1	2 7 2 4 2 0 6 4 7 5 3 4 6 3 5 7 3 0 6 5
7	2	1 1 1 3 6 7 3 4 5 3 6 4 7 5 0 4 6 0 5 7
7	3	1 1 1 6 2 7 2 0 2 6 4 5 7 0 4 6 5 0 4 7 0 5
7	4	1 1 1 6 7 0 2 5 2 6 2 7 5 3 0 6 3 5 7 3
7	5	6 7 1 1 1 2 6 2 7 2 4 0 6 3 4 7 3 0 4 3
7	6	1 1 1 2 7 2 3 2 0 3 5 7 3 4 0 5 0 4 7 0 5 4
8	1	2 4 2 7 2 4 8 6 0 4 7 5 3 6 8 3 5 7 3 6 0 5 8
8	2	1 1 1 3 8 0 3 6 7 3 4 5 8 6 4 7 5 0 4 6 8 5 7
8	3	1 1 1 2 4 2 8 2 4 6 7 0 4 5 8 6 0 7 5 0 0 6 8 5 7
8	4	1 1 1 8 6 2 7 2 0 2 6 8 5 7 0 3 6 5 3 8 7 3 5
8	5	1 1 1 8 2 4 2 7 2 4 6 8 3 4 7 3 6 0 3 8 0 7 6
8	6	1 1 1 2 7 2 3 2 8 3 5 7 3 4 0 5 8 4 7 0 5 4 0 0 8
8	7	1 1 1 2 8 2 4 2 5 6 4 0 8 5 4 6 3 0 5 3 8 6 3
9	1	2 5 2 9 2 4 5 8 6 4 7 5 9 4 6 8 3 7 0 3 6 9 3 8 7
9	2	1 1 1 7 9 3 6 8 3 5 7 3 6 9 5 8 4 7 6 5 4 0 9 8 4
9	3	1 1 1 2 8 2 9 2 0 5 6 7 8 4 5 9 6 4 7 5 8 4 6 0 9 7
9	4	7 8 9 5 1 1 1 7 5 8 6 9 3 5 7 3 6 8 3 2 9 2 6 2
9	5	9 7 1 1 1 6 8 3 7 9 3 6 4 3 8 7 4 6 9 2 4 2 8 2
9	6	8 1 1 1 9 7 4 2 8 2 4 2 7 9 4 5 8 0 3 7 5 3 9 0 3 5
9	7	1 1 1 9 6 2 8 2 5 2 6 4 9 5 8 4 6 3 5 4 3 9 8 3
9	8	1 1 1 2 9 2 3 2 6 3 7 5 3 9 6 4 5 7 0 4 6 5 9 4 7

Table 4

4 Conclusions and Open Problems

Initial computational results suggest that the necessary conditions for the existence of the generalized Skolem, extended, hooked and near-Skolem sequences are sufficient. Thus it is easy to conjecture that all these sequences exist for $n >$ constant. However, the constructive techniques that were used to prove the sufficiency for $s = 2$ seem to be difficult to implement for $s > 2$. Levine (1968) [8] made a similar conjecture for the perfect Langford sequences with $s = 3$. This conjecture still open.

We hope that some applications for the generalized Skolem, extended, hooked, and near-Skolem sequences will be discovered in the areas of designs, codes, and graph factorizations.

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