A Note on 1 – Tough Hamiltonian Graphs

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ABSTRACT. Let G be a 1 – tough graph of order n. If $| N(S) | \geq (n+ | S | -1)/3$ for every non – empty subset S of vertex set $V(G)$ of G, then G is hamiltonian.

The terminology and notation used in this note are standard and we consider only simple graphs.

Theorem 1 *Let G be a 1 – tough graph of order n. If $| N(S) | \geq (n+ | S | -1)/3$ for every non – empty subset S of vertex set $V(G)$ of G, then G is hamiltonian.*

In order to prove Theorem 1, we will use the following results as our Lemmas.

Lemma 1 [1] *Let G be a 1 – tough graph of order n such that $\delta \geq n/3$. If G is nonhamiltonian and C is a longest cycle in G, then every component of $G - C$ is an isolated vertex.*

Lemma 2 [2] (the Hopping Lemma). *Let $a_1, a_2, \ldots, a_m, a_1$ be the vertices in order round a cycle C in a graph G, where the suffices of a_i are reduced modulo m. Suppose that G contains no cycle of length $m+1$, and no cycle C_1 of length m such thah $G - C_1$ has fewer components than $G - C$. Suppose that a is an isolated vertex of $G - C$. Let $Y_0 := \emptyset$ and, for $j \geq 1$, define*

$$X_j := N(Y_{j-1} \cup \{a\}),$$

$$Y_j := \{a_i \in C : a_{i-1} \in X_j \text{ and } a_{i+1} \in X_j\}.$$

Then, for all $j \geq 1$, $X_j \subseteq C$, and X_j does not contain two consecutive vertices of C.

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Lemma 3 [2] *Let C be a cycle of length m. Let X be a set of vertices of C that contains no two consecutive vertices of C. Let Y be the set of vertices of C whose two neighbors round C are both in X. Then $|Y| \geq 3|X| - m$.*

Proof of Theorem 1: Suppose G is a graph satisfying the conditions in Theorem 1 and it is nonhamiltonian. Let C be a longest cycle in G and no cycle C_1 of length $|C|$ such that $G - C_1$ has fewer components than $G - C$. We first note that $\delta \geq n/3$. So by Lemma 1, each component of $G - C$ is an isolated vertex. Let a be one of the components of $G - C$. Let $Y_0 := \emptyset$ and, for $j \geq 1$, define

$$X_j := N(Y_{j-1} \cup \{a\}),$$

$$Y_j := \{a_i \in C : a_{i-1} \in X_j \text{ and } a_{i+1} \in X_j\}.$$

By Lemmas 2 and 3, $|Y_j| \geq 3|X_j| - |C|$ $(j = 1, 2, \ldots)$. By the hypothesis, $|X_j| \geq (n + |Y_{j-1}|)/3$, $(j = 1, 2, \ldots)$. So we have

$$|Y_j| \geq n + |Y_{j-1}| - |C| \geq |Y_{j-1}| + 1, (j = 1, 2, \ldots).$$

Therefore we have $|Y_j| \geq j$, $(j = 0, 1, \ldots)$. Hence $|Y_{n+1}| \geq n + 1$, a contradiction. So we complete proof of Theorem 1.

References

[1] A. Bigalke and H.A. Jung, Über Hamiltonsche Kreise und unabhängige Ecken in Graphen, *Monatsh. Math.* **88** (1979), 195–210.

[2] D.R. Woodall, The binding number of a graph and its Anderson number, *J. Combinatorial Theory (B)* **15** (1973), 225–255.

[3] D.R. Woodall, A sufficient condition for hamiltonian circuits, *J. Combinatorial Theory (B)* **25** (1978) 184–186.