On Three Conjectures of GRAFFITI

Peter Dankelmann, Henda C. Swart
University of Natal
Durban, South Africa

Ortrud R. Oellermann
Brandon University
Brandon, MB Canada

ABSTRACT. In this paper we consider three conjectures of the computer program GRAFFITI. Moreover, we prove that every connected graph with minimum degree δ and diameter dm contains a matching of size at least $\delta(dm+1)/6$. This inequality improves one of the conjectures under the additional assumption that $\delta \geq 6$.

Introduction

The computer program GRAFFITI was developed by S. Fajtlowicz as an attempt to discover hidden relations between graph invariants. Many conjectures of GRAFFITI led in fact to relations between parameters that seemed to have no obvious inter-dependence. The most famous example is the inequality

$$\mu(G) \leq \alpha(G),$$

where $\mu(G)$ is the average distance and $\alpha(G)$ is the independence number of the graph G. This GRAFFITI conjecture was published in 1986 [5] and was proved two years later by Chung [2].

The related GRAFFITI conjecture $\text{rad}(G) \leq \alpha(G)$, where $\text{rad}(G)$ denotes the radius of G, later turned out to be already known (see [4]), but various proofs of this conjecture ([7], [9)]) gave more insight into the relationship between the radius and the independence number. Interesting relations between $R(G)$, the sum of the inverse degrees of the vertices in G, and μ(G) were given in [3], though the original GRAFFITI conjecture $\mu(G) \leq R(G)$ was disproved.

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In this paper we consider three conjectures of GRAFFITI [6] that relate, among others, the above-mentioned distance parameters. We disprove the conjecture [6]

$$\mathrm{rad}(G) \leq \mu(G) + R(G)$$

and prove a conjecture by Shearer (see [11]) stating that there are triangle free graphs G with almost all distances odd. This conjecture goes back to the GRAFFITI conjecture 111 (see [8]) that in each triangle free graph at least half of the distances are even, which turned out to be false [8].

Finally we turn to the conjecture 61 (see [6])

$$D'(G) \leq \beta(G),$$

where $\beta(G)$ is the matching number of G and $D'(G)$ is the entry that appears most often in the distance matrix of G. We will prove that with the additional assumption that the minimum degree $\delta(G)$ is at least 6, the latter conjecture is true and that even the considerably stronger inequality

$$\frac{(dm(G)+1)\delta(G)}{6} \leq \beta(G)$$

holds for not necessarily regular graphs. Here $dm(G)$ denotes the diameter of G. For notation not defined here see [1].

Results

We first disprove the following conjecture, stated as Conjecture 25 in [6]. The counter example is based on the idea that attaching a large complete graph to a graph neither changes the radius nor R significantly, but μ approaches 1. A similar idea has been used by Plesnik [10] to prove that, apart from the restriction $1 \leq \mu \leq dm(G)$, the average distance is independent of the radius and the diameter.

Conjecture 25: [6] *If G is a connected graph then*

$$\mathrm{rad}(G) \leq \mu(G) + R(G).$$

Definition 1: For natural numbers r, k, ℓ, where $r \geq 2$, let $G_{rk,\ell}$ be the graph obtained from the cycle with $2r$ vertices as follows. Replace one vertex of the cycle by a complete graph of order k, replace each of the remaining vertices by a complete graph of order ℓ, and join vertices of different complete graphs if their original vertices are adjacent in the cycle.

The radius of the graph $G_{r,k,\ell}$ is obviously r. Now let r be fixed and let k and ℓ tend to infinity such that $\ell = o(k)$. Then we have

$$R(G_{r,k,\ell}) = \sum_{v \in V} d(v)^{-1} = \frac{k+2\ell}{k+2\ell-1} + \frac{(2r-3)\ell}{3\ell-1} = \frac{2}{3}r + o(1).$$

Since $G_{r,k,\ell}$ contains a clique of order k, at least $k(k-1)$ ordered pairs of vertices have distance one. The remaining distances are at most $r = dm(G_{r,k,\ell})$. Hence

$$\mu(G_{r,k,\ell}) \le \frac{k(k-1) + r[(k+(2r-1)\ell)(k+(2r-1)\ell-1) - k(k-1)]}{(k+(2r-1)\ell)(k+(2r-1)\ell-1)}$$
$$= 1 + o(1).$$

Hence $G_{r,k,\ell}$ is a counter example to Conjecture 25. Moreover it indicates that there is no upper bound of the form $c\mu(G) + R(G)$ for the radius, where c is a constant.

Several GRAFFITI conjectures involve the parameter Even: Let G be a connected graph of order p and v be a vertex of G; then $E(v)$ denotes the number of vertices at an even distance from v and Even(G) denotes an $p \times 1$ vector whose components are $E(v)$ for $v \in V(G)$. The mean of Even(G), i.e. $\frac{1}{p}\sum_v E(v)$, is denoted by $ME(G)$. It gives the proportion of the number of even distances to the total number of distances in G. In [8] GRAFFITI conjectured that in every triangle free graph at least half of the distances are even.

Conjecture 101: *Let G be a connected triangle free graph of order p and let $ME(G)$ be the mean of* Even(G), *then*

$$ME(G) \ge \frac{p}{2}.$$

That this conjecture had been disproved by Staton [12] and independently by Shearer [11] was reported in [8] in which a new conjecture by Shearer was mentioned.

Shearer's conjecture: [11]. *There exist connected triangle free graphs G of order p for which $ME(G) = o(p)$.*

That Shearer's conjecture is true is shown by consideration of the following class of graphs G_i, defined inductively as follows: Let k be a positive integer and let G_1 be the graph obtained from the union of two disjoint 5-cycles C_1': $a_0'b_0'd_0'a_1c_0'a_0'$ and C_1'': $a_0''b_0''d_0''b_1c_0''a_0''$ by the introduction of $4k$ new vertices, $w_1, w_2, \ldots, w_{4k}$ as well as the edges $a_0'w_i$ $(i = 1, \ldots, k)$, $b_0'w_i$ $(i = k+1, \ldots, 2k)$, $a_0''w_i$ $(i = 2k+1, \ldots, 3k)$, $b_0''w_i$ $(i = 3k+1, \ldots, 4k)$, and a_1b_1.

For $i \in N$, when G_i has been defined, G_{i+1} is obtained from two disjoint copies of G_i, viz. G_i' and G_i'', in which the vertices of degree three are denoted by a_i', b_i' and a_i'' and b_i'', respectively, by the introduction of six new vertices c_i', c_i'', d_i', d_i'', a_{i+1}, b_{i+1} and nine new edges $b_i'd_i'$, $d_i'a_{i+1}$, $a_{i+1}c_i'$, $c_i'a_i'$, $b_i''d_i''$, $d_i''b_{i+1}$, $b_{i+1}c_i''$, $c_i''a_i''$ and $a_{i+1}b_{i+1}$. (We note that $a_i'b_i'd_i'a_{i+1}c_i'a_i'$ and $a_i''b_i''d_i''b_{i+1}c_i''a_i''$ are induced 5-cycles of G_{i+1}, joined by the edge $a_{i+1}b_{i+1}$.)

Letting p_i denote the order of G_i, we note that $p_{i+1} = 2p_i+6$, $p_1 = 4k+10$ and thus $p_i = (2k+8)2^i - 6$. Let $X_i \subset V(G_i)$ be the set of the $k2^{i+1}$ end vertices of G_i. Each vertex $v \in X_i$ has k vertices having even distance to v. Denote for subsets $A, B \subset V(G_i)$ the number of ordered pairs of vertices (a, b) with $a \in A$ and $b \in B$ by Even(A, B). Then we have for fixed i and $k \to \infty$

$$\begin{aligned}\frac{ME(G_i)}{p_i} &= \frac{1}{p_i^2}(\text{Even}(X_i, X_i) + 2\text{Even}(X_i, V(G_i) - X_i) + \text{Even}(V(G_i) \\ &\quad -X_i, V(G_I) - X_i) \\ &\leq \frac{1}{p_i^2}(k|X_i| + 2|X_i||V((G_i) - X_i| + |V(G_i) - X_i||V(G_i) - X_i|) \\ &= \frac{1}{((2k+8)2^i - 6)^2}(k^2 2^{i+1} + k2^{i+2}(2^{i+3} - 6) + (2^{i+3} - 6)^2) \\ &= \frac{1}{2^{i+1}} + o(k).\end{aligned}$$

For any $\varepsilon > 0$, the integers k and i may be chosen such that $ME(G_i) \leq \varepsilon p_i$, as required.

We remark that a slight modification of the above construction in which the 5-cycles are replaced by larger odd cycles yields not only triangle free graphs but graphs of arbitrarily large girth in which almost all distances are odd.

We now turn to conjecture 61 [6], relating $\beta(G)$, the matching number of G, and $D'(G)$, the distance that appears most often in the distance matrix of G.

Conjecture 61: *Let G be a regular, connected graph with matching number $\beta(G)$, then*

$$D'(G) \leq \beta(G).$$

Fajtlowicz reports in [8] that Saks, Seymour, and Shearer disproved this conjecture. They also proved that under the additional assumption that the maximum degree of G is at least 10 the inequality of the conjecture holds. The following theorem shows that under the additional assumption $\delta(G) \geq 6$ not only Conjecture 61 but even a considerably stronger statement holds.

Theorem 1. *If G is a connected graph with minimum degree δ, diameter dm and matching number β then*

$$\beta \geq \frac{(dm+1)\delta}{6}.$$

Proof: Let $a, b \in V(G)$ be a pair of diametrical vertices, i.e. $d(a, b) = dm$.

For integers i, j we define

$$L_{i,j} = \{v \in V(G) \mid i \leq d(a,v) \leq j\}$$

and for $L_{i,i}$ we will write L_i.

Now let $M \subset E(G)$ be a maximal matching such that

$$\sum_{v \in V - V(M)} d(a,v) \text{ is maximal,} \tag{*}$$

where $V(M)$ denotes the set of vertices incident with an edge of M.

We will show that for each i with $1 \leq i \leq dm$ at least one of the following statements holds:

$$|L_{i-1,i+1} \cap V(M)| \geq \delta. \tag{1}$$

$$|L_{i-4,i+1} \cap V(M)| \geq 2\delta. \tag{2}$$

Suppose that (1) does not hold. Then we have

Claim 1: $L_i \subset V(M)$.

Suppose there exists a vertex $v \in L_i$ not incident with M. Since all the neighbours of v are incident with M we have

$$|L_{i-1,i+1} \cap V(M)| \geq |N(v) \cap V(M)| = d(v) \geq \delta.$$

Thus (1) holds, contradicting our assumption.

Claim 2: $L_{i+1} \subset V(M)$.

Suppose to the contrary that there exists a vertex $v \in L_{i+1} \cap V(M)$. We show that

$$|(L_{i+1} - N(v)) \cap V(M)| \geq |L_{i+2} \cap V(v)|. \tag{3}$$

Since (1) does not hold $L_{i+2} \cap N(v)$ is not empty. Let $w \in L_{i+2} \cap N(v)$. Since $v \notin V(M)$, there exists a vertex w' with $ww' \in M$. Then w' must be in L_{i+1} since otherwise $w' \in L_{i+2,i+3}$ and $M' = M - ww' + vw$ is a matching with $V(M') = V(M) - w' + v$ contradicting (*). Moreover w' cannot be a neighbour of v since otherwise $M' = M - ww' + vw'$ would contradict (*). Hence for each $w \in L_{i+2} \cap N(v)$ there exists a $w' \in L_{i+1} - N(v)$ with $ww' \in M$, implying (3).

From (3) and $N(v) \subset V(M)$ we have

$$\begin{aligned}
|L_{i-1,i+1} \cap V(M)| &\geq |L_{i,i+1} \cap N(v)| + |(L_{i+1} - N(v)) \cap V(M)| \\
&\geq |L_{i,i+1} \cap N(v)| + |L_{i+2} \cap N(v)| \\
&= d(v)
\end{aligned}$$

and thus (1). This contradiction proves claim 2.

Now let $v \in L_i$ and $w \in V$ with $vw \in M$. Since (1) does not hold there exists a vertex $u \in N(v)$ not incident with M. Claims 1 and 2 yield $u \in L_{i-1}$. The vertex w is in L_{i-1} since otherwise the matching $M' = M - vw + uv$ is a matching with $V(M') = V(M) - w + u$ contradicting (*).

A similar argument yields $N(w) \subset V(M)$.

We show that for each $xy \in M$ holds

$$|N(u) \cap \{x, y\}| + |N(w) \cap \{x, y\}| \leq 2. \tag{4}$$

Suppose there exists an edge $xy \in M$ with $|N(u) \cap \{x, y\}| + |N(w) \cap \{x, y\}| \geq 3$ and thus, say, $ux, wy \in E(G)$. The matching $M' = M - vw - xy + ux + yw$ has $V(M') = V(M) - w + u$ contradicting (*). Thus (4) holds.

Summation over all $xy \in M$ with $\{x, y\} \subset L_{i-3,i+1}$ yields now

$$\begin{aligned} d(u) + d(w) &= |N(u) \cap V(M)| + |N(w) \cap V(M)| \\ &\leq 2|\{xy \in M \mid x, y \in L_{i-3,i+1}\}| \\ &\leq |V(M) \cap L_{i-3,i+1}|, \end{aligned}$$

implying (2). Hence at least one of the statements (1) or (2) holds.

Since claims 1 and 2 hold also for $i = 0$, we have $|L_{0,1} \cap V(M)| \geq \delta$. Using (1) or (2), respectively, it is easily proved by induction on k that for $3k \leq dm$

$$|V(M) \cap L_{0,3k+1}| \geq (k+1)\delta.$$

Since we can choose $k \geq (dm-2)/3$, this completes the proof of the theorem. □

The bound given in Theorem 1 is asymptotically best possible, as shown by the graph

$$G = K_1 + K_\delta + (K_1 + K_1 + K_{\delta-1})^k + K_1 + K_1 + K_\delta + K_1, k \geq 1.$$

where $K_r + K_s + K_t + \ldots$ is the graph obtained from the union of $K_r, K_s, K_t, \ldots$ by joining every vertex of K_r to every vertex of K_s, every vertex of K_s to every vertex of K_t, and so on. The notation $(K_r + K_s + K_t)^k$ is shorthand for $K_r + K_s + K_t + K_r + K_s + K_t + \ldots$, where K_r, K_s, and K_t are repeated k times each. The matching number of G is $\beta(G) = (dm+1)(\delta+1)/6$.

We remark that the inequality of Theorem 1 can be strengthened to

$$\beta(G) \geq \frac{(dm+2)\delta}{5} \tag{+}$$

if G is bipartite.

To see this let a, b, $L_{i,j}$, and L_i be as above and let $A \subset V(G)$ be a minimum covering set. For each i with $0 \leq i \leq dm$ we have $|A \cap L_{i-2,i+2}| \geq$

δ since either $L_{i-1,i+1} \subset A$ and thus $|A \cap L_{i-1,i+1}| \geq \delta$ or there exists a vertex $v \in L_{i-1,i+1} - A$ all whose neighbours are in A, again implying that $|A \cap L_{i-1,i+1}| \geq \delta$. Using König's Theorem, we derive (+) as above.

Inequality (+) is almost best possible as the following graph G indicates.

$$G = K_1 + \delta K_1 + (\delta - 1)K_1 + (K_1 + K_1 + (\delta - 1)K_1 + (\delta - 1)K_1)^k$$
$$+ K_1 + K_1 + (\delta - 1)K_1 + \delta K_1 + K_1,$$

where the "+" and the brackets are to be read as above. For even k the graph G has matching number $\beta(G) = (dm + 3)(2\delta + 1)/10$.

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