# On Circuit Polynomials and Determinants of Matrices

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ABSTRACT. It is shown that the determinant of the variable adjacency matrix, and hence the determinant of the adjacency matrix of a graph, are circuit polynomials. From this, it is deduced that determinants of symmetric matrices are indeed circuit polynomials of associated graphs. The results are then extended to general matrices.

## 1 Introduction

The graphs considered here are all finite, Let $G$ be such a graph. A **circuit** or **cycle cover** of $G$, is a spanning subgraph of $G$, whose components are all cycles. We take a cycle with one node to be a component node (and sometimes a loop). A cycle with two nodes is an edge. These cycles are called trivial. Cycles with more than two nodes are called **proper**. With each cycle $\alpha$ in $G$ let us associate an indeterminate or weight $w_\alpha$ and with each cycle cover $S$ in $G$, the weight

$$w(S) = \prod w_\alpha,$$

where the product is taken over all the components of $S$. Then the **circuit polynomial** of $G$ is

$$C(G; \underline{w}) = \sum w(S),$$

where $\underline{w}$ is a vector of the weights assigned to the cycles, and the summation is taken over all the cycle covers in $G$. We refer the reader to Farrell [3] for the basic results on circuit polynomials.

Let $G$ be a graph without multiple edges, but possibly with loops. Let the node set of $G$ be $\{v_1, v_2, \ldots, v_p\}$ and let the edges of $G$ be labelled

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$x_1, x_2, \ldots, x_q$. Then the **variable adjacency matrix** of $G$, denoted by $A(G;\underline{x})$, is the $p \times p$ symmetric matrix defined by $A(G;\underline{x}) = (a_{ij})$, where

$$a_{ij} = \begin{cases} x_k & \text{if } v_i \text{ and } v_j \text{ are joined by the edge labelled } x_k \\ 0 & \text{otherwise} \end{cases}$$

If $\underline{x} = (x, x, x, \ldots)$, then $A(G;x)$ will be called the **simple variable adjacency matrix** of $G$. Notice that if we put $x_k = 1$, for all $k$, then $A(G;\underline{x})$ becomes the usual adjacency matrix $A(G)$. The matrix $A(G;x)$ is also defined in Harary [5]. The reader may also wish to consult Clarke [2] for another application of $A(G;\underline{x})$.

First of all, we will establish a relation between $C(G;\underline{w})$ and $A(G;\underline{x})$. From this, we will deduce the determinant of a symmetric matrix as the circuit polynomial of an associated graph. The result will then be extended to general matrices.

We will assume that the graph $G$ contains $p$ nodes and $q$ edges, unless otherwise specified. We will normally assign a weight $w_n$ to a cycle with $n$ nodes. Therefore $\underline{w}$ will be of the form $(w_1, w_2, w_3, \ldots, w_p)$. This form of $\underline{w}$ will be assumed, unless otherwise specified. The cycle with $r$ nodes will be denoted by $Z_r$. We denote the determinant of the matrix $M$ by $|M|$.

## 2 The Determinants of the Simple Variable Adjacency Matrix

First of all, we define a **circuital graph** to be a graph with components that are either proper cycles or edges. Let $G$ be a loopless graph with node set $\{v_1, v_2, \ldots, v_p\}$ and with its $q$ edges labelled $x_k$ $(k = 1, 2, \ldots, q)$. Let $S$ be a circuital subgraph of $G$ such that the edges $x_{k_i}$ are in proper cycles and the edges $x_{k_j}$ are components. We define the function $\sigma$ by

$$\sigma(S;\underline{x}) = (-1)^e 2^c \prod x_{k_i} \prod x_{k_j}^2, \tag{1}$$

where $e$ is the number of even cycles and $c$ is the number of proper cycles in $S$. For a non-circuital graph $G$, we define $\sigma(G;x)$ as 0, and for the null graph $\emptyset$, we define $\sigma(\emptyset;\underline{x})$ as 1. Both the circuital graph and the function $\sigma$ were defined by Harary (See [5]). The following lemma is taken from [5].

**Lemma 1.** *Let $S_i$ $(1, 2, \ldots, m)$, be spanning circuital subgraphs of $G$ and let $A(G;\underline{x})$ be the variable adjacency matrix of $G$, then*

$$|A(G;\underline{x})| = \sum_{i=1}^{m} \sigma(S_i;\underline{x}).$$

This lemma together with Equation (1) yields

$$|A(G;\underline{x})| = \sum_{S_i} (-1)^e 2^c \prod x_{k_i} \prod x_{k_j}^2. \tag{2}$$

Let us put $\underline{x} = (x, x, x, \dots, x)$. Then we get the determinant

$$|A(G;\underline{x})| = \sum_{S_i} (-1)^c 2^c \prod x \prod x^2. \qquad (3)$$

where the first product is taken over all the proper cycles and the second, over all the isolated edges in $S_i$. Let $n_k$ be the number of cycles in $S_i$ with $k$ nodes. Let $a$ be the number of even proper cycles and $b$ the number of odd proper cycles. Then $c = a + b$. Equation (3) yields

$$\begin{aligned} |A(G;\underline{x})| &= \sum_{S_i} (-1)^{n_2} (1)^a 2^a 2^b \prod_{k=2} (x^{2k})^{n_{2k}} (x^{2k-1})^{n_{2k-1}} (x^2)^{n_2} \\ &= \sum_{S_i} (-x^2)^{n_2} \prod_{k=2} (2x^{2k})^{n_{2k}} \prod_{k=2} (2x^{2k-1})^{n_{2k-1}} \qquad (4) \end{aligned}$$

Now, every circuital subgraph of $G$ (including the empty graph) can be uniquely extended to a circuit cover of $G$ by adding isolated nodes. Conversely every cover of $G$ can be uniquely reduced to a circuital subgraph of $G$ by removing its isolated nodes. Hence the circuital subgraphs of $G$ are equinumerous with the circuits covers of $G$. We can therefore extend the summation in Equation (4) to all the circuital subgraphs of $G$ (instead of only the spanning ones) and therefore to all the circuit covers of $G$, if we associate 0 with isolated nodes. Thus we have the following theorem.

**Theorem 1.** *Let $G$ be a loopless graph with variable adjacency matrix $A(G;\underline{x})$. Then*

$$|A(G;x)| = C(G;(0, -x^2, 2x^3, -2x^4, \dots))$$

*i.e. the determinant of the simple variable adjacency matrix of $G$ obtained from $C(G;\underline{w})$ by putting*

$$w_1 = 0, w_2 = -x^2 \text{ and } w_r = (-1)^{r+1} 2x^r, \text{ for } r > 2.$$

By putting $x = 1$ in the above theorem, we obtain the following results for the determinant of the adjacency matrix of a graph.

**Corollary 1.1.**

$$|A(G)| = C(G;(0, -1, 2, -2, 2, -2, \dots)).$$

This corollary is also given in Farrell and Grell [4]. In [4] it was derived from a result which established a connection between $C(G;\underline{w})$ and a polynomial due to Clarke (See[2]).

We can extend our results to graphs which contain loops. In this application, we take a loop to be a cycle with one node in a "circuital graph". We can extend the definition of $A(G;\underline{x})$, by putting $a_{ii} = z_i$, if there is a loop at node $v_i$. However, in order to maintain consistency in assigning weights, we will have to assume that there is a loop on every node of $G$. Lemma 1 then becomes

$$|A(G;\underline{x},\underline{z})| = \sum_{i=1}^{m} \sigma(S_i;\underline{x},\underline{z}).$$

Theorem 1 and Corollary 1.1 lead to the following results.

**Theorem 2.** *Let $G$ be a graph with loops at each node. Then*

$$|A(G;x,z)| = C(G;(z,-x^2,2x^3,-2x^4,\dots)).$$

**Corollary 2.1.**

$$|A(G)| = C(G;1,-1,2-2,2,\dots)).$$

## 3 Determinants of Symmetric Matrices

Let $M = (m_{ij})$ be a $p \times p$ symmetric matrix, defined by

$$m_{ij} = \begin{cases} x \text{ or } 0 & \text{for } i \neq j \\ z & \text{for } i = j \end{cases}.$$

Let us associate with $M$, a graph $G_M$ defined as follows. Label the nodes of $G_M$ as $v_i$ $(i = 1,2,\dots,p)$. Nodes $v_i$ and $v_j$ are joined by an edge if and only if $m_{ij}$. Put a loop at each node of $G_M$. Then clearly

$$M = A(G_M;x,z),$$

where edge $v_iv_j$ is labelled $x$ and each loop is labelled $z$. Hence from Theorem 2, we get the following result.

**Theorem 3.** *Let $M = (m_{ij}$ be a $p \times p$ symmetric matrix defined by $m_{ij} = x$, or 0, for $i \neq j$ and $m_{ii} = z$. Then*

$$|M| = C(G_M;(z,-x^2,2x^3,-2x^4,\dots).$$

Theorem 3 shows that the determinants of certain restricted kinds of symmetric matrices can be obtained from the circuit polynomial of an associated graph.

In order to obtain a connection with the determinants of general symmetric matrices we must keep our analysis as general as possible. Equation

(2) may be written as

$$|A(G;\underline{x})| = \sum_{S_i} (-1)^{q_i} \prod (x_{k_j})^2 \left[(-1)^a 2^a \prod (x_{k_i})\right] \left[2^b \prod (x_{k_i})\right]$$

$$= \sum_{S_i} \prod_{j=1}^{q_i} (-x_{k_j}^2) \prod_{i=1}^{a} (-2 \prod_{Z_{e_i}} x_{k_i}) \prod_{j=1}^{b} (2 \prod_{Z_{o_j}} x_{k_j}),$$

where $q_i$ is the number of component edges in $S_i$ and $a$ and $b$ are the numbers of even and odd proper cycles respectively, in $S_i$. The third and fifth products are taken over the even and odd cycles respectively; i.e. $e_i$ and $o_i$ are even and odd numbers respectively. This is a general form of Equation(4).

Following a similar extension which lead to Theorem 2 we can add to each node $v_i$ a loop labelled $z_i$. The effect of this will be to put $a_{ii} = z_i$ in the variable adjacency matrix. Thus we will have

$$|A(G;\underline{x},\underline{z})| = \sum_{S_i} \prod_{j=1}^{p_i} Z_i \prod_{j=1}^{q_i} (-x_{k_j}^2) \prod_{i=1}^{a} (-2 \prod_{Z_{e_i}} x_{k_i}) \prod_{j=1}^{b} (2 \prod_{Z_{o_j}} x_{k_j})$$

where $p_i$ is the number of loops in $S_i$. Here we allow loops to be components of circuital subgraphs. Our discussions lead to the following theorem, which is a generalization of Theorems 1 and 2.

**Theorem 4.**

$$|A(G;\underline{x},\underline{z})| = C(G;\underline{w}),$$

*where the weights are assigned as follows. A loop at node $v_i$ labelled $z_i$ will have a weight $w_{v_i} = z_i$. For the edge $e_j$ labelled $x_{k_j}$, $w_{e_j} = -x_{k_j}^2$. The $r$-cycle $Z_r$ containing edges labelled $x_{k_1}, x_{k_2}, \ldots, x_{k_r}$, will receive weight*

$$w(Z_r) = (-1)^{r+1} 2 \prod_{i=1}^{r} x_{k_i}, \text{ for } r > 2.$$

Theorem 4 provides us with the necessary tools for attacking the general symmetric matrix. Let $M = (m_{i,j})$ be a symmetric matrix of order $p$. Let us associate with $M$, the graph $G_M$, defined as follows: $G_M$ will have $p$ nodes labelled $v_i$ $(i = 1, 2, \ldots, p)$ with a loop at each node. Nodes $v_i$ and $v_j$ will be joined by an edge if and only if $m_{ij} \neq 0$. Finally, label the edge $v_i v_j$ with $m_{ij}$ and the loop at $v_i$, with $m_{ii}$. Then we get $M = A(G_M;\underline{m})$, where $\underline{m} = (m_{11}, m_{12}, \ldots)$.

The following theorem is immediate.

**Theorem 5.** *Let $M = (m_{ij})$ be a $p \times p$ symmetric matrix. Then $|M| = C(G_M, \underline{x})$, where $w_{v_i} = m_{i,i}$, $w_{v_i v_i} = -(m_{i,j}^2)$ and $w(Z_r) = (-1)^{r+1} 2 \prod_{k=1}^{r} m_{i_k j_k}$, for $r > 2$, and the product is taken over all the edges in $Z_r$.*

## 4 Determinants of General Matrices

It is not difficult to see that in order to extend our results to general $p \times p$ matrices, the graph $G_M$ must be directed (and will be denoted by $\overline{G}_M$). Since $a_{ij}$ is not necessarily equal to $a_{ji}$, different weights must be assigned to the edges $v_iv_j$ and $v_jv_i$. Notice that if $a_{ij} = a_{ji}$, then the two directed edges joining nodes $v_i$ and $v_j$ can be replaced by a single undirected edge. If this is true for all the edges in the directed graph $\overline{G}_M$, then we can use the undirected graph $G_M$. However, corresponding to each proper cycle $Z_r$ in $G_M$ there will be two proper directed cycles $\overline{Z}_r$ in $\overline{G}_M$. We can compensate for this by multiplying the weight assigned to $Z_r$ by 2. Corresponding to each loop in $G_M$ will be one loop in $\overline{G}_M$. Therefore the weights of all loops will be equal. Corresponding to each edge component in $G_M$ there will be one directed 2-cycle in $\overline{G}_M$ with the square of the weight of an edge. Therefore edge components in $G_M$ should be weighted $-x_k^2$, where $x_k$ is the label on the directed edges in $\overline{G}_M$. Hence the assignment of weights to the covers of $G_M$ will be identical with the assignment defined in Theorem 5.

The more general form of Theorem 5 can now be easily extrapolated from the above discussion. In the following result, $\overline{G}_M$ is the directed graph constructed as follows. $\overline{G}_M$ has $p$ nodes labelled $v_i$ $(i = 1, 2, \ldots, p)$. Nodes $v_i$ and $v_j$ are joined by an edge $v_iv_j$ if and only if $m_{ij} \neq 0$. This edge will then be labelled $m_{ij}$. $\overline{G}_M$ will have a directed loop at each node $v_i$, with a label $m_{ii}$.

**Theorem 6.** *Let $M = (m_{ij})$ be a $p \times p$ matrix. Then $|M| = C(\overline{G}_M; \underline{w})$, where $w_{v_i} = m_{ii}$, and $\overline{Z}_r = (-1)^{r+1} \prod_{k=1}^{r} m_{i_k, j_k}$, for $r > 1$,where the product is taken over all the edges in the directed cycle $\overline{Z}_r$.*

A formal proof of Theorem 6 can be established by (i) extending the definitions of $A(G; \underline{x})$ and the function $\sigma$ to include directed graphs (ii) establishing the directed graph version of Lemma 1 and then (iii) following a proof similar to that of Theorem 1.

The following corollary is immediate for the theorem.

**Corollary 6.1.** *Let the weight $w_{ij} = m_{ij}$ be associated with the edge $(i, j)$ of the digraph $\overline{G}_M$. Then,*

$$|M| = \sum (-1)^{\mu(F)} \omega(F),$$

*where the sum is taken over all the cycle covers $F$ and $\mu(F)$ is the number of even cycles in $F$.*

This corollary is a well known result given in Berman and Fryer [1] (Theorem 3.8).

## 5 Some Remarks

We can use the ideas developed above, in order to establish some familiar properties of determinants. For example,

**Property 1.**

Let $D = \text{diag}\ (d_1, d_2, \ldots, d_p)$. Then $|D| = d_1, d_2, \ldots, d_p$.

**Property 2.**

Let $B$ be the matrix obtained from $A$ by multiplying row $i$ by a scalar $k$. Then $|B| = k|A|$.

Other properties of determinants of matrices can be established by using circuit polynomials. It will be worthwhile to investigate which operations on $G_M$ leave $C(G_M; \underline{w})$ invariant. Such operations when transformed to $A(G_M; \underline{x})$ will also leave $|A(G_M; \underline{x})|$ invariant.

The results given here can be regarded, in some ways, to be a continuation of the ideas developed in [4]. One of the specific comments in [4] (immediately after Corollary 1.1) was the following ; "We suspect that determinants of general matrices could be evaluated by suitable weighted circuit polynomials." In this paper we have shown that our suspicion was correct.

The establishment of a connection between the circuit polynomial and determinants of matrices is an important one, if only because of the marriage of Graph Theory and Linear Algebra. Some other interesting applications of Combinatorics to Linear Algebra can be found in Stanton and White [6] and Zeilberger [7].

## References


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