Real Domination in Graphs

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ABSTRACT. A function $f: V \to \mathbb{R}$ is defined to be an $\mathbb{R}$-dominating function of graph $G = (V, E)$ if the sum of the function values over any closed neighbourhood is at least 1. That is, for every $v \in V$, $f(N(v) \cup \{v\}) \geq 1$. The $\mathbb{R}$-domination number $\gamma_{\mathbb{R}}(G)$ of G is defined to be the infimum of $f(V)$ taken over all $\mathbb{R}$-dominating functions f of G. In this paper, we investigate necessary and sufficient conditions for $\gamma_{\mathbb{R}}(G) = \gamma(G)$ where $\gamma(G)$ is the standard domination number.

1 Introduction

All our graphs are finite and without loops or multiple edges. A *block* of a graph is a maximal nontrivial connected subgraph of the graph with no cut-vertices. A *complete block graph* is a connected graph in which every block is complete. A *trivial block* is isomorphic to K_2. If each block is a trivial block, then the complete block graph is a tree. A vertex that belongs to exactly one block we will call a *special* vertex. Two blocks are said to be adjacent if they have a common cut-vertex. A graph is *chordal* if it contains no cycle of length greater than three as an induced subgraph. A *strongly chordal graph* is a chordal graph that contains no induced trampoline, where

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a *trampoline* consists of a $2n$-cycle $v_1, v_2, \dots, v_{2n}, v_1$ in which the vertices v_{2i} of even subscript form a complete graph on $n \geq 2$ vertices.

For a graph $G = (V, E)$ with vertex set V and edge set E, the open neighbourhood of $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighbourhood of v is $N[v] = \{v\} \cup N(v)$. For a set S of vertices, we define the open neighbourhood $N(S) = \bigcup_{v \in S} N(v)$, and the closed neighbourhood $N[S] = N(S) \bigcup S$. A *dominating set* $S \subseteq V$ for a graph $G = (V, E)$ is such that each $v \in V$ is either in S or adjacent with a vertex of S. (That is, $N[S] = V$.) The *domination number* of G, $\gamma(G)$, equals the minimum cardinality of a dominating set. The domination number has received considerable attention in the literature. Haynes, Hedetniemi, and Slater [15] have written a book on domination theory.

For a real-valued function $f: V \to \mathbb{R}$ the *weight* of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$ we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. For a vertex v in V, we denote $f(N[v])$ by $f[v]$ for notational convenience. Let $f: V \to \{0, 1\}$ be a function which assigns to each vertex of a graph an element of the set $\{0, 1\}$. We say f is a *dominating function* if for every $v \in V$, $f[v] \geq 1$. Then the domination number of a graph G can be defined as $\gamma(G) = \min\{w(f) \mid f \text{ is a dominating function on } G\}$.

Several authors have suggested changing the allowable weights. Well-known is fractional domination where the weights are allowed to be in the range [0,1]. Reporting on results in [17], at the Eighteenth Southeastern International Conference on Combinatorics, Graph Theory and Computing S.T. Hedetniemi formally defined fractional domination as follows. For a graph $G = (V, E)$, a function $f: V \to [0, 1]$ is called a *fractional dominating function* of G if $f[v] \geq 1$ for each $v \in V$. The *fractional domination number* of G is given by $\gamma_f(G) = \min\{w(f) \mid f \text{ is a fractional dominating function for } G\}$. For r-regular graphs on n vertices the fractional domination number is $n/(r+1)$, which is attained by placing weights $1/(r+1)$ on each vertex. This fractional version of domination has been studied in [4, 5, 10, 11, 12, 14, 17] and elsewhere.

Recently, the idea of allowing negative weights was put forward. This resulted in minus domination, where f has codomain $\{-1, 0, 1\}$, and signed domination, where f has codomain $\{-1, 1\}$. A *minus dominating function* is defined in [7] as a function $f: V \to \{-1, 0, 1\}$ such that $f[v] \geq 1$ for all $v \in V$. The *minus domination number* for a graph G is $\gamma^-(G) = \min\{w(f) \mid f \text{ is a minus dominating function on } G\}$. A *signed dominating function* is defined in [9] as a function $f: V \to \{-1, 1\}$ such that for every $v \in V$, $f[v] \geq 1$. The *signed domination number* for a graph G is $\gamma_s(G) = \min\{w(f) \mid f \text{ is a signed dominating function on } G\}$. These parameters are similar in many ways to ordinary domination, but also have different properties. Minus and signed domination have been studied in [6, 7, 8, 9, 16, 18, 19, 20] and elsewhere.

Bange et al. [1] introduced the generalization to $\mathcal{P}$-domination for an arbitrary subset $\mathcal{P}$ of the reals $\mathbb{R}$. A function $f: V \to \mathcal{P}$ is a $\mathcal{P}$-dominating function if the sum of its function values over every closed neighbourhood is at least 1. That is, for every $v \in V$, $f[v] \geq 1$. The $\mathcal{P}$-domination number of a graph G, denoted $\gamma_{\mathcal{P}}(G)$, is defined to be the infimum of $w(f)$ taken over all $\mathcal{P}$-dominating functions f. Of course this might be $-\infty$. For example, if $\mathcal{P} = \mathbb{Z}$ and α is a positive integer, then for the tree T shown in Figure 1, $\gamma_{\mathcal{P}}(T) \leq 4 - 2\alpha$. As we can make α as large as we like, it is evident that $\gamma_{\mathcal{P}}(T) = -\infty$.

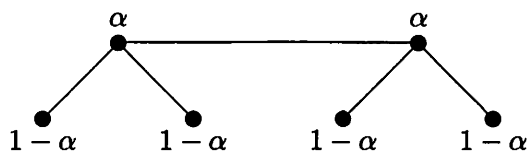


Figure 1. A tree T with $\gamma_{\mathbb{Z}} = -\infty$.

When $\mathcal{P} = \{0, 1\}$ we obtain the standard domination number. When $\mathcal{P} = [0, 1]$, $\{-1, 0, 1\}$ or $\{-1, 1\}$ we obtain the fractional, minus or signed domination numbers, respectively. When $\mathcal{P} = \mathbb{R}$, we obtain the real domination number $\gamma_{\mathbb{R}}(G)$. A trivial observation is that if $\mathcal{P} \subseteq T$, then $\gamma_{\mathcal{P}}(G) \geq \gamma_T(G)$. In particular, $\gamma(G) \geq \gamma_{\mathbb{R}}(G)$.

2 Real domination

Let $\mathcal{P}$ be a subset of the reals $\mathbb{R}$. We say a function $f: V \to \mathcal{P}$ is an *efficient $\mathcal{P}$-dominating function* if for every vertex v it holds that $f[v] = 1$. A function is *nonnegative* if all the function values are nonnegative. We denote a function which is both nonnegative and efficient $\mathcal{P}$-dominating as an $NE\mathcal{P}D$-function. For example, if G is a regular graph of degree r, then the function f that assigns to each vertex the value $1/(r+1)$ is an $NE\mathcal{P}D$-function for G. If G is a complete bipartite graph of order at least 3 with one partite set $\mathcal{L}$ of cardinality ℓ and the other $\mathcal{R}$ of cardinality r, then the function f that assigns to each vertex of $\mathcal{L}$ the value $(r-1)/(lr-1)$ and to each vertex of $\mathcal{R}$ the value $(l-1)/(lr-1)$ is an $NE\mathcal{P}D$-function for G. Goddard and Henning [13] showed that the property of possessing a $NE\mathbb{R}D$-function is the key to the real domination number of a graph.

Theorem A. (Goddard, Henning) *For any graph G,*

$$\gamma_{\mathbb{R}}(G) = \begin{cases} w(f) & \text{if } G \text{ has a } NE\mathbb{R}D\text{-function } f, \\ -\infty & \text{otherwise.} \end{cases}$$

If f is an $NE\mathcal{P}D$-function of G, then $w(f) \geq \gamma_{\mathcal{P}}(G) \geq \gamma_{\mathbb{R}}(G) = w(f)$. Hence we have the following corollary of Theorem A.

Corollary 1. *For any subset $\mathcal{P}$ of $\mathbb{R}$, if a graph G has an $NE\mathcal{P}D$-function f, then $\gamma_{\mathcal{P}}(G) = \gamma_{\mathbb{R}}(G) = w(f)$.*

When $\mathcal{P} = \{0, 1\}$ an efficient $\mathcal{P}$-dominating function of a graph G is the characteristic function of a so-called *efficient dominating set* D of G: $|N[v] \cap D| = 1$ for every $v \in V$. (Equivalently, D dominates G and $u, v \in D$ implies $d(u, v) \geq 3$.) Efficient dominating sets were introduced by Bange, Barkauskas, and Slater [2, 3]. As a special case of Corollary 1, we have the following result.

Corollary 2. *If a graph G has an efficient dominating set D, then $\gamma_{\mathbb{R}}(G) = \gamma(G)$.*

3 $NE\mathbb{R}D$-functions and efficient dominating sets

If a graph G has an efficient dominating set D, then G has a $NE\mathbb{R}D$-function (simply take the characteristic function of D). However, the converse is not true. Many graphs that do not have efficient dominating sets will have a $NE\mathbb{R}D$-function. For example, the graph G shown in Figure 2 has a $NE\mathbb{R}D$-function as illustrated, but does not have an efficient dominating set. Hence, the existence of a $NE\mathbb{R}D$-function does not necessarily imply that G has an efficient dominating set.

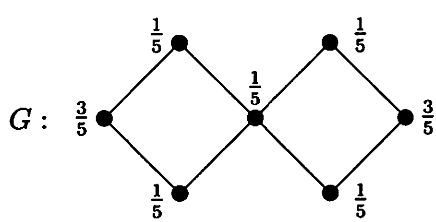


Figure 2. A $NE\mathbb{R}D$-function of a graph G.

Goddard and Henning [13] proved the following result.

Theorem B. (Goddard, Henning) *For any tree T, T has a $NE\mathbb{R}D$-function if and only if it has an efficient dominating set.*

We extend Theorem B to complete block graphs. We shall need the following lemmas:

Lemma 1. *Let f be a $NE\mathbb{R}D$-function of a complete block graph or a unicyclic graph $G = (V, E)$. If $N[u] \subseteq N[v]$ for some vertices u and v of G, then every vertex at distance 2 from u of G has weight 0 under f.*

Proof: Suppose G is a complete block graph. Then the subgraph induced by $N[u]$ is complete. Thus $N[u] \subseteq N[x]$ for every vertex x adjacent with u. Let w be an arbitrary vertex at distance 2 from u in G, and let u, x,

w be a $u - w$ path. Then, since f is a nonnegative function, $1 = f[x] \geq f[u] + f(w) = 1 + f(w)$, so $f(w) \leq 0$. Thus, $f(w) = 0$.

If G is a unicyclic graph, then u has degree 1 or 2. If u has degree 2, then the subgraph induced by $N[u]$ is complete and the result follows as before. If u is a leaf, then let $S = N[v] - N[u]$. Then $1 = f[v] = f[u] + f(S) = 1 + f(S)$, so $f(S) = 0$. But f is a nonnegative function, so $f(w) = 0$ for all $w \in S$. □

Lemma 2. *If a complete block graph has a $NE\mathbb{R}D$-function, then it has an efficient dominating set.*

Proof: We prove that if a complete block graph $G = (V, E)$ has a $NE\mathbb{R}D$-function f, then it has an efficient dominating set S such that $S \subseteq \{v \in V \mid f(v) > 0\}$. We proceed by induction on the number m of blocks in the complete block graph. The base case when the complete block graph has one block is trivial. So, assume for all complete block graphs $G' = (V', E')$ with less than m blocks that if G' has a $NE\mathbb{R}D$-function f, then it has an efficient dominating set S' such that $S' \subseteq \{v \in V' \mid f(v) > 0\}$. Let $G = (V, E)$ be a complete block graph with m blocks with diameter d that has a $NE\mathbb{R}D$-function f.

If $d = 2$, then G consists of m end-blocks and a single cut-vertex v that belongs to every block. By Lemma 1, all vertices other than the central vertex have weight 0, and thus the central vertex v has weight 1. Letting $S = \{v\}$ we have an efficient dominating set S satisfying $S \subseteq \{v \in V \mid f(v) > 0\}$.

If $d = 3$, then G has at least two cut-vertices and contains $m - 1$ end-blocks and one other block that is adjacent with all the other blocks. It follows from Lemma 1 that every vertex of this central block has weight 0. Hence the sum of the weights of all the special vertices in each end-block is 1 (so each end-block contains a special vertex of positive weight). We know then from Lemma 1 that no two end-blocks can be adjacent. Hence every cut-vertex belongs to exactly two blocks, namely to an end-block and to the central block, so any two special vertices from distinct end-blocks are at distance 3 apart. Letting S consist of a special vertex of positive weight from each end-block, we have an efficient dominating set S satisfying $S \subseteq \{v \in V \mid f(v) > 0\}$.

For $d \geq 4$, let u and v be two vertices of G at distance d apart, and consider a $u - v$ path $u = v_0, v_1, v_2, \ldots, v_d = v$ in G of length d. For $i = 1, 2, \ldots, d$, let B_i be the block containing the edge $v_{i-1}v_i$. The removal of the edges v_2w for all vertices $w \in B_3 - \{v_2\}$ yields two complete block graphs G_u (containing u) and G_v. By Lemma 1, $f(w) = 0$ for all $w \in V(B_2) - \{v_1\}$. In particular, $f(v_2) = 0$. So the restriction of f to G_v is a $NE\mathbb{R}D$-function of G_v. Hence, by induction, there exists an efficient dominating set S_v of G_v such that $S_v \subseteq \{w \in V \mid f(w) > 0\}$. It remains to extend S_v to the desired dominating set of G.

If v_2 has a neighbour x in G_u with weight 1 under f, then every other neighbour of v_2, including those vertices in $B_3 - \{v_2\}$, has weight 0 under f. So no vertex of $B_3 - \{v_2\}$ is in S_v. Let B_x be the block containing the edge v_2x. Then by Lemma 1, every vertex of B_x different from x and v_2 is a special vertex. Since v_2 belongs to at least three blocks, then by Lemma 1 each neighbour of v_2 that does not belong to B_x or B_3 belongs to exactly two blocks, namely an end-block and a block containing v_2. Furthermore, the sum of the weights of the special vertices in each end-block of G_u not containing x is 1 under f, so each such end-block has a special vertex of positive weight. Let S_u consist of a special vertex of positive weight from each end-block of G_u that does not contain x. Then $S = S_u \cup S_v \cup \{x\}$ is an efficient dominating set of G satisfying $S \subseteq \{w \in V \mid f(w) > 0\}$.

Suppose every neighbour of v_2 in G_u has weight less than 1 under f. If v_2 is contained in a block B_x in G_u the sum of the weights of whose vertices is 1 under f, then every neighbour of v_2, not in B_x, has weight 0 under f. In particular, the vertices in $B_3 - \{v_2\}$ have weight 0 under f. So no vertex of $B_3 - \{v_2\}$ is in S_v. By Lemma 1, every vertex of B_x different from v_2 is a special vertex. Hence B_x is different from the block B_2, so v_2 belongs to at least three blocks. By Lemma 1 each neighbour of v_2 that does not belong to B_x or B_3 belongs to exactly two blocks, namely an end-block and a block containing v_2. Let S_u consist of a special vertex of positive weight from each end-block of G_u. Then $S = S_u \cup S_v$ is an efficient dominating set of G satisfying $S \subseteq \{w \in V \mid f(w) > 0\}$.

Suppose the sum of the weights of the vertices in every block that contains v_2 in G_u is less than 1 under f. If v_2 is contained in a nontrivial block B_x in G_u, then, by Lemma 1, each vertex of B_x different from v_2 has weight 0 under f and belongs to exactly two blocks, namely an end-block and a block containing v_2. The sum of the weights of the special vertices in each such end-block of G_u is 1 under f, so each such end-block has a special vertex of positive weight. If v_2 is contained in a trivial block B_x in G_u, then the vertex, x say, adjacent with v_2 in B_x belongs to exactly two blocks, namely an end-block and the trivial block B_x containing v_2. The sum of the weights of the special vertices in the end-block containing x is $1 - f(x) > 0$, so this end-block must contain a special vertex of positive weight under f. Hence every end-block in G_u contains a special vertex of positive weight under f.

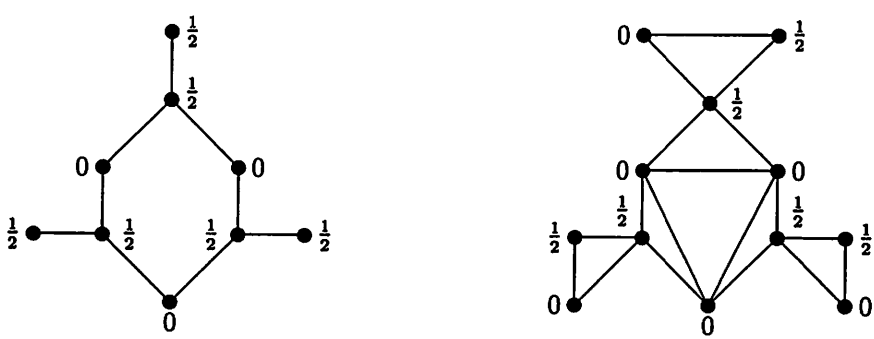
If S_v contains a vertex $w \in B_3 - \{v_2\}$, then let S be S_v together with a special vertex of positive weight from each end-block of G_u. If S_v contains no vertex $w \in B_3 - \{v_2\}$, then since v_3 is dominated by S_v, one of its neighbours in G_v that does not belong to B_3 must have positive weight under f and hence the sum of the weights of the vertices in $B_3 - \{v_2\}$ must be less than 1. From this it follows that at least one neighbour x of v_2 in G_u has positive weight under f. So let S be $S_v \cup \{x\}$ together with a special vertex of positive weight from each end-block of G_u that does not contain x. In both cases we produce an efficient dominating set S of G satisfying $S \subseteq \{w \in V \mid f(w) > 0\}$. □

If a graph has an efficient dominating set, then it has a $NE\mathbb{R}D$-function. Hence for a complete block graph we have the following:

Theorem 1. *For any complete block graph G, G has a $NE\mathbb{R}D$-function if and only if it has an efficient dominating set.*

For what other classes of graphs does the existence of a $NE\mathbb{R}D$-function imply the existence of an efficient dominating set? As pointed out in Section 2, every regular graph has a $NE\mathbb{R}D$-function. However, for every integer $r \geq 2$, there exist regular graphs G of degree r that do not possess efficient dominating sets. For example, if $r = 2$, then let $G \cong C_n$ where $n \not\equiv 0 \pmod 3$, while for $r \geq 3$, let $G \cong K_{r-1} \times K_2$. The bipartite graph G shown in Figure 2 or 3(i) has a $NE\mathbb{R}D$-function as illustrated, but no efficient dominating set. The unicyclic graph G shown in Figure 3(i) has a $NE\mathbb{R}D$-function as illustrated, but does not possess an efficient dominating set. The graph H shown in Figure 3(ii) is both chordal and outerplanar in which every bounded region is a triangle. Although H has a $NE\mathbb{R}D$-function as illustrated, it has no efficient dominating set. Hence we have the following result.

Theorem 2. *If a graph is bipartite, chordal, unicyclic, regular, or outerplanar in which every bounded region is a triangle, then the existence of a $NE\mathbb{R}D$-function does not necessarily imply the existence of an efficient dominating set.*



(*i*) A unicyclic graph G. (*ii*) A chordal graph H.

Figure 3. $NE\mathbb{R}D$-functions of the graphs G and H.

Next we consider the family $\mathcal{U}$ of unicyclic graphs with the property that the vertices of the cycle dominate all other vertices. Equivalently, if $G \in \mathcal{U}$, then every vertex of G either lies on the cycle or is adjacent with a vertex of the cycle. We characterize those graphs G from $\mathcal{U}$ for which the following two conditions are equivalent:

(1) G has a $NE\mathbb{R}D$-function;

(2) G has a efficient dominating set.

Every regular graph has a $NE\mathbb{R}D$-function, so if G is a cycle, then it satisfies condition (1). A cycle has an efficient dominating set if and only if its length is a multiple of 3. Therefore, if $G \cong C_n$, then the conditions (1) and (2) are equivalent if and only if $n \equiv 0 \pmod 3$. Hence in what follows we restrict our attention to those graphs of $\mathcal{U}$ that are not cycles. We will refer to the edges that do not belong to the cycle as *legs*. Two legs are adjacent if they are incident with a common vertex. Two nonadjacent legs are called *consecutive legs* if they are joined by a path every internal vertex of which has degree 2. The following two lemmas will be useful.

Lemma 3. *If $G \in \mathcal{U}$ has a $NE\mathbb{R}D$-function f which assigns weight 1 to some vertex v of G, then f has codomain $\{0,1\}$.*

Proof: Suppose, to the contrary, that there is a vertex of G whose weight under f is strictly between 0 and 1. Among all such vertices, let u be one whose distance from v is minimum. Notice that $d(u,v) \geq 3$. Let $v = v_0, v_1, \ldots, v_{d-1}, v_d = u$ be a shortest $u-v$ path. Since $f[v_{d-1}] = 1$, we must have $f(v_{d-1}) < 1$ and $f(v_{d-2}) < 1$. Hence, by our choice of u, we have $f(v_{d-1}) = f(v_{d-2}) = 0$. By assumption $f(u) < 1$, so v_{d-1} must be adjacent with some vertex x, different from u, whose weight under f is strictly between 0 and 1. Therefore, at least one of u and x is an end-vertex and either $f[u] < 1$ or $f[x] < 1$. This contradicts that fact that f is a $NE\mathbb{R}D$-function of G. □

Lemma 4. *If $G \in \mathcal{U}$ has a $NE\mathbb{R}D$-function f, then f has codomain $[0,1)$ if and only if G satisfies the following three conditions:* (i) *G has maximum degree 3,* (ii) *if G has exactly one leg, then the length of the cycle is congruent to 2 (mod 3), and* (iii) *if G has at least two legs, then any path that joins two consecutive legs, the internal vertices of which have degree 2, has length congruent to 2 (mod 3).*

Proof: To prove the sufficiency, suppose G has the structure as described in the lemma. Then a $NE\mathbb{R}D$-function f of G with codomain $[0,1)$ can be defined as follows: let $f(v) = 0$ if v is a vertex of degree 2 whose distance from a closest vertex of degree 3 is congruent to 1 (mod 3), and let $f(v) = 1/2$ otherwise.

Next we prove the necessity. Suppose that G has a $NE\mathbb{R}D$-function f with codomain $[0,1)$. First we show that G has at most one leg attached to every vertex of the cycle. If this is not the case, then there are end-vertices u and v that are adjacent with a common vertex x on the cycle. By Lemma 1, every vertex at distance 2 from u (respectively, v) has weight 0 under f. Therefore, every neighbour of x has weight 0 under f. Consequently $f(x) = 1$, contrary to our assumption that no vertex of G has weight 1 under f. Hence G has maximum degree 3.

If G has exactly one leg u_1v_1 with u_1 on the cycle, then let $u_1, u_2, \ldots, u_d, u_1$ be the cycle. If $f(v_1) = 0$, then $f(u_1) = 1$, a contradiction. Hence $f(v_1) = \alpha$ where $0 < \alpha < 1$. The weights of the vertices along P are now determined, namely, $f(u_1) = 1 - \alpha$, $f(u_2) = 0$, $f(u_3) = \alpha$, $f(u_4) = 1 - \alpha$, $f(u_5) = 0$, $f(u_6) = \alpha, \ldots$, $f(u_{d-2}) = \alpha$, $f(u_{d-1}) = 1 - \alpha$, $f(u_d) = 0$. Hence $1 = f[u_d] = 2(1 - \alpha)$, so $\alpha = 1/2$, and the length d of the cycle satisfies $d \equiv 2 \pmod 3$.

If G has at least two legs, then consider two consecutive legs u_1v_1 and u_2v_2 along the cycle with u_1 and u_2 on the cycle. Let P: $u_1 = x_0, x_1, \ldots, x_{d-1}, x_d = u_2$ be a $u_1 - u_2$ path every internal vertex of which has degree 2. If $f(v_1) = 0$, then $f(u_1) = 1$, a contradiction. Hence $f(v_1) = \alpha$ where $0 < \alpha < 1$. The weights of the vertices along P are now determined, namely, $f(x_0) = 1 - \alpha$, $f(x_1) = 0$, $f(x_2) = \alpha$, $f(x_3) = 1 - \alpha$, $f(x_4) = 0$, $f(x_5) = \alpha, \ldots$, $f(x_{d-2}) = 1 - \alpha$, $f(x_{d-1}) = 0$, $f(x_d) = \alpha$. Hence the length d of P satisfies $d \equiv 2 \pmod 3$. □

Some immediate corollaries of Lemma 4 now follow.

Corollary 3. *Let f be a $NE\mathbb{R}D$-function of $G \in \mathcal{U}$ with codomain $[0, 1)$. Then the weights assigned under f are 0, α, or $1 - \alpha$ where $0 < \alpha < 1$.*

Corollary 4. *Let f be a $NE\mathbb{R}D$-function of $G \in \mathcal{U}$ with codomain $[0, 1)$. If u_1v_1 and u_2v_2 are consecutive legs along the cycle with u_1 and u_2 on the cycle, then $f(v_1) = \alpha$ and $f(v_2) = 1 - \alpha$ where $0 < \alpha < 1$.*

Theorem 3. *Let $G = (V, E)$ be a graph from $\mathcal{U}$ that is not a cycle. Then G has a $NE\mathbb{R}D$-function if and only if G has an effcient dominating set, unless G satisfies the following three conditions:* (i) *G has maximum degree 3,* (ii) *if G has exactly one leg, then the length of the cycle is congruent to 2 (mod 3), and* (iii) *if G has at least two legs, then it has an odd number of legs and any path that joins two consecutive legs, the internal vertices of which have degree 2, has length congruent to 2 (mod 3).*

Proof: If G has an efficient dominating set, then it has a $NE\mathbb{R}D$-function. For the converse, let f be a $NE\mathbb{R}D$-function of G. If f assigns weight 1 to some vertex of G, then, by Lemma 3, f has codomain $\{0, 1\}$ and the set $\{v \in V \mid f(v) = 1\}$ is an efficient dominating set of G. So we may assume that f has codomain $[0, 1)$. By Lemma 4, G has maximum degree 3.

If G has exactly one leg, then it follows from the proof of Lemma 4 that the length of the cycle is congruent to 2 (mod 3). Such a graph G has no efficient dominating set. On the other hand, if G has at least two legs, then, by Lemma 4, no two legs are adjacent and a path that joins two consecutive legs, the internal vertices of which have degree 2, has length congruent to 2 (mod 3). Let $u_0v_0, u_1v_1, \ldots, u_{m-1}v_{m-1}$ be the legs of G in clockwise ordering, with end-vertices $v_0, v_1, \ldots, v_{m-1}$, and let S be an efficient dominating set of G. Exactly one of u_i and v_i belongs to S. If $u_i \in S$, then $v_{i-1}, v_{i+1} \in S$ while if $v_i \in S$, then $u_{i-1}, u_{i+1} \in S$ (where addition is taken modulo m). Hence the number m of legs must be even. Thus if m is odd, then G has no efficient dominating set. If m is even, then let S_1 (S_2) denote the set of vertices u_i with i odd (respectively, even) and vertices v_j with j even (respectively, odd). Then the set S_1 (respectively, S_2) together with all vertices on the cycle whose distance from S_1 (respectively, S_2) is a multiple of 3 forms an efficient dominating set of G. □

4 Graphs for which $\gamma_{\mathbb{R}} = \gamma$ is equivalent to the existence of an efficient dominating set

Due to Corollary 2, we need only consider those graphs for which $\gamma_{\mathbb{R}} = \gamma$ implies the existence of an efficient dominating set. As an immediate consequence of Theorem A, Corollary 2, and Theorem 1 we have the following result.

Corollary 5. *For any complete block graph G, $\gamma_{\mathbb{R}}(G) = \gamma(G)$ if and only if G has an efficient dominating set.*

We show next that for the class of regular graphs G, $\gamma_{\mathbb{R}}(G) = \gamma(G)$ implies that G has an efficient dominating set.

Lemma 5. *If G is a regular graph of degree r satisfying $\gamma_{\mathbb{R}}(G) = \gamma(G)$, then G has an efficient dominating set.*

Proof: Let $G = (V, E)$ have order n. Then $\gamma(G) = \gamma_{\mathbb{R}}(G) = n/(r+1)$, so $n = (r+1)k$ for some positive integer k. Thus, $\gamma(G) = k$. Hence there exists a minimum dominating set $D = \{v_1, v_2, \ldots, v_k\}$ of G of cardinality k. Each vertex of D can dominate at most r vertices not in D, so $n - \gamma(G) = |V - D| \leq r|D| = r\gamma(G)$. Thus, $\gamma(G) \geq n/(r+1)$ with equality if and only if $|V - D| = r|D|$; that is, if and only if D is an efficient dominating set of G. Since $\gamma(G) = n/(r+1)$, the result now follows. □

As a consequence of Corollary 2 and Lemmas 5, we have the following result.

Corollary 6. *For any regular graph G, $\gamma_{\mathbb{R}}(G) = \gamma(G)$ if and only if G has an efficient dominating set.*

However, in general a graph G satisfying $\gamma_{\mathbb{R}}(G) = \gamma(G)$ does not necessarily possess an efficient dominating set. For example, the graph G shown in Figure 3(i), which is bipartite, unicyclic, and a cactus, has a $NE\mathbb{R}D$-function f as illustrated, and so $\gamma_{\mathbb{R}}(G) = w(f) = 3$. Furthermore, it is evident that $\gamma(G) = 3$. However, the graph G does not possess an efficient dominating set. The graph H shown in Figure 3(ii), which is both chordal and outerplanar in which every bounded region is a triangle, has a $NE\mathbb{R}D$-function f as illustrated, and so $\gamma_{\mathbb{R}}(H) = w(f) = 3$. Furthermore, it is evident that $\gamma(H) = 3$. However, the graph H does not possess an efficient dominating set. Hence we have the following result.

Theorem 4. *If a graph G is bipartite, chordal, a cactus, unicyclic, or outerplanar in which every bounded region is a triangle, then $\gamma_{\mathbb{R}}(G) = \gamma(G)$ does not necessarily imply that G has an efficient dominating set.*

We close this section with the following. Is it true that if G is a strongly chordal graph, then $\gamma_{\mathbb{R}}(G) = \gamma(G)$ implies that G has an efficient dominating set?

5 Graphs for which $\gamma_{\mathbb{R}} = \gamma$ is equivalent to the existence of a $NE\mathbb{R}D$-function

If G is a graph for which $\gamma_{\mathbb{R}}(G) = \gamma(G)$, then G has a $NE\mathbb{R}D$-function. This follows since if G has no $NE\mathbb{R}D$-function, then, by Theorem A, $\gamma_{\mathbb{R}}(G) = -\infty$. Hence in this section we consider graphs for the existence of a $NE\mathbb{R}D$-function implies $\gamma_{\mathbb{R}} = \gamma$. As an immediate consequence of Corollary 5 and Theorem 1 we have the following result.

Corollary 7. *For any complete block graph G, $\gamma_{\mathbb{R}}(G) = \gamma(G)$ if and only if G has a $NE\mathbb{R}D$- function.*

Suppose that a strongly chordal graph $G = (V, E)$ has a $NE\mathbb{R}D$-function f. Since Farber [10] showed that for strongly chordal graphs their fractional domination number is equal to their domination number, it follows that $w(f) \geq \gamma_f(G) = \gamma(G) \geq \gamma_{\mathbb{R}}(G) = w(f)$. Thus we must have equality throughout. In particular, $\gamma_{\mathbb{R}}(G) = \gamma(G)$. Hence we have the following result.

Corollary 8. *For any strongly chordal graph G, $\gamma_{\mathbb{R}}(G) = \gamma(G)$ if and only if G has a $NE\mathbb{R}D$-function.*

We conjecture that Corollary 8 can be extended to the family of all chordal graphs. Corollaries 7 and 8 may be restated as follows:

Corollary 9. *For any complete block graph or strongly chordal graph G,*

$$\gamma_{\mathbb{R}}(G) = \begin{cases} \gamma(G) & \text{if } G \text{ has a } NE\mathbb{R}D\text{-function,} \\ -\infty & \text{otherwise.} \end{cases}$$

However it is not true in general that if a graph G has a $NE\mathbb{R}D$-function, then $\gamma_{\mathbb{R}}(G) = \gamma(G)$. We know that every regular graph of order n and of degree r satisfies $\gamma_{\mathbb{R}}(G) = n/(r+1)$. However, for every integer $r \geq 2$, there exist regular graphs G of degree r for which $\gamma(G) \neq n/(r+1)$. For example, if $r = 2$, then let $G \cong C_n$ where $n \equiv 0 \pmod 3$, while for $r \geq 3$, let $G \cong K_{r-1} \times K_2$. The bipartite graph G shown in Figure 2 has a $NE\mathbb{R}D$-function f as illustrated, so $\gamma_{\mathbb{R}}(G) = w(f) = 11/5 < 3 = \gamma(G)$. The unicyclic graph G shown in Figure 4 has a $NE\mathbb{R}D$-function f as illustrated, so $\gamma_{\mathbb{R}}(G) = w(f) = 7/3$ while $\gamma(G) = 3$. Hence we have the following result.

Theorem 5. *If a bipartite, unicyclic or regular graph G has a $NE\mathbb{R}D$-function, then it is not necessarily true that $\gamma_{\mathbb{R}}(G) = \gamma(G)$.*

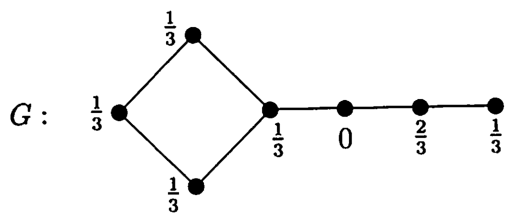


Figure 4. A unicycle graph G with $\gamma_{\mathbb{R}}(G) < \gamma(G)$.

As a consequence of Corollary 3 and Theorem 3, we have the following result.

Corollary 10. *If $G \in \mathcal{U}$ is not a cycle, then $\gamma_{\mathbb{R}}(G) = \gamma(G)$ if and only if G has a $NE\mathbb{R}D$-function.*

Theorem 6. *If a unicyclic graph G has a $NE\mathbb{R}D$-function f which assigns weight 1 to some vertex v of the cycle, then $\gamma_{\mathbb{R}}(G) = \gamma(G)$.*

Proof: Let u be a neighbour of v on the cycle, and let w be the neighbour of u on the cycle different from v. Then $f(u) = f(w) = 0$. Let $e = uw$. Then f is a $NE\mathbb{R}D$-function of the tree $G - e$, so by Corollary 7, $\gamma_{\mathbb{R}}(G-e) = \gamma(G-e)$. Hence $\gamma(G) \leq \gamma(G-e) = \gamma_{\mathbb{R}}(G-e) = w(f) = \gamma_{\mathbb{R}}(G) \leq \gamma(G)$. Thus we must have equality throughout. In particular, $\gamma_{\mathbb{R}}(G) = \gamma(G)$. □

6 Summary

Let $\mathcal{F}$ denote a family of graphs, and let P_1 and P_2 denote the following two properties of the family $\mathcal{F}$.

P_1 For every $G \in \mathcal{F}$, $\gamma_{\mathbb{R}}(G) = \gamma(G)$ implies that G has an efficient dominating set;

P_2 For every $G \in \mathcal{F}$, the existence of a $NE\mathbb{R}D$-function for G implies that $\gamma_{\mathbb{R}}(G) = \gamma(G)$.

The following table summarizes our results.

Family of graphs	complete block graphs	unicyclic graphs	bipartite graphs	regular graphs	chordal graphs	strongly chordal graphs
Property P_1	yes	no	no	yes	no	?
Property P_2	yes	no	no	no	?	yes

We close with the following question which we have yet to settle: does P_2 imply P_1?

References

[1] D.W. Bange, A.E. Barkauskas, L.H. Host, and P.J. Slater, Generalized domination and efficient domination in graphs, *Discrete Math.* **159** (1996), 1–11.

[2] D.W. Bange, A.E. Barkauskas, and P.J. Slater, Disjoint dominating sets in trees, *Sandia Laboratories Report* **SAND 78-1087J** (Sandia Laboratories, Albuquerque, 1978).

[3] D.W. Bange, A.E. Barkauskas, and P.J. Slater, Efficient dominating sets in graphs, in: R.D. Ringeisen and F.S. Roberts, eds., *Applications of Discrete Mathematics* (SIAM, Philadelphia, 1988), 189–199.

[4] G.S. Domke, G. Fricke, S.T. Hedetniemi, and R. Laskar, Relationships between integer and fractional parameters of graphs, *Graph Theory, Combinatorics, and Applications*, John Wiley & Sons, Inc., **2** (1991), 371–387.

[5] G.S. Domke, S.T. Hedetniemi, and R. Laskar, Fractional packings, coverings and irredundance in graphs, *Congressus Numer.* **66** (1988), 227–238.

[6] J.E. Dunbar, W. Goddard, S.T. Hedetniemi, M.A. Henning, and A. McRae, On the algorithmic complexity of minus domination in graphs, *Discrete Applied Math.* **68** (1996), 73–84.

[7] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, and A.A. McRae, Minus domination in graphs, Submitted for publication.

[8] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, and A. McRae, Minus domination in regular graphs, *Discrete Math.* **49** (1996), 311–312.

[9] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning, and P.J. Slater, Signed domination in graphs, *Graph Theory, Combinatorics, and Applications*, John Wiley & Sons, Inc., **1** (1995), 311–322.

[10] M. Farber, Domination, independent domination, and duality in strongly chordal graphs, *Discrete Applied Math.* **7** (1984), 115–130.

[11] D.C. Fisher, J. Ryan, G. Domke, and A. Majumdar, Fractional domination of strong direct products, *Discrete Applied Math.* **50** (1994), 89–91.

[12] G. H. Fricke, Upper domination on double cone graphs, *Congressus Numer.* **72** (1990), 199–207.

[13] W. Goddard and M.A. Henning, Real and integer domination in graphs, Submitted for publication.

[14] D.L. Grinstead and P.J. Slater, Fractional domination and fractional packing in graphs, *Congressus Numer.* **71** (1990), 153–172.

[15] T.W. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel-Dekker, Inc., to appear in 1997.

[16] J.H. Hattingh, M.A. Henning, and P.J. Slater, Three-valued k-neighbourhood domination in graphs, *Australasian J. Combin.* **9** (1994), 233–242.

[17] S.M. Hedetniemi, S.T. Hedetniemi, and T.V. Wimer, Linear time resource allocation for trees, manuscript (1987).

[18] M.A. Henning, Domination in regular graphs, *Ars Combin.* **43** (1996), 263–271.

[19] M.A. Henning and P.J. Slater, Inequalties relating domination parameters in cubic graphs, *Discrete Math.* **158** (1996), 87–98.

[20] B. Zelinka, Some remarks on domination in cubic graphs, *Discrete Math.* **158** (1996), 249–255.