Aperiodic Complementary Quadruples of Binary Sequences

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ABSTRACT. We have carried out a large number of computer searches for the base sequences $BS(n+1,n)$ as well as for three important subsets known as Turyn sequences, normal sequences, and near-normal sequences. In the Appendix we give an extensive list of $BS(n+1,n)$ for $n \leq 32$. The existence question for Turyn sequences in $BS(n+1,n)$ was resolved previously for all $n \leq 41$, and we extend this bound to $n \leq 51$. We also show that the sets $BS(n+1,n)$ do not contain any normal sequences if $n = 27$ or 28. To each set $BS(n+1,n)$ we associate a finite graph Γ_n and determine these graphs completely for $n \leq 27$. We show that $BS(m,n) = \emptyset$ if $m \geq 2n$, $n > 1$, and $m+n$ is odd, and we also investigate the borderline case $m = 2n-1$.

Introduction

In this paper we discuss several conjectures of combinatorial nature which are closely related to the Hadamard matrix conjecture (*HMC*). Recall that *HMC* asserts that for every positive integer n there exists a Hadamard matrix of order $4n$. This notoriously difficult combinatorial problem remains open for more than hundred years, in spite of many hundreds of papers written on the subject. There is a plethora of partial results that construct some infinite families of Hadamard matrices or just some particular ones for special orders $4n$ (see e.g. the survey papers [5, 25]), but as yet there is no decisive result.

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A very promising new method has been introduced by R. J. Turyn [27]. This is the method of base sequences which is used in many recent papers, e.g. [4, 17, 18, 19, 20, 21, 22, 28]. The collection of all base sequences $(A;B;C;D)$ with A and B of length m, and C and D of length n will be denoted by $BS(m,n)$ (see Section 3 for precise definition). In applied areas (such as signal processing, image coding, etc.) base sequences with $m=n$ are called sets of (four) aperiodic complementary binary sequences (see e.g. [3, 26]). The most important contribution to the theory of base sequences, after Turyn's paper [27], was made by C. H. Yang [28]. Many subsequent papers are essentially elaborations of his ideas.

One of the main objectives of this paper is to attract the attention of the readers to the following two intriguing conjectures.

Aperiodic Complementary Quadruples Conjecture ($ACQC$). *The sets $BS(n,n)$ are non-empty for all $n \geq 0$.*

Base Sequence Conjecture (BSC). *The sets $BS(n+1,n)$ are non-empty for all $n \geq 0$.*

Although these are not new conjectures, I have not seen them stated in print. We point out that HMC is a consequence of $ACQC$ (see Section 3, or [25, 28] for more details). It is known (see Section 5) that $ACQC$ is true for all $n \leq 100$ such that $n \neq 67, 73, 79, 83, 89, 97$.

It is easy to show that BSC implies $ACQC$. At the present time it is known that BSC holds for all $n \leq 32$ and for all integers $n = 2^a 10^b 26^c$ where a, b, c are arbitrary non-negative integers.

In order to gain some insight into the structure of the sets $BS(n+1,n)$, we introduce a graph Γ_n. The vertices of this graph are all partitions of $2(2n+1)$ into four squares. To each $(A;B;C;D) \in BS(n+1,n)$ we can associate a set consisting of one or two vertices of Γ_n. If this set is a singleton, we introduce a loop at that vertex, and otherwise an edge joining the two vertices in the set. We postulate that there is at most one loop at any vertex and at most one edge between any two distinct vertices. We say that $(A;B;C;D)$ is a witness for the edge of Γ_n that it defines. Some elementary properties of these graphs are proved in Section 7. We have determined completely these graphs for $n \leq 27$ (see Section 8).

By imposing various symmetry conditions on base sequences, one obtains three important subsets $TU(n), NS(n)$, and $NN(n)$ of $BS(n+1,n)$. We refer to their members as Turyn sequences, normal sequences, and near-normal sequences, respectively (for the precise definitions see Sections 10, 11, and 12). The normal and near-normal sequences were originally introduced by C. H. Yang [28] in a different (but equivalent) form.

One knows that $TU(n) \neq \emptyset$ for $n \leq 7$ and $n = 12$ and 14. It appears that $TU(n) = \emptyset$ for all other values of n. This was confirmed for $n \leq 29$ in [21], and for $n \leq 41$ in the very recent paper [6]. Our computations show that $TU(n) = \emptyset$ also for $41 < n \leq 51$.

Particular normal and/or near-normal sequences were constructed in [20, 28]. C. H. Yang has stated [28] that he has enumerated all equivalence classes of $NN(n)$ for even $n \leq 18$. He has also shown (see [20]) that $NN(30) \neq \emptyset$. We have carried out exhaustive computer search for both $NS(n)$ and $NN(n)$ for all $n \leq 28$. An extensive list of base sequences $BS(n+1, n)$ for $n \leq 32$ (including normal and near-normal sequences) is given in the Appendix.

In Section 13 we show that the $BS(m, n)$ are empty if (i) $m \geq 2n$, $n > 1$, and $m + n$ is odd; or (ii) $m = 2n - 1$ and n is odd. In a certain sense, $m = 2n - 1$ is a borderline case. Indeed it is known that $BS(2n-1, n)$ is not empty for even $n \leq 24$.

In Section 14 we discuss an important class of sequences called T-sequences which are closely related to $ACQC$ and BSC. The set of all T-sequences of length n is denoted by $TS(n)$. There is also a conjecture regarding these sequences.

T-sequence Conjecture (*TSC*). *The sets $TS(n)$ are non-empty for all $n \geq 0$.*

The relationship between the four conjectures mentioned above can be summarized as follows :

$$BSC \quad \Rightarrow \quad TSC \Rightarrow ACQC \Rightarrow HMC.$$

In Section 15 we describe essential features of our computer program which was used to carry out the searches mentioned in the body of the paper.

1 Aperiodic Auto-Correlation Functions

We shall denote finite sequences of integers by capital letters. If (say) A is such a sequence of length n, then its elements will be denoted by the corresponding lower case letter with indices $1, 2, \ldots, n$. Thus

$$A = a_1, a_2, \ldots, a_n \ . \tag{1.1}$$

The *aperiodic auto-correlation function* N_A of A is defined by:

$$N_A(i) = \sum_{j \in Z} a_j a_{i+j} \ , \ i \in Z \ ,$$

where $a_k = 0$ if $k < 1$ or $k > n$. Observe that $N_A(-i) = N_A(i)$ for all $i \in Z$.

To the sequence (1.1) we associate the polynomial

$$A(x) = a_1 + a_2 x + \ldots + a_n x^{n-1} \ ,$$

which we view as an element of the Laurent polynomial ring $Z[x, x^{-1}]$. A simple computation shows that

$$A(x)A(x^{-1}) = \sum_{i \in Z} N(i)x^i \ .$$

For later use, it will be convenient to define the *norm* of A to be

$$N(A) = A(x)A(x^{-1}) \in Z[x, x^{-1}].$$

We shall be especially interested in $\{\pm 1\}$-sequences A, i.e., those for which $a_i = \pm 1$. When displaying such sequences, we shall write $+$ for $+1$ and $-$ for -1.

If A is the sequence (1.1), then $-A$ will denote the *negated* sequence:

$$-A = -a_1, -a_2, \ldots, -a_n.$$

By A, B we denote the concatenation of sequences A and B.

2 Golay Sequences

Let $(A; B)$ be an ordered pair of $\{\pm 1\}$-sequences of length $n \geq 1$. They are called *Golay sequences* if

$$N(A) + N(B) = 2n \ . \tag{2.1}$$

Let $GS(n)$ denote the set of all Golay sequences of length n. If $GS(n) \neq \emptyset$, we say that n is a *Golay number*.

By evaluating the Laurent polynomials $N(A)$ and $N(B)$ at $x = 1$, we obtain from (2.1) the equality

$$a^2 + b^2 = 2n \ , \tag{2.2}$$

where a and b are the sums of A and B, respectively. Thus $2n$, and consequently n, must be a sum of two squares. It is also easy to show that if $n > 1$ is a Golay number, then n must be even. Furthermore, Eliahou, Kervaire, and Saffari [8] have shown that Golay numbers are not divisible by any prime $p \equiv 3 \pmod 4$. For a simpler proof of this result see [7]. These negative results show that relatively few natural numbers are Golay numbers. On the positive side, it is known that all numbers

$$n = 2^a \cdot 10^b \cdot 26^c \ , \tag{2.3}$$

where a, b, c are non-negative integers, are Golay numbers. Golay himself [11, 12] has shown that 2,10 and 26 are Golay numbers. Turyn [27] has constructed a map

$$GS(m) \times GS(n) \rightarrow GS(mn) .$$

It follows that all numbers (2.3) are indeed Golay numbers.

If $(A;B) \in GS(n)$, $n > 1$, one can easily show that

$$a_i a_{n+1-i} + b_i b_{n+1-i} = 0 \ , \ 1 \leq i \leq n \, .$$

This condition is useful when programming a computer to search for Golay sequences (see [2]).

After applying the above mentioned existence results and non-existence conditions to the integers $n \leq 100$, only the six cases $n = 34$, 50, 58, 68, 74, 82 remain undecided. Exhaustive computer searches carried out by Andres [1], James [16], and more recently by Eliahou, Kervaire, and Saffari [9] have shown that 34, 50, 58 and 68 are not Golay numbers.

3 Base Sequences

Let $m, n \geq 0$ and let $(A;B;C;D)$ be a quadruple of $\{\pm 1\}$-sequences with A and B of length m, and C and D of length n. We say that they are *base sequences* if

$$N(A) + N(B) + N(C) + N(D) = 2(m+n) \, . \tag{3.1}$$

Let $BS(m,n)$ denote the set of all such base sequences. If $(A;B) \in GS(m)$ and $(C;D) \in GS(n)$, then $(A;B;C;D) \in BS(m,n)$. This gives an embedding :

$$GS(m) \times GS(n) \rightarrow BS(m,n), \tag{3.2}$$

and in particular a bijection

$$GS(n) \rightarrow BS(n,0). \tag{3.3}$$

There is another important embedding

$$GS(n) \rightarrow BS(n+1,n) \tag{3.4}$$

defined by

$$(A;B) \mapsto (A,+;A,-;B;B). \tag{3.5}$$

Because of the bijection (3.3) we may view Golay sequences as a special case of base sequences. Base sequences are used extensively in various constructions of Hadamard matrices. We refer the interested reader to the survey paper [25] and also the papers [4, 18, 28]. The most important case for these applications is the case of $BS(m,n)$ with $m+n$ odd.

By evaluating the Laurent polynomials in (3.1) at $x = 1$, we obtain the equality

$$a^2 + b^2 + c^2 + d^2 = 2(m+n) \tag{3.6}$$

where a, b, c, d are the sums of the sequences A, B, C, D, respectively. Thus every quadruple $(A; B; C; D) \in BS(m, n)$ determines the quadruple of integers a, b, c, d satisfying (3.6). The squares a^2, b^2, c^2, d^2 form a partition of $2(m+n)$. We shall say that this partition is *associated* with $(A; B; C; D)$. Two partitions which differ only in the order in which the squares are listed will not be considered as different.

Note that $BS(0, 0)$ is a singleton. There is a map

$$BS(m, n) \to BS(m+n, m+n) \tag{3.7}$$

defined by

$$(A; B; C; D) \mapsto (A, C; A, -C; B, D; B, -D).$$

By taking $m = n$ in (3.7), we see that it suffices to prove $ACQC$ for odd n only.

It is well known that $ACQC$ implies HMC. Indeed, if $(A; B; C; D) \in BS(n, n)$, construct four circulant matrices of order n having these sequences as their first rows, and insert these circulants into the Goethals–Seidel array (see [10]) to obtain a Hadamard matrix of order $4n$.

4 Equivalence Classes in $BS(m, n)$

If A is as in (1.1), we write A' for the *reversed* sequence :

$$A' = a_n, a_{n-1}, \ldots, a_1.$$

It is easy to see that

$$N(-A) = N(A') = N(A). \tag{4.1}$$

We also introduce the *alternated* sequence A^* by :

$$A^* = a_1, -a_2, a_3, -a_4, \ldots, (-1)^{n-1}a_n.$$

Note that

$$N_{A^*}(i) = (-1)^i N_A(i), \tag{4.2}$$

for all $i \in Z$.

It follows from (4.1) and (4.2) that if $(A; B; C; D) \in BS(m, n)$, then we can construct another member of $BS(m, n)$ by performing one of the following *elementary transformations* :

(i) negate one of A, B, C, D;

(ii) reverse one of A, B, C, D;

(iii) interchange two of the sequences of the same length;

(iv) alternate all of A, B, C, D.

Two members of $BS(m,n)$ are said to be *equivalent* if one can be obtained from the other by a finite sequence of elementary transformations.

In Section 3 we have associated to each $(A;B;C;D) \in BS(m,n)$ a partition of $2(m+n)$ into four squares. The elementary transformations of types (i), (ii) and (iii) do not change the associated partition. On the other hand, the elementary transformation of type (iv) may change this partition. For instance the base sequences in $BS(6,5)$ given by :

$$\begin{aligned} A = & \quad +,+,-,+,-,+; \\ B = & \quad +,+,+,-,-,-; \\ C = & \quad +,+,-,+,+ \ = D; \end{aligned}$$

have associated partition

$$22 = 2^2 + 0^2 + 3^2 + 3^2. \tag{4.3}$$

The partition associated with the alternated base sequences $(A^*;B^*;C^*;D^*)$ is

$$22 = 4^2 + 2^2 + 1^2 + 1^2. \tag{4.4}$$

5 Base Sequences $BS(n+1,n)$

Recall that it suffices to prove *ACQC* for sequences of odd length only. By taking $m = n+1$ in (3.7), we obtain the map

$$BS(n+1,n) \to BS(2n+1, 2n+1).$$

Hence *ACQC* is a consequence of *BSC*. It is known that *BSC* is true for all $n \leq 32$ (see [25] and our Table 1). Consequently *ACQC* holds for all $n \leq 66$. It is also known that *BSC* holds for all integers n given by (2.3). Indeed these numbers are Golay numbers and it suffices to apply the map (3.5). It follows that *ACQC* is true for all numbers $n = 2k+1$ where k is a Golay number.

Another important construction due to Yang [28, Theorem 4], gives a map :

$$BS(m+1,m) \times BS(n+1,n) \to BS(t,t),$$

where $t = (2m+1)(2n+1)$. Since we can take m and n to be arbitrary Golay numbers, this construction provides an infinite collection of odd integers for which *ACQC* holds.

It was shown in [19] that the sets $BS(2n-1,n)$ are non-empty for even $n \leq 24$. By taking $n = 24$ and by applying the map (3.7), it follows that $BS(71,71) \neq \emptyset$.

By using the results mentioned above, it is easy to check that $ACQC$ is true for all integers $n \leq 100$ except for the cases $n = 67, 73, 79, 83, 89, 97$ that remain undecided. The claim made in [5] that it is known that $BS(67,67) \neq \emptyset$ is in error.

A number of particular base sequences in $BS(n+1,n)$ for small n have been constructed in many places. For instance in [21] several members of $BS(n+1,n)$ are constructed for each n in the interval $19 \leq n \leq 24$. The claim made there, that the authors have exhibited enough elements of $BS(n+1,n)$ so that the associated partitions of $2(2n+1)$ into four squares exhaust all possibilities, is erroneous. For instance, if $n = 19$ they do not obtain the partition $76 = 8^2 + 3^2 + 2^2 + 1^2$.

All of these constructions, with a few exceptions, have been achieved by using a computer search. For that purpose the following fact established in [27] (see also [21]) is very useful. If $(A;B;C;D) \in BS(n+1,n)$, then

$$a_i a_{n+2-i} = b_i b_{n+2-i}\ ,\ 2 \leq i \leq n; \tag{5.1}$$

$$c_i c_{n+1-i} = d_i d_{n+1-i}\ ,\ 1 \leq i \leq n. \tag{5.2}$$

6 The Graphs Γ_n

The computational results indicate that the sets $BS(n+1,n)$ increase rapidly in size with n (although not monotonically). In this section we introduce a graph Γ_n which serves as a very crude measure for the richness of $BS(n+1,n)$.

The vertices of Γ_n are partitions of $2(2n+1)$ into four squares. More formally, the vertices of Γ_n are ordered quadruples (a,b,c,d) of integers such that $a \geq b \geq c \geq d \geq 0$ and $a^2+b^2+c^2+d^2 = 2(2n+1)$. For instance, Γ_5 has only two vertices : the partitions (4.3) and (4.4).

Two vertices, say v and w, will be joined by an edge (if $v = w$ this means that we are introducing a loop at the vertex v) if there exists $(A;B;C;D) \in BS(n+1,n)$ such that v and w are the partitions associated with $(A;B;C;D)$ and $(A^*;B^*;C^*;D^*)$, respectively. We shall also say that $(A;B;C;D)$ is a *witness* for the edge joining v and w. An edge of Γ_n may have many witnesses.

For simplicity we allow only one edge between any two different vertices and only one loop at each vertex. There is an obstruction of arithmetical nature to the existence of certain edges in Γ_n. This will be discussed in the next section.

We conclude this section with the description of Γ_5. The base sequences given in Section 4 show that the two vertices of Γ_5 are joined by an edge.

The base sequences

$$\begin{aligned} A &= +,+,+,-,-,+; \\ B &= +,+,-,+,-,-; \\ C &= +,+,+,-,+ \;= D; \end{aligned}$$

and

$$\begin{aligned} A &= +,+,+,-,+,+; \\ B &= +,+,+,-,+,-; \\ C &= +,+,+,-,-; \\ D &= +,-,+,+,-; \end{aligned}$$

show that Γ_5 has a loop at each of its vertices. Thus Γ_5 has the maximum number of possible edges.

7 Some General Facts about Γ_n

We establish here some general properties of the graphs Γ_n.

Proposition 7.1. *Let $(A, B, C, D) \in BS(n+1, n)$ with n even. If a, b, c, d are the sums of A, B, C, D, and a^*, b^*, c^*, d^* those of A^*, B^*, C^*, D^*, then*

(i) $c \equiv d$ *and* $c^* \equiv d^* \pmod 4$;

(ii) $c^* \equiv c + n \pmod 4$.

Proof: As n is even and a and b are odd, we have

$$a^2 + b^2 \equiv 2(2n+1) \equiv 2 \pmod 8.$$

By applying the equality (3.3) with $m = n+1$, we conclude that

$$c^2 + d^2 \equiv 0 \pmod 8.$$

As c and d are even, it follows that $c \equiv d \pmod 4$. Thus (i) is proved. To prove (ii) let

$$c' = c_1 + c_3 + \cdots + c_{n-1}\ , \quad c'' = c_2 + c_4 + \cdots + c_n\ ,$$

and observe that $c' \equiv c'' \equiv n/2 \pmod 2$. Since $c = c' + c''$ and $c^* = c' - c''$, we have $c - c^* = 2c'' \equiv n \pmod 4$. Thus (ii) holds. □

We can partition the set of vertices (a, b, c, d) of Γ_n, n even, into two classes : the *even vertices* and the *odd vertices* according to whether the two even integers among a, b, c, d are congruent to 0 or 2 (mod 4).

Corollary 7.2. *If $n \equiv 0 \pmod 4$, then no edge of Γ_n can be incident with an even vertex and an odd vertex.*

Corollary 7.3. *If $n \equiv 2 \pmod 4$, then every edge of Γ_n is incident with an even vertex and an odd vertex. Thus Γ_n is a bipartite graph, and in particular it has no loops.*

Both corollaries follow immediately from Proposition (7.1). □

There is no simple formula for the number of vertices ν of Γ_n as a function of n. A more general question has been studied by E. Grosswald [14]. He denotes by $P_k(n)$ the number of partitions of n into k squares (allowing zeros). He shows how to determine the sets $S_{k,m}$ of positive integers n such that $P_k(n) = m$. By using his notation, we have $\nu = P_4(4n + 2)$. D. H. Lehmer [23] (see also [13, p. 85]) has shown that $P_4(n) > n/48$, if n is not divisible by 4. It follows that $n < 12\nu$. Hence $\nu = 1$ only for $n = 0, 1, 3$; $\nu = 2$ only for $n = 2, 5, 7, 11$; and $\nu = 3$ only for $n = 4, 6, 9, 15$.

8 Description of Γ_n for $n \leq 27$

In graph theory one denotes by K_m the complete graph on m vertices. This means that every pair of vertices is joined by a single edge and that there are no loops. We shall denote by K_m^0 the graph which is obtained from K_m by adding a loop at each of the vertices. Recall also that graph theorists denote by $K_{m,n}$ the complete bipartite graph on $m + n$ vertices. If Γ and Δ are finite graphs, we shall denote by $\Gamma + \Delta$ their disjoint union. The notation $\Gamma \simeq \Delta$ will be used to indicate that the graphs Γ and Δ are isomorphic.

We can now state our result.

Theorem 8.1. *Let ν (resp. ν_0, ν_1) be the number of vertices (resp. even, odd vertices) of Γ_n, $n \leq 27$. Then*

$$\begin{aligned}
&\Gamma_n \simeq K_\nu^0 \text{ if } n \text{ is odd;}\\
&\Gamma_n \simeq K_{\nu_0,\nu_1} \text{ if } n = 2m,\ m \text{ odd;}\\
&\Gamma_n \simeq K_{\nu_0}^0 + K_{\nu_1}^0 \text{ if } n = 0, 16, 20, 24;\\
&\Gamma_4 \simeq K_1^0 + \Gamma' \ ;\ \Gamma_8 \simeq K_1^0 + \Gamma'' \ ;\ \Gamma_{12} \simeq K_1^0 + \Gamma''';
\end{aligned}$$

where Γ' is obtained from K_2^0 by deleting both loops, Γ'' is obtained from K_3^0 by deleting one of the three edges which are not loops, and Γ''' is obtained from K_4^0 by deleting one of the four loops.

Proof: In the Appendix we give the list (in encoded form) of witnesses for all edges of Γ_n whose existence is asserted in the theorem. The encoding used there will be explained in the next section. Corollary (7.3) determines the partition of Γ_n into two connected components when $n > 0$ is divisible by 4. Similarly, the partition of the complete bipartite graphs Γ_n when $n = 2m$, m odd, is determined by Corollary (7.2).

It remains to show that Γ_n has no other edges. This is clear if n is odd since K_m^0 has maximum number of edges. If $n \equiv 2 \pmod 4$, the validity of the theorm follows from Corollary (7.2). If $n \equiv 0 \pmod 4$ and $n \neq 4, 8, 12$, then the assertion follows from Corollary (7.3).

If $n = 4$, Corollary (7.3) shows that there is no edge joining the vertex $(3,2,2,1)$ to $(3,3,0,0)$ or $(4,1,1,0)$. One still has to show that there are no loops at the last two vertices. Assume that there is a loop at $(3,3,0,0)$ and let $(A;B;C;D) \in BS(5,4)$ provide such a loop. Using the notations from the previous section, we would have

$$a' + a'' = \pm 3 \ , \ a' - a'' = \pm 3.$$

Since a' is odd, it follows that $a' = \pm 3$ and similarly $b' = \pm 3$. This implies that $N_A(4) + N_B(4) = 2$, a contradiction. One can show similarly that there is no loop at $(4,1,1,0)$. In the remaining two cases $n = 8, 12$ we rely on our computer search. □

9 Encoding of $BS(n+1, n)$

It is inconvenient to display in full the base sequences $(A;B;C;D) \in BS(n+1, n)$ for large n. For that purpose we shall use the following encoding scheme. Consider first the pair $(A;B)$. We decompose this pair into quads

$$\begin{bmatrix} a_i & a_{n+2-i} \\ b_i & b_{n+2-i} \end{bmatrix}, \ i = 1, 2, \ldots, \left\lceil \frac{n+1}{2} \right\rceil \tag{9.1}$$

and, if $n = 2m$ is even, the central column

$$\begin{bmatrix} a_{m+1} \\ b_{m+1} \end{bmatrix}. \tag{9.2}$$

By replacing $(A;B;C;D)$ with an equivalent quadruple, we may assume that the first quad in (9.1) is the following

$$\begin{bmatrix} + & + \\ + & - \end{bmatrix}. \tag{9.3}$$

We shall encode this quad with 0. The restriction (5.1) implies that all other quads in (9.1) must be one of the following which we encode with integers $1, \ldots, 8$:

$$1 = \begin{bmatrix} + & + \\ + & + \end{bmatrix}, \quad 2 = \begin{bmatrix} + & + \\ - & - \end{bmatrix}, \quad 3 = \begin{bmatrix} - & + \\ - & + \end{bmatrix}, \quad 4 = \begin{bmatrix} + & - \\ - & + \end{bmatrix},$$

$$5 = \begin{bmatrix} - & + \\ + & - \end{bmatrix}, \quad 6 = \begin{bmatrix} + & - \\ + & - \end{bmatrix}, \quad 7 = \begin{bmatrix} - & - \\ + & + \end{bmatrix}, \quad 8 = \begin{bmatrix} - & - \\ - & - \end{bmatrix}.$$

The central column (if present) will be encoded as follows:

$$0 = \begin{bmatrix} + \\ + \end{bmatrix}, \ 1 = \begin{bmatrix} + \\ - \end{bmatrix}, \ 2 = \begin{bmatrix} - \\ + \end{bmatrix}, \ 3 = \begin{bmatrix} - \\ - \end{bmatrix}.$$

If $n = 2m$ is even, then the pair $(A; B)$ will be encoded by the symbol

$$\delta_1\delta_2 \cdots \delta_m\delta_{m+1} \tag{9.4}$$

where δ_i, $1 \le i \le m$, is the digit (1-8) representing the i-th quad in (9.1), and δ_{m+1} is the digit (0-3) representing the central column (9.2). If $n = 2m+1$ is odd, then the pair $(A; B)$ will be encoded by (9.4) where now, for each i, δ_i represents the i-th quad in (9.1).

We shall use the same scheme to encode the pair $(C; D)$. The only difference is that the quad 0 will not occur because of (5.2). The quadruple $(A; B; C; D)$ will be encoded by a pair of symbols of type (9.4). For instance, the base sequences from Section 4 are encoded as 065 ; 113. In the Appendix we list certain base sequences using this encoding scheme.

10 Turyn Sequences

Recall that A' denotes the reverse of the sequence A (see Section 4). We say that a sequence A is *symmetric* (resp. *skew-symmetric*) if $A' = A$ (resp. $A' = -A$). Note that a skew-symmetric $\{\pm 1\}$-sequence must have even length.

We say that $(A; B; C; D) \in BS(n+1, n)$ are *Turyn sequences* if :

(i) $A' = A$ and $C' = -C$, if n is even;

(ii) $A' = -A$ and $C' = C$, if n is odd.

Then the conditions (5.1) and (5.2) imply that also :

(iii) $b_{n+2-i} = b_i$ for $2 \le i \le n$ and $D' = -D$, if n is even;

(iv) $b_{n+2-i} = -b_i$ for $2 \le i \le n$ and $D' = D$, if n is odd.

Our definition is a slight modification of the usual one (e.g. see [6]) where it is also required that

$$a_1 = b_1 = c_1 = d_1 = a_2 = b_2 = + \,. \tag{10.1}$$

Given Turyn sequences (in the sense of our definition) it is easy to apply a few elementary transformations in order to achieve normalization (10.1).

We denote by $TU(n)$ the subset of $BS(n+1, n)$ consisting of Turyn sequences. Let $TU^*(n) \subset BS(n+1, n)$ consist of all quadruples $(A^*; B^*; C^*; D^*)$ where $(A; B; C; D) \in TU(n)$. The set $TU^*(n)$ has a very simple characterization : it consists of all base sequences $(A; B; C; D) \in BS(n+1, n)$ such that all four sequences $A; B; C; D$ are symmetric except that $b_{n+1} = -b_1$.

It is known that $TU(n) \neq \emptyset$ for $n \le 7$ and also for $n = 12$ and 14 (see [6, 25]). On the other hand, if n is odd and $2n + 1$ is not a sum of two squares, then $TU(n)$ and $TU(n+1)$ are empty (see e.g. [6, Lemma 8]). It is also known that $TU(n) = \emptyset$ for all n such that $8 \leq n \leq 51$ and $n \neq 12, 14$. For $n \leq 29$ this was reported in [21], and for $n \leq 41$ in [6]. We have carried out exhaustive computer search in the remaining cases $n = 42, 43, 44, 45, 48, 49, 50, 51$ and did not find any Turyn sequences.

11 Normal Sequences

We view base sequences as fundamental objects, and so we shall include under the same umbrella some other important classes of sequences. In this section and the next we describe two such classes which were introduced originally (in a different form) by C. H. Yang [28].

We say that the base sequences $(A; B; C; D) \in BS(n+1, n)$ are *normal sequences* if $a_{n+1} = +$, $b_{n+1} = -$, and $a_i = b_i$ for $1 \leq i \leq n$. Thus normal sequences have the form

$$(A = X, +; B = X, -; C; D). \tag{11.1}$$

We denote by $NS(n)$ the subset of $BS(n+1, n)$ consisting of normal sequences. From normal sequences (11.1) we can extract the following triple of sequences of length n:

$$X; \; Y = (C+D)/2 \; ; \; Z = (C-D)/2. \tag{11.2}$$

This means that

$$y_i = (c_i + d_i)/2 \; , \; z_i = (c_i - d_i)/2$$

for all $i = 1, 2, \ldots, n$. As C and D are $\{\pm 1\}$-sequences, Y and Z are $\{0, \pm 1\}$-sequences. Furthermore Y and Z are *disjoint* in the sense that $y_i z_i = 0$ for each i.

Since $(A; B; C; D)$ satisfy (3.1) with $m = n + 1$, it is easy to see that

$$N(X) + N(Y) + N(Z) = 2n, \tag{11.3}$$

(see e.g. [19, Theorem 5] for details).

Conversely, given sequences $(X; Y; Z)$ of length n, with X a $\{\pm 1\}$-sequence and Y and Z disjoint $\{0, \pm 1\}$-sequences, satisfying (11.3), then

$$(A = X, +; B = X, -; C = Y + Z; D = Y - Z) \in NS(n) \; .$$

(The usual practise is to refer to the triple (11.2) as normal sequences. In essentially this form they were introduced by Yang [28].)

If $NS(n) \neq \emptyset$, then (11.3) implies that $2n$ is a sum of three squares. On the other hand, the embedding (3.4) maps $GS(n)$ into $NS(n)$, and so $NS(n) \neq \emptyset$ for all numbers n given by (2.3). There is also an embedding

$$TU(n) \rightarrow NS(2n+1) \; .$$

In order to describe this embedding we define interleaved sequence A/C by

$$A/C = a_1, c_1, a_2, c_2, \ldots, c_n, a_{n+1}$$

if A has length $n+1$ and C length n. Then the above embedding is defined by

$$(A; B; C; D) \mapsto (A/C, +; A/C, -; B/D; B/-D) \qquad (11.4)$$

if n is even, and

$$(A; B; C; D) \mapsto (B/C, +; B/C, -; A/D; A/-D) \qquad (11.5)$$

if n is odd.

Consequently $NS(2n+1) \neq \emptyset$ for all $n \leq 7$ and $n = 12$ and 14. For $n \leq 30$ it is known that $NS(n) = \emptyset$ only for

$$n = 6, 14, 17, 21, 22, 23, 24, 27, 28, 30 .$$

This claim for $n = 17, 21, 22, 23$ is established in [6], and for $n = 24$ it was shown by M. Gysin (ibid). The search for $n = 27$ and 28 was carried out by us.

12 Near-Normal Sequences

We define *near-normal sequences* to be base sequences $(A; B; C; D) \in BS(n+1, n)$, with n even, such that

$$b_i = (-1)^{i-1} a_i, \ 1 \leq i \leq n,$$

and

$$a_{n+1} = +, \ b_{n+1} = - .$$

We denote by $NN(n)$ the set of all near-normal sequences in $BS(n+1, n)$. Again the original definition of C. H. Yang [28] is different. For the relationship between this definition and the usual one we refer the reader to [20]. Yang has stated in the above paper that it is likely that $NN(n) \neq \emptyset$ for all even integers $n \geq 0$. This is known to be true for $n \leq 30$, but nothing is known about $NN(n)$ for even $n > 30$. For even $n \leq 22$ examples of near-normal sequences are given in [28], and for $n = 24, 26, 28$ and 30 see [19]. We have enumerated (up to equivalence) all near-normal sequences in $BS(n+1, n)$ for all even integers $n \leq 28$. In our Appendix the reader will find examples of near-normal sequences $NN(n)$ for all even $n \leq 30$.

13 Some $BS(m,n)$ are empty

So far we have discussed mainly the base sequences $BS(m,n)$ with $m = n+1$. Other cases (with $m+n$ odd) are also important for the construction of Hadamard matrices. Turyn [27] proposed a method for constructing base sequences in $BS(2n-1,n)$. It turns out (see [19]) that this method is feasible only for n even (and $n=1$). By using this method, it was shown that $BS(2n-1,n) \neq \emptyset$ for even $n \leq 24$. (The claim made in [19, Remark 2], is in error. For instance, when $n = 12$, the number $2(3n-1) = 70$ has five partitions into four squares, while only two of them are listed in their Table 1.)

Proposition 13.1. *If $m \geq 2n$, $n > 1$, and $m+n$ is odd, then $BS(m,n)$ is empty.*

Proof: Assume that there exist base sequences $(A;B;C;D) \in BS(m,n)$. If $x, y \in \{\pm 1\}$, then $xy \equiv x+y-1 \pmod 4$. Hence for $s = n, n+1, \ldots, m-1$ we have

$$\begin{aligned} N_A(s) + N_B(s) &= \textstyle\sum_{i=1}^{m-s}(a_i a_{i+s} + b_i b_{i+s}) \\ &\equiv \textstyle\sum_{i=1}^{m-s}(a_i + b_i + a_{i+s} + b_{i+s} - 2) \pmod 4 \\ &= \textstyle\sum_{i=1}^{m-s}(a_i + b_i + a_{m+1-i} + b_{m+1-i} - 2). \end{aligned}$$

Since (3.1) implies that $N_A(s) + N_B(s) = 0$ for the above values of s, we conclude that

$$\sum_{i=1}^{m-s}(a_i + b_i + a_{m+1-i} + b_{m+1-i} - 2) \equiv 0 \pmod 4 \tag{13.2}$$

for $s = n, n+1, \ldots, m-1$. It follows that

$$a_i + b_i + a_{m+1-i} + b_{m+1-i} \equiv 2 \pmod 4 \tag{13.3}$$

for $i = 1, 2, \ldots, m-n$ and, by symmetry, also for $i = m, m-1, \ldots, n+1$. As $m \geq 2n$, (13.3) holds for all $i = 1, 2, \ldots, m$.

If m is odd, we obtain a contradiction from (13.3) by setting $i = (m+1)/2$. Hence m must be even, and so n is odd and $n \geq 3$. By using the congruences (13.3), we find that

$$\begin{aligned} N_A(s) + N_B(s) &= \textstyle\sum_{i=1}^{m-s}(a_i a_{i+s} + b_i b_{i+s}) \\ &\equiv \textstyle\sum_{i=1}^{m-s}(a_i + b_i + a_{i+s} + b_{i+s} - 2) \pmod 4 \\ &= \textstyle\sum_{i=1}^{m-s}(a_i + b_i + a_{m+1-i} + b_{m+1-i} - 2) \\ &\equiv 0 \pmod 4, \end{aligned}$$

for $s = 1, 2, \ldots, m-1$. Now (3.1) implies that

$$N_C(s) + N_D(s) \equiv 0 \pmod 4$$

for $s = 1, 2, \ldots, n-1$. By the same argument as in the proof of (13.3), we can show that

$$c_i + d_i + c_{n+1-i} + d_{n+1-i} \equiv 2 \pmod 4 \tag{13.4}$$

for $i = 1, 2, \ldots, n$. By setting $i = (n+1)/2$, we obtain again a contradiction. □

In [19, Theorem 7] the authors show that their method cannot be used to construct base sequences $BS(2n-1, n)$ for odd $n > 1$. This is not surprising, as we can show that these sets are empty.

Proposition 13.5. *If $n > 1$ is odd, then $BS(2n-1, n)$ is empty.*

Proof: Assume that there exist base sequences $(A; B; C; D) \in BS(2n-1, n)$. As in the proof of Proposition (13.1), we can show that

$$a_i + b_i + a_{2n-i} + b_{2n-i} \equiv 2 \pmod 4$$

for $i = n+1, n+2, \ldots, 2n-1$ (and, by symmetry, also for $i = 1, 2, \ldots, n-1$). Hence for $s = 1, 2, \ldots, n-1$ we have

$$\begin{aligned} N_A(s) + N_B(s) &= \textstyle\sum_{i=1}^{2n-1-s} (a_i a_{i+s} + b_i b_{i+s}) \\ &\equiv \textstyle\sum_{i=1}^{2n-1-s} (a_i + b_i + a_{i+s} + b_{i+s} - 2) \pmod 4 \\ &= \textstyle\sum_{i=1}^{2n-1-s} (a_i + b_i + a_{2n-i} + b_{2n-i} - 2) \\ &\equiv 2 \pmod 4. \end{aligned}$$

By (3.1), we conclude that

$$N_C(s) + N_D(s) \equiv 2 \pmod 4$$

for $s = 1, 2, \ldots, n-1$. Since

$$\begin{aligned} N_C(s) + N_D(s) &= \textstyle\sum_{i=1}^{n-s} (c_i c_{i+s} + d_i d_{i+s}) \\ &\equiv \textstyle\sum_{i=1}^{n-s} (c_i + d_i + c_{i+s} + d_{i+s} - 2) \pmod 4 \\ &= \textstyle\sum_{i=1}^{n-s} (c_i + d_i + c_{n+1-i} + d_{n+1-i} - 2) \end{aligned}$$

we have

$$\sum_{i=1}^{n-s} (c_i + d_i + c_{n+1-i} + d_{n+1-i} - 2) \equiv 2 \pmod 4$$

for $s = 1, 2, \ldots, n-1$. It follows that (13.4) holds for $i = 2, 3, \ldots, n-1$. By setting $i = (n+1)/2$ in (13.4), we obtain a contradiction. □

14 T-sequences

T-sequences are ordered quadruples $(A;B;C;D)$ of $\{0,\pm1\}$-sequences of the same length, say n, such that :

(i) $N(A)+N(B)+N(C)+N(D)=n$;

(ii) for each $i=1,2,\ldots,n$ exactly one of a_i,b_i,c_i,d_i is non-zero.

We denote by $TS(n)$ the set of all T-sequences of length n. They are used extensively for the construction of orthogonal designs $OD(4n;n,n,n,n)$ (see [4, 10, 18, 24, 25]). The embedding $TS(n)\to BS(n,n)$, given by

$$(A;B;C;D)\to(A+B+C+D;A+B-C-D;A-B+C-D;A-B-C+D),$$

shows that TSC implies $ACQC$. More generally, Yang [28, Theorem 2*] has constructed a map :

$$NS(m)\times TS(n)\to BS(mn,mn)\ .$$

The problem of constructing T-sequences was originally considered by R. J. Turyn [27]. He observed that there is a map

$$BS(m,n)\to TS(m+n) \tag{14.1}$$

given by

$$(A;B;C;D)\to((A+B)/2\,,\,0_n;(A-B)/2,0_n;0_m,(C+D)/2;0_m,(C-D)/2)$$

where 0_k denotes the zero sequence of length k. More generally, C. H. Yang [28, Theorems 1 and 3] has constructed maps :

$$NS(s)\times BS(m,n)\to TS((2s+1)(m+n)), \tag{14.2}$$

and

$$NN(s)\times BS(m,n)\to TS((2s+1)(m+n)), \tag{14.3}$$

known as Yang multiplications. In view of these results, it is convenient to define a *Yang number* to be a positive odd integer $2s+1$ such that $NS(s)$ or $NN(s)$ is not empty. Hence if t is a Yang number and $BS(m,n)\neq\emptyset$, then we can conclude that $TS(t(m+n))\neq\emptyset$.

From Sections 11 and 12 it follows that :

(a) if $t\leq 61$ is a positive odd integer, then t is a Yang number if and only if $t\neq 35,43,47,55$;

(b) if s is a Golay number, then $2s+1$ is a Yang number.

In view of the map $TS(n) \to TS(2n)$ given by

$$(A;B;C;D) \to (A+B,0_n;C+D,0_n;0_n,A-B;0_n,C-D),$$

we are mainly interested in constructing T-sequences of odd length. By setting $m = n+1$ in (14.1) and by taking into account the above map, we conclude that *BSC* implies *TSC*.

By using known results about $BS(n+1,n)$ and $BS(2n-1,n)$, mentioned in previous sections, and by applying the maps (14.1-3), it is easy to show that *TSC* holds for all odd integers $n < 200$ except perhaps $n = 67$ and the primes $n > 71$.

15 Outline of the Algorithm

We represent the known terms of the sequences $(A;B;C;D)$ by using the quad notation of Section 9. We assume that A and B have length $n+1$, while C and D have length n.

After k steps the following elements are known:

$$\begin{array}{l} a_1,a_2,\ldots,a_{k+1},a_{n-k+1},a_{n-k+2},\ldots,a_n,a_{n+1}; \\ b_1,b_2,\ldots,b_{k+1},b_{n-k+1},b_{n-k+2},\ldots,b_n,b_{n+1}; \\ c_1,c_2,\ldots,c_k,c_{n-k+1},c_{n-k+2},\ldots,c_n; \\ d_1,d_2,\ldots,d_k,d_{n-k+1},d_{n-k+2},\ldots,d_n; \end{array}$$

and we also know that the equation

$$N_A(s)+N_B(s)+N_C(s)+N_D(s)=0 \tag{15.1}$$

is satisfied for $s = n, n-1, \ldots, n-k$. We proceed to choose next two quads

$$\begin{bmatrix} a_{k+2} & a_{n-k} \\ b_{k+2} & b_{n-k} \end{bmatrix} \text{ and } \begin{bmatrix} c_{k+1} & c_{n-k} \\ d_{k+1} & d_{n-k} \end{bmatrix}$$

so that the equation (15.1) is also satisfied for $s = n-k-1$.

At the same time we keep track of the partial values of the first autocorrelation, i.e., the expression

$$\begin{array}{ll} \sum_{i=1}^{k} & (a_ia_{i+1}+b_ib_{i+1}+a_{n+1-i}a_{n+2-i}+b_{n+1-i}b_{n+2-i}) \\ & +\sum_{i=1}^{k-1}(c_ic_{i+1}+d_id_{i+1}+c_{n-i}c_{n+1-i}+d_{n-i}d_{n+1-i}) \, . \end{array}$$

If the absolute value of this expression becomes too big, we switch to our backtracking procedure. When $k+1$ becomes equal to $n-k-1$ or $n-k$, we choose the remaining unknown elements and test whether the remaining equations (15.1) are satisfied or not. Although one can determine in advance the possible values for the sums a, b, c, d of the sequences $(A;B;C;D)$, we do not make any use whatsoever of that fact.

In addition to the partial values of the first auto-correlation, we also store the values of the expressions

$$\sum_{i=2}^{k} (a_i a_{i+n-k} + b_i b_{i+n-k}) + \sum_{i=2}^{k-1} (c_i c_{i+n-k} - d_i d_{i+n-k}),$$

in order to avoid the task of re-computing them after backtracking.

The search for normal or near-normal sequences is easier because at each step the number of possible branches is smaller.

Appendix

For each n in the range $0, 1, 2, \ldots, 27$ we list at least one witness $(A; B; C; D) \in BS(n+1, n)$ for each edge of Γ_n. For $n = 28, \ldots, 32$ we list witnesses for all known edges of Γ_n. The integer n is given in the first column of Table 1 below. The second (resp. third) column gives, in encoded form, the first (resp. last) two sequences of the witness. The encoding scheme is explained in Section 9.

As an example let us consider the last set of base sequences given below for $n = 8$. We see that the first two sequences A and B are encoded as 06142. Since $n = 8$ is even the digit 2 is encoding the central elements of A and B. In this case we have

$$A = +, +, +, +, -, -, +, -, + ;$$
$$B = +, +, +, -, +, +, +, -, - .$$

Similarly the last two sequences, C and D are encoded as 1675, and so

$$C = +, +, -, -, +, -, -, + ;$$
$$D = +, +, +, +, -, +, -, + .$$

The fourth column gives the sums a, b, c, d of these four sequences. In the above case they are :

$$a = 3,\ b = 3,\ c = 0,\ d = 4.$$

The fifth column gives the sums a^*, b^*, c^*, d^*, of the alternated sequences A^*, B^*, C^*, D^*. In our example,

$$a^* = 3,\ b^* = 3,\ c^* = 0,\ d^* = -4.$$

These base sequences show that there is a loop at the vertex $(4, 3, 3, 0)$ of Γ_8.

In the last column, the symbol *ns* resp. *nn* indicates that the base sequences are normal resp. near-normal. If an edge has a witness in $NS(n)$ or $NN(n)$ we always give such a witness. Consequently some edges have two witnesses in the table.

Table 1
Some base sequences $BS(n+1, n)$

n	A & B	C & D	a, b, c, d	a^*, b^*, c^*, d^*	
0	1		$1, -1, 0, 0$	$1, -1, 0, 0$	*ns*, *nn*
1	0	0	$2, 0, 1, 1$	$0, 2, 1, 1$	*ns*
2	00	6	$3, 1, 0, 0$	$1, -1, 2, 2$	*ns*
	01	6	$3, -1, 0, 0$	$1, 1, 2, 2$	*nn*
3	06	11	$2, 0, 3, 1$	$-2, 0, 1, 3$	*ns*
4	060	16	$3, 1, 2, 2$	$3, 1, -2, -2$	*ns*
	073	12	$-1, 1, 4, 0$	$3, -3, 0, 0$	*nn*
5	016	640	$4, 2, 1, 1$	$2, 4, 1, 1$	*ns*
	065	113	$2, 0, 3, 3$	$-4, 2, -1, -1$	
	064	160	$2, 0, 3, 3$	$0, -2, 3, 3$	
6	0512	127	$3, 3, 2, 2$	$5, 1, 0, 0$	*nn*
	0760	167	$1, 3, 0, 4$	$3, -3, -2, -2$	
7	0616	1232	$4, 2, 3, 1$	$-4, -2, 1, 3$	*ns*
	0613	1673	$4, 2, -1, 3$	$0, 2, 1, 5$	*ns*
	0618	1261	$2, 0, 5, -1$	$-2, 0, -1, 5$	*ns*
8	06633	1163	$1, -1, 4, 4$	$1, -1, 4, 4$	*ns*
	05850	1163	$1, -1, 4, 4$	$1, -1, 4, 4$	*nn*
	08110	1866	$5, 3, 0, 0$	$5, 3, 0, 0$	*ns*
	06183	1271	$1, -1, 4, 4$	$5, 3, 0, 0$	*ns*
	07643	1641	$-1, 1, 4, 4$	$3, -3, 0, -4$	*nn*
	06151	1618	$5, 1, 2, 2$	$5, 1, -2, -2$	
	06142	1675	$3, 3, 0, 4$	$3, 3, 0, -4$	
9	06136	16650	$4, 2, 3, 3$	$2, 4, 3, 3$	*ns*
	06581	11671	$2, 0, 3, 5$	$-4, 2, 3, -3$	
	06583	11631	$0, -2, 5, 3$	$-6, 0, 1, -1$	
	06187	16131	$0, 2, 5, 3$	$-2, 0, 5, 3$	
	01246	66540	$6, 0, 1, 1$	$0, 6, 1, 1$	
	01675	61530	$2, 4, 3, 3$	$0, 6, -1, -1$	
10	061633	12671	$3, 1, 4, 4$	$5, 3, 2, 2$	*ns*
	056732	11726	$-1, 3, 4, 4$	$5, -3, 2, 2$	
	058511	11635	$3, -1, 4, 4$	$1, 1, 2, 6$	
	061740	12685	$3, 5, 2, -2$	$5, -1, 0, 4$	
	064240	16573	$5, -1, 0, 4$	$-1, 1, -6, -2$	

n	A & B	C & D	a, b, c, d	a^*, b^*, c^*, d^*	
11	061618	126232	$4, 2, 5, -1$	$-4, -2, -1, 5$	ns
	061774	126353	$0, 6, 3, -1$	$-4, 2, 1, 5$	
	061624	126332	$6, 0, 3, 1$	$-6, 0, 1, 3$	
12	0686130	115763	$3, 1, 2, 6$	$3, 1, 2, 6$	ns
	0585140	115763	$3, 1, 2, 6$	$3, 1, 2, 6$	nn
	0764870	167162	$-3, 3, 4, 4$	$5, -5, 0, 0$	nn
	0685871	117266	$-3, -3, 4, 4$	$1, -7, 0, 0$	
	0686240	116723	$3, -3, 4, 4$	$3, -3, 4, 4$	
	0612760	128287	$5, 3, 0, -4$	$1, 7, 0, 0$	
	0617220	126876	$7, 1, 0, 0$	$7, 1, 0, 0$	
	0617212	126857	$5, 5, 0, 0$	$5, -3, 0, 4$	
	0647373	126716	$-3, 3, 4, 4$	$5, -5, 0, 0$	
	0737510	186672	$1, 7, 0, 0$	$5, -5, 0, 0$	
	0716872	187651	$-3, 5, 0, 4$	$5, -3, -4, 0$	
13	0618616	1613551	$4, 2, 5, 3$	$2, 4, 5, 3$	ns
	0161633	6484150	$6, 4, 1, 1$	$4, 6, 1, 1$	ns
	0614642	1286351	$6, 0, 3, -3$	$-4, 6, -1, 1$	
	0617874	1271662	$-2, 4, 3, 5$	$0, -2, -5, 5$	
	0618824	1265152	$2, -4, 5, 3$	$0, -2, 1, 7$	
	0617824	1265620	$2, 0, 7, -1$	$0, -2, -1, 7$	
	0616358	1267113	$2, 0, 5, 5$	$-4, -6, 1, 1$	
	0616234	1265760	$6, 0, 3, 3$	$0, -2, -1, 7$	
	0615146	1268562	$6, 4, 1, -1$	$0, 2, 1, 7$	
	0618557	1613540	$0, 2, 5, 5$	$-2, 0, 5, 5$	
	0617413	1618861	$4, 6, 1, -1$	$-2, -4, 5, 3$	
	0613647	1678373	$2, 4, -5, 3$	$0, 6, 3, 3$	
	0612646	1675230	$6, 0, 3, 3$	$0, 6, 3, 3$	
	0648276	1262282	$0, -2, 5, -5$	$2, 0, 1, 7$	
	0165243	6151653	$6, 0, 3, 3$	$0, 2, -5, -5$	
14	05673512	1172336	$1, 5, 4, 4$	$7, -1, 2, 2$	nn
	05821712	1182236	$7, 3, 0, 0$	$5, 1, 4, 4$	nn
	05123512	1678524	$7, 3, 0, 0$	$5, 5, -2, -2$	nn
	06814171	1187667	$3, 3, -2, 6$	$-7, -3, 0, 0$	
	06484861	1617326	$-1, -5, 4, 4$	$5, 5, -2, -2$	
	06421270	1653857	$7, 1, -2, 2$	$-3, 7, 0, 0$	
15	06163118	12676761	$6, 4, 1, 3$	$-6, -4, 3, 1$	ns
	06188366	12626262	$0, -2, 7, -3$	$0, 2, -3, 7$	ns
	06147226	12876712	$6, 0, -1, 5$	$-6, 0, 5, -1$	

n	A & B	C & D	a, b, c, d	a^*, b^*, c^*, d^*	
15	06177646	12716733	$0, 6, 1, 5$	$-4, -6, -1, 3$	
	06173682	12776732	$2, 0, -3, 7$	$-6, -4, 3, 1$	
	06172486	12777652	$2, 0, -3, 7$	$-6, 0, -1, 5$	
16	061611880	12537165	$5, 3, 4, 4$	$5, 3, 4, 4$	*ns*
	076534120	12556713	$5, 3, 4, 4$	$5, 3, 4, 4$	*nn*
	051564173	12276715	$3, 5, 4, 4$	$7, 1, 4, 0$	*nn*
	076517353	12441318	$-1, 5, 6, 2$	$7, -3, 2, 2$	*nn*
	066821450	11186754	$5, -1, 2, 6$	$5, -1, -2, 6$	
	066814363	11187637	$1, -1, 0, 8$	$5, 3, -4, -4$	
	066814222	11186725	$5, -3, 4, 4$	$5, 5, 4, 0$	
	066387422	11676547	$-1, -1, 0, 8$	$-1, -1, 8, 0$	
	066236780	11673753	$1, -1, 0, 8$	$1, 7, 04,$	
	066427850	11262185	$1, -1, 8, 0$	$5, -5, 0 - 4,$	
	078425223	16781722	$3, -7, 2, 2$	$3, -7, -2, -2$	
	078451111	16765381	$5, 5, 0, 4$	$1, -7, -4, 0$	
	078434872	16653727	$-7, 1, 0, 4$	$1, -7, 0, -4$	
	072462243	18768555	$5, -5, -4, 0$	$5, -5, 0, -4$	
17	066275153	117182163	$4, 2, 5, 5$	$2, -4, -7, 1$	
	066217543	117654642	$4, 2, 1, 7$	$-6, 4, -3, 3$	
	066424181	116754232	$6, 0, 3, 5$	$-4, 6, 3, -3$	
	066423683	116535720	$2, -4, 5, 5$	$0, 6, -3, 5$	
	066424726	116373250	$4, -2, 5, 5$	$-2, 8, 1, 1$	
	066388475	116762180	$-4, -2, 5, 5$	$2, 4, 5, 5$	
	066385122	116536813	$6, -4, 3, 3$	$4, 2, -5, -5$	
	066358141	116546771	$4, 2, 1, 7$	$-2, 8, 1, -1$	
	065327141	118675781	$6, 4, -3, 3$	$-4, 6, -3, 3$	
	065838623	116576181	$0, -6, 3, 5$	$-6, 0, 3, 5$	
	018763577	613775481	$-4, 6, -3, 3$	$2, 8, 1, -1$	
	018258836	617443151	$0, -6, 3, 5$	$2, 8, -1, 1$	
	018253521	615752482	$8, -2, -1, 1$	$-2, 8, -1, 1$	
	018433826	643685872	$2, -4, -7, -1$	$0, 6, -3, -5$	
	018466371	643735873	$2, 4, -7, 1$	$-4, 2, 1, -7$	
18	0616138163	126575621	$5, 3, 6, 2$	$3, 1, 8, 0$	*ns*
	0517848731	125352156	$-3, 1, 8, 0$	$3, -5, 2, 6$	*nn*
	0512876462	164341136	$1, 1, 6, 6$	$3, -1, 8, 0$	*nn*
	0767846432	126627155	$-5, 3, 6, 2$	$5, -7, 0, 0$	*nn*
	0664281361	117652676	$5, -3, 2, 6$	$7, 3, 0, 4$	

n	A & B	C & D	a, b, c, d	a^*, b^*, c^*, d^*	
18	0662457272	117653214	1, 1, 6, 6	−5, 7, 0, 0	
	0664248713	116337215	1, −1, 6, 6	7, −3, 0, 4	
19	0118636816	6653441710	4, 2, 3, 7	4, 6, 5, 1	*ns*
	0668563745	1117265342	−2, 0, 5, 7	−2, −4, 7, −3	
	0614454413	1288767580	6, 4, −5, −1	2, 0, −7, 5	
	0614461254	1288656450	8, 2, 1, −3	−8, 2, −1, 3	
	0614461272	1288635571	8, 2, −1, −3	−4, −2, −3, 7	
	0614736578	1287526211	−2, 4, 7, −3	2, 4, −3, 7	
	0614744631	1286852741	4, 6, 1, −5	−8, −2, −1, −3	
	0614724631	1286856641	6, 4, 1, −5	−6, −4, −5, 1	
	0614587446	1286265260	0, 2, 7, −5	−8, 2, −3, 1	
	0644176818	1677121322	0, 2, 5, 7	0, 2, 7, 5	
20	08311616133	1883131336	7, 5, 2, 2	7, 5, 2, 2	*ns*
	05173534120	1275533663	7, 5, 2, 2	7, 5, 2, 2	*nn*
	06613118360	1133831863	7, 5, 2, 2	3, 1, 6, 6	*ns*
	06811616383	1183131366	3, 1, 6, 6	3, 1, 6, 6	*ns*
	05153487123	1616571625	3, 1, 6, 6	3, 1, −6, −6	*nn*
	05146784840	1663611547	−1, 1, 4, 8	3, −3, 8, 0	*nn*
	05126532340	1286556373	9, −1, 0, 0	1, 7, 4, 4	*nn*
	05673282320	1166536724	5, −5, 4, 4	−3, 3, 0, 8	*nn*
	06484827832	1617552435	−5, −5, 4, 4	7, −1, −4, −4	
	06483552752	1617867574	−1, −1, −4, 8	−1, 7, −4, −4	
	06482715432	1617677855	1, 1, −4, 8	9, 1, 0, 0	
	06481556532	1617554561	1, 1, 4, 8	5, 5, 4, −4	
	06484174433	1616377215	−1, 1, 4, 8	−1, 1, −4, −8	
	06484821640	1615724773	3, −3, 0, 8	7, 1, 4, −4	
	06482768562	1615731478	−3, −3, 0, 8	9, 1, 0, 0	
	06452512173	1615874387	7, 1, −4, 4	7, 1, 4, −4	
	06442456280	1677534728	5, −5, −4, 4	9, −1, 0, 0	
	06424687240	1677355451	3, −3, 0, 8	3, −3, −8, 0	
	06432357223	1667576385	5, −5, −4, 4	5, −5, 4, 4	
	06152512273	1276533843	9, −1, 0, 0	9, −1, 0, 0	
21	06842717113	11876737623	4, 6, −3, 5	−6, 0, 1, −7	
	06844824235	11865726460	2, −8, 3, 3	−4, 6, 3, −5	
	06844842738	11863357211	−4, −6, 5, 3	2, 0, 1, −9	
	06842348538	11868657722	−2, −8, −3, 3	0, 6, −7, −1	
	06837848463	11866377853	−6, −4, −5, 3	0, −6, −5, −5	

n	A & B	C & D	a,b,c,d	a^*,b^*,c^*,d^*	
21	06838772746	11867666312	−6, 0, 1, 7	0, 6, −7, −1	
	06842346121	11868537272	8, −2, −3, 3	−2, 0, 1, −9	
	06818622553	11817655241	4, −6, 5, 3	−6, −4, −3, −5	
	06824574641	11863375712	2, 0, −1, 9	0, 2, −9, 1	
	06821537745	11865476710	0, 2, 1, 9	−6, 0, −7, 1	
	06877635856	11766781543	−6, 0, −1, 7	0, −6, −5, −5	
	06875863837	11765543280	−8, −2, 3, 3	2, 8, 3, 3	
	06886577813	11765412380	−6, 0, 5, 5	0, −2, 1, 9	
	06864822184	11762355813	2, −8, 3, 3	0, −6, −5, −5	
	06875624284	11661534580	0, −6, 5, 5	−6, 0, 5, 5	
22	051532351482	12653363142	5, 1, 8, 0	3, 3, 6, 6	*nn*
	051535148732	12631554424	1, 5, 8, 0	7, −1, 6, 2	*nn*
	078212153261	16778255254	9, −3, 0, 0	−1, 7, 2, −6	*nn*
	076487121512	16337381132	3, 7, 4, 4	9, 1, 2, 2	*nn*
	076537321212	16156871224	5, 5, 6, 2	7, 3, 4, −4	*nn*
	076435857863	12871616562	−7, −1, 6, 2	3, 1, −8, −4	
	076434883831	12876423125	−5, −5, 6, −2	5, 1, 0, 8	
	076414341780	12876773668	1, 7, −6, 2	7, −3, 4, 4	
	076411654773	12876151586	−1, 9, 2, 2	5, −1, 0, 8	
	076445318823	12866242834	−1, −3, 4, −8	5, −5, 6, −2	
	076448413133	12865264125	1, 3, 8, −4	3, −3, −6, 6	
	076438642411	12867121282	3, −1, 8, −4	9, 1, −2, −2	
	076424341510	12866352873	7, 5, 0, −4	9, −1, −2, 2	
	064411463722	12863315525	5, 5, 6, −2	7, −5, −4, 0	
	064384226811	12868354577	5, −7, −4, 0	3, 3, −6, 6	
	064256153521	12816563847	9, −3, 0, 0	3, 3, 6, 6	
	064213758243	12875838384	3, −3, −6, −6	−3, 7, −4, −4	
	064221664181	12858637416	9, −3, 0, 0	−5, −5, 2, −6	
	064423632142	12828778675	7, −1, −6, −2	5, −7, 0, 4	
	064238484243	12876854464	1, −9, −2, −2	3, 9, 0, 0	
23	061588351872	161234284320	0, −2, 9, −3	0, −6, 7, 3	
	011876765672	668385722453	0, 6, −3, −7	−8, −2, −5, −1	
	011876734544	668371721610	0, 6, 3, 7	4, −2, −7, 5	
	011865653637	668387782181	2, 4, −7, −5	2, −4, −5, −7	
	011834576387	668856384471	−2, 4, −7, −5	−2, 8, −1, 5	
	011843854727	668755387713	0, 2, −9, 3	8, −2, 5, 1	
	011844723445	668717154763	4, 2, −5, 7	0, −2, 9, −3	

n	A & B	C & D	a, b, c, d	a^*, b^*, c^*, d^*	
23	011835823773	668736358482	$0, 2, -9, -3$	$0, -2, -3, -9$	
	011824565625	668757528711	$8, -2, -5, 1$	$-8, 2, 1, -5$	
	011824564276	668751123170	$6, 0, 3, 7$	$-6, 0, -7, -3$	
24	0515373265143	126265241457	$5, 3, 8, 0$	$5, 3, 0, 8$	nn
	0512653237623	165353436747	$7, -3, -2, 6$	$-1, 5, 6, 6$	nn
	0761232522583	162738637128	$7, -7, 0, 0$	$7, -7, 0, 0$	
	0618441731220	161824654782	$9, 3, 2, -2$	$9, 3, 2, -2$	
	0644381822863	128652453571	$1, -9, 4, 0$	$5, 3, 0, -8$	
	0785384835831	161835855841	$-7, -7, 0, 0$	$5, -3, 0, -8$	
	0785231641223	161871657538	$7, -3, -2, 6$	$3, -7, 2, -6$	
	0785848265132	161785487748	$-3, -3, -8, 4$	$-3, -3, 4, -8$	
	0785823385781	161785243834	$-7, -7, 0, 0$	$-3, -3, -8, -4$	
	0785864557880	161758373262	$-9, -3, 2, 2$	$3, -7, -6, -2$	
	0785866386572	161757832366	$-9, -1, 0, 4$	$-1, -9, 0, 4$	
	0785866347221	161726576241	$-1, -5, 6, 6$	$3, -9, -2, -2$	
	0785675761323	161755421233	$-3, 3, 8, 4$	$1, -9, 0, -4$	
	0785552453682	161772115438	$-3, -3, 4, 8$	$5, -3, -8, 0$	
	0785584522330	161755454162	$1, -5, 6, 6$	$5, -1, 6, -6$	
	0785577341573	161754832843	$-7, 7, 0, 0$	$1, -9, 0, 4$	
25	0161633813118	6414148485143	$8, 6, 1, 1$	$6, 8, 1, 1$	ns
	0615136477647	1618836524732	$0, 10, -1, 1$	$2, 8, 3, 5$	
	0615135164456	1618856557451	$8, 6, -1, 1$	$6, 4, 7, 1$	
	0615614824146	1614556627431	$8, 2, 5, 3$	$2, 4, 9, -1$	
	0615621857446	1615271224643	$4, 2, 9, 1$	$2, 4, 9, 1$	
	0615735785244	1616751436260	$0, 2, 7, 7$	$2, 0, 7, 7$	
	0614576515556	1675341263153	$4, 6, 5, 5$	$2, 4, 1, 9$	
	0614724178515	1676251355261	$4, 6, 7, 1$	$-6, 4, 7, 1$	
	0615712245624	1615876732523	$10, 0, 1, 1$	$8, -6, 1, 1$	
	0614742453467	1676575435610	$2, 4, 1, 9$	$-8, 6, 1, 1$	
	0615643274811	1615563275621	$6, 4, 7, 1$	$-4, 2, -1, 9$	
	0615622781175	1615434228350	$6, 4, 7, -1$	$0, 2, 7, 7$	
	0615642466136	1615277334622	$8, 2, 3, 5$	$2, 8, 3, 5$	
	0615564245823	1615337721820	$6, -4, 5, 5$	$0, -10, 1, 1$	
	0614614367622	1675245351341	$8, 2, 5, 3$	$6, 8, 1, -1$	
	0615612742235	1614725488440	$10, 0, 1, 1$	$4, 2, 1, 9$	
	0615511258745	1614834521760	$6, 4, 5, 5$	$-8, 6, 1, 1$	
	0615671377685	1613276538312	$-2, 8, 3, 5$	$4, 6, 7, 1$	

n	A & B	C & D	a, b, c, d	a^*, b^*, c^*, d^*	
25	0615714268347	1612864262711	2, 4, 9, −1	0, −2, −7, 7	
	0615774715633	1612576388522	0, 10, 1, −1	6, −4, 1, 7	
	0614284161471	1678385227450	8, 6, −1, −1	2, 0, 7, 7	
	0614741227336	1676617221543	6, 4, 5, 5	−4, 6, −7, 1	
	0614725168638	1675812411440	2, 0, 7, 7	0, 10 − 1, −1	
	0614778264623	1675415381210	2, 0, 7, 7	−4, 6, −5, −5	
	0614774641627	1675421382312	2, 8, 5, 3	0, −2, −7, 7	
	0614622163581	1674576223833	10, 0, −1, −1	0, 10 − 1, −1	
	0174617668216	6167252374160	4, 6, 5, 5	2, 8, 5, −3	
	0172116382817	6166757871282	6, 4, −5, 5	4, 6, −5, 5	
26	06663168818110	1113811681836	5, 3, 6, 6	−5, −7, 4, 4	*ns*
	05126265841481	1287432361571	7, −5, 4, 4	−3, 5, 6, 6	*nn*
	06642512781532	1176758554637	5, 1, −4, 8	7, 7, 2, 2	
	06462876132573	1611252645646	1, −1, 10, 2	7, 5, 4, −4	
	06347845157852	1681715562551	−5, 3, 6, 6	5, 9, 0, 0	
	06337112714573	1685756153533	3, 9, 0, 4	−7, 7, 2, 2	
	06338372427540	16847857	, −1, −10, −2	−9, 5, 0, 0	
	06336277161612	1683252617578	5, 9,	7, 7, −2, −2	
	0646328743	1611325452831	−1, 1,	1, −5, −8, 4	
	07742445466113	1688225363621	6, −6	5, 4, 8	
	07741624145363	1688654231522	3, 5, 6, −6	9, 3, 0, −4	
	07745533541130	1686154242817	3, 9, 4, 0	1, −1, 2, −10	
	07868434286832	1617885275223	−7, −7, 2, −2	7, −5, −4, −4	
27	01747385847264	61683574385882	−4, 2, −9, −3	0, 10, 1, 3	
	06824788121163	11876252325230	4, −2, 9, −3	4, 2, −9, 3	
	01747376332571	61674586812243	0, 10, 1, −3	0, 2, −5, −9	
	01648381282472	61633885385753	4, −6, −7, −3	0, 10, 3, −1	
	01747836688263	61673761181783	−4, −2, −3, 9	0, 6, 7, −5	
	01287663641511	66383725412172	8, 6, 1, 3	4, 6, 3, −7	
	01743118358613	61665711252813	4, 6, 7, 3	−4, −2, 9, −3	
	01741618677553	61667358723653	0, 10, −3, 1	0, 6, −5, 7	
	01741633516525	61661725682342	8, 6, 3, 1	0, 2, 9, −5	
	01746165128335	61683567215170	6, 4, 3, 7	6, 4, −3, −7	
	01748338354572	61673821152130	−2, 0, 9, 5	2, −4, 3, −9	
	01748157537837	61673562483473	−6, 8, −3, 1	−6, 8, −1, 3	
	01747465342683	61678388854363	0, 2, −9, −5	−4, 6, −3, −7	
	01747244633685	61678716533732	0, 2, −5, 9	0, 2, 5, −9	

n	A & B	C & D	a,b,c,d	a^*,b^*,c^*,d^*	
27	01747346137354	61675674228411	0, 10, 3, 1	0, 10, −3, −1	
	01747258378538	61675231544141	−6, 0, 7, 5	−6, 0, −7, −5	
	01746325133176	61677258628230	6, 8, 1, −3	2, 4, −9, 3	
	01746183123277	61677276651240	4, 6, 3, 7	0, 6, −7, 5	
	01746332341224	61675822457212	10, 0, 3, 1	−6, 8, 1, 3	
	01746166227166	61675483213571	8, 6, 1, 3	0, 6, −5, −7	
	06174741583364	16185625225443	0, 6, 7, −5	0, 2, 9, 5	
28	076514146435673	12566715632821	1, 7, 8, 0	9, −1, 4, 4	*nn*
	051567121285343	12256358721165	7, 1, 8, 0	3, 5, 8, 4	*nn*
	078517356737323	12747162866717	−5, 5, 0, 8	7, −7, 0, 4	*nn*
	078582621567150	12456332286115	3, 1, 10, −2	3, 1, 10, −2	*nn*
	076534321432170	16128847836248	7, 5, −2, −6	7, 5, 6, −2	*nn*
	063442242645720	16878675223565	9, −5, −2, −2	1, 3, 2, −10	
	063843754227283	12877268658522	−1, −7, 0, −8	7, 1, 8, 0	
	064276387717651	16214382134382	−358 − 4	5580	
	064442136178270	16772717864644	5, 3, −4, 8	5, 3, −4, −8	
	064442171647233	16774372735546	5, 3, −4, 8	9, −1, 4, −4	
	064843722816531	16178757852578	3, −5, −8, 4	7, 7, 0, −4	
	071286286424863	18877615344454	1, −9, −4, 4	5, −5, −8, 0	
	072161633841562	18332113414382	5580	1 − 780	
	076185788565812	16215443325247	−7180	−7 − 7 − 40	
	076441234324112	12876513727844	9, 5, −2, 2	5, −7, 6, 2	
	076441322271811	12876352552655	9, 1, 4, −4	−7, −7, 0, 4	
	076815711426771	16143822334818	194 − 4	91 − 4 − 4	
	077442312346813	16888578675634	3, 1, −10, −2	−5, −7, −6, −2	
	077658617271583	12852541333416	−5, 5, 8, 0	−5, 5, 0, −8	
29	016186616313366	641515851514853	8, 6, 3, 3	6, 8, 3, 3	*ns*
30	0641462126585640	164711856213678	9, −1, 2, 6	3, −7, 8, 0	*nn*
31	0164482648131672	6488874517518730	6, 0, −9, 3	2, 0, 1, −11	
	0164483618165471	6488874628615542	4, 6, −7, −5	−8, 2, 3, −7	
	0653131761458613	1185857125332351	6, 8, 5, −1	6, 4, −5, −7	
32	01113181831663860	6666818111883663	9, 7, 0, 0	−7, −9, 0, 0	*ns*
	01836183616638333	6116611661833816	1, −1, 8, 8	−7, −9, 0, 0	*ns*
	06668636113881680	1111363633661881	1, −1, 8, 8	1, −1, −8, 8	*ns*

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