Colorability, Frequency and Graffiti - 119

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ABSTRACT. Conjecture 119 in the file "Written on the Wall", which contains the output of the computer program "Graffiti" of Fajtlowicz, states: If G has girth 5 then its chromatic number is not more than the maximum frequency of occurrence of a degree in G. Our main result provides an affirmative solution to this conjecture if $|G| = n$ is sufficiently large. We prove:

Theorem. Let $k \geq 2$ be a positive integer and let G be a C_{2k}-free graph (containing no cycle of length $2k$).

1) There exists a constant $c(k)$, depending on k only, such that $\chi(G) \leq c(k)^{k-1}\sqrt{f(G)}/\log|G|$, where $f(G)$ is the frequency of the mode of the degree sequence of G.

2) There exists a constant $c(k)$, depending on k only, such that $\chi(G) \leq c(k)|G|^{1/k}/\log|G|$.

3) If girth $(G) \geq 5$ then $\chi(G) \leq f(G)$ if $|G| \geq e^{49}$.

1 Introduction

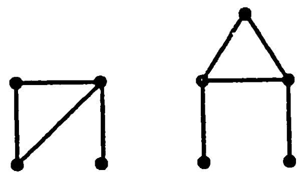
Let G be a simple graph and denote by $f(G)$ the frequency of the mode of the degree sequence of G.

The computer program "Graffiti" of Siemion Fajtlowicz of the University of Houston has produced several conjectures concerning possible relations between $\chi(G)$ the chromatic number of G and $f(G)$ the frequency of the mode of the degree sequence (namely, the maximum frequency of occurrence of a degree in G).

Conjecture 67 which states that if G is C_3-free graph then $\chi(G) \leq f(G)$ was disproved in [5] where a counter-example on 19 vertices is given, and an infinite family of counter-examples is known.

Conjecture 119 of Graffiti states that if G has girth 5 (i.e. contains no C_3 and no C_4) then $\chi(G) \leq f(G)$.

A stronger version of Conjecture 119 to the case when G is only C_4-free but not necessarily C_3-free is false as seen by the graphs:



The role played by even cycles is explored and the following result is proved:

Theorem 1. *Let $k \geq 2$ be a positive integer and let G be a C_{2k}-free graph. There exists a positive constant $c(k)$, depending on k only, such that $\chi(G) \leq c(k)^{k-1}\sqrt{f(G)}/\log|G|$. Thus Graffiti 119 holds true if $|G|$ is sufficiently large just by taking $k = 2$ above.*

We shall give first some elementary constructions showing that in general $\chi(G)$ and $f(G)$ are uncorrelated graph parameters (extending some constructions given in [5]).

We shall then list some known results concerning the density of C_{2k}-free graphs, due to Bondy and Simonovits [2]. We shall use a deep new result of Alon [1] to deduce an upper bound for the chromatic number of C_{2k}-free graphs. Lastly using a simple lemma relating the number of edges and $f(G)$ we shall deduce Theorem 1. It is worth mentioning that the result of Alon is used only to get the $\log|G|$ factor in Theorem 1 which is crucial to the original $\chi(G) \leq f(G)$ relation. Also we mention that for the girth-5 a result of Ajtai-Komlós-Szemerédi [6] in an improved version by Shearer [7] could be used to replace the result of Alon.

Lastly, we shall follow the definition, and notation of [3] and write $|G| = n$.

2 Some constructions and basic lemmas

Below we give some constructions aimed to show that in general $\chi(G)$ and $f(G)$ are uncorrelated.

Construction 1. $\chi(G) = k$ is fixed, $f(G) = n$.

Just take G to be t vertex disjoint copies of K_k, namely $G = tK_k$. Clearly $\chi(G) = k$ and $f(G) = |G| = n$. □

Construction 2. $f(G) = 2$, $\chi(G) = \frac{|G|+2}{2}$.

Consider two disjoint sets A, B such that $|A| = |B| = 2n - 1$. Label the elements of A by $u_1, \dots, u_{2n-1}$ and of B by $v_1 \dots v_{2n-1}$. Set $V(G) = A \cup B$, then $|G| = 4n - 2$.

Now $(u_i, v_j) \in E(G)$ if $i + j \geq 2n$ also $(u_i, u_j) \in E(G)$ if $i, j \geq n$ and $(v_i, v_j) \in E(G)$ if $i, j \geq n$.

It is easy to see that $\chi(G) = \frac{|G|+2}{2}$ while $f(G) = 2$.

It is also easy to modify this construction to produce a graph G such that $f(G) = k$ and $\chi(G) = \frac{|G|+2}{k}$. □

Construction 3. $f(G) = \chi(G) = k$, $k \geq 2$.

Let $A_1, A_2, \dots A_k$ be k sets of cardinality n each.

Label the elements of A_i by $u_1^i, \dots, u_n^i$.

Set $V(G) = \bigcup_{i=1}^{k} A_i$ and define $E(G)$ as follows: for $1 \leq i < j \leq k$ $(u_r^i, u_t^j) \in E(G)$ if $r + t \geq n + 1$.

Again it is easy to see that $\chi(G) = f(G) = k$ and $|G| = n$ can be made arbitrarily large. □

Define $m(n, t, \delta)$ the minimum number of edges in a graph G on n vertices, with minimum degree δ and $f(G) = t$.

Proposition 1. *Suppose $n = tq + r$, $0 \leq r \leq t-1$. Then $m(n, t, \delta) \geq \frac{(\frac{n}{t}-1)n - (\frac{r}{t}-1)r + 2\delta n}{4}$. In particular* $m(n,t,\delta) \geq \begin{cases} \frac{1}{4}\left(\frac{n^2}{t} - n\right) & \delta = 0 \\ \frac{1}{4}\left(\frac{n^2}{t} + n\right) & \delta \geq 1 \end{cases}$

Proof: Counting degrees we set: $e(G) = \frac{1}{2}\sum_{v \in V} \deg v \geq \frac{1}{2}[t\delta + \dots + t(\delta + q - 1) + r(\delta + q)] = \frac{1}{4}(t(2\delta + q - 1)\cdot q + 2r(\delta + q))$ = by replacing $q = \frac{n-r}{t}$ and rearranging $\frac{1}{4}\left(\left(\frac{n}{t} - 1\right)n - \left(\frac{r}{t} - 1\right)r + 2\delta n\right)$.

Now since $0 \leq r \leq t - 1$ we infer that $\frac{r}{t} - 1 \leq 0$. Hence we get

$$m(n,t,\delta) \geq \begin{cases} \frac{1}{4}\left(\frac{n^2}{t} - n\right) & \delta = 0 \\ \frac{1}{4}\left(\frac{n^2}{t} + n\right) & \delta \geq 1 \end{cases}$$

□

Recall that a graph G is called k-degenerate if $\max_{H \subset G} \delta(H) \leq k$ and by a well known theorem of Szekeres and Wilf [3, page 221] a k-degenerate graph is $k+1$ colorable.

We can now prove a weaker version of Theorem 1 relating colorability to frequency via degeneracy.

Proposition 2. *Let G be a graph on n vertices which is $c_1 n^\alpha$ degenerate, for some constants $c_1 > 0$ and $0 \leq \alpha < 1$. Then there exists $c_2 > 0$ such that $\chi(G) \leq c_2(f(G))^{\alpha/1-\alpha} + 1$.*

Proof: Since G is $c_1 n^\alpha$ degenerate then clearly $\chi(G) \leq 1 + c_1 n^\alpha$. Also counting edges ([3, p. xvii]) by inductively deleting vertices of minimum degree) we find that $e(G) \leq c_1 n^{1+\alpha}$. Suppose now that $f(G) \leq \frac{n^{1-\alpha}}{4c_1+2}$

then by $m(n,t,\delta)$ we get that $e(G) \geq \frac{1}{4}\left(\frac{n^2}{f(G)} - n\right) \geq \frac{1}{4}\left(\frac{n^2}{\frac{n^{1-\alpha}}{4c_1+2}} - n\right) = \frac{1}{4}\left(\frac{(4c_1+2)n^2}{n^{1-\alpha}} - n\right) = \frac{1}{4}\left(4c_1 n^{1+\alpha} + 2n^{1+\alpha} - n\right) > c_1 n^{1+\alpha} \geq e(G)$ which is impossible.

Hence $f(G) \geq \frac{1}{4c_1+2} n^{1-\alpha}$ and $\chi(G) \leq c_1 n^{\alpha} + 1 \leq c_1 (f(G)(4c_1+2))^{\frac{\alpha}{1-\alpha}} + 1$ and we are done by taking $c_2 = c_1(4c_1+2)^{\frac{\alpha}{1-\alpha}}$. □

Computing the constants precisely we get:

Proposition 3. *If G is $\frac{(\sqrt{5}-1)}{4} \cdot n^{1/2} - 1$ degenerate then $\chi(G) \leq f(G)$.* □

Recall now the famous theorem of Bondy and Simonovits [2] saying that if G is C_{2k}-free graph then $e(G) \leq 90kn^{1+\frac{1}{k}}$. Since C_{2k}-freeness is a hereditary property it follows that a graph containing no copy of C_{2k} is $180kn^{\frac{1}{k}}$ degenerate and hence also $180kn^{\frac{1}{k}}+1$ colorable. By Proposition 2 if $k \geq 2$ we get that $\chi(G) \leq c(k)^{k-1}\sqrt{f(G)}+1$ where $c(k) = 180k(720k+2)^{\frac{1}{k-1}}$. In particular a very rough estimate comparing the bounds in proposition 3 and the Bondy-Simonovits bound shows that for $k \geq 3$ if $|G| = n > \left(\frac{720k+2}{\sqrt{5}-1}\right)^{\frac{2k}{k-2}}$ then $\chi(G) \leq f(G)$, solving Graffiti 119 for large n and C_{2k}-free graphs for $k \geq 3$.

Yet we know from extremal graph theory that

(1) if G has girth-5 then $e(G) \leq \frac{1}{2}n\sqrt{n-1}$ (see e.g. [8] page 31) and hence it is $\sqrt{n-1}$ degenerate and we cannot use the above reasoning.

(2) if G is C_4-free then $e(G) \leq \frac{1}{4}\left(n + n\sqrt{4n-3}\right)$ ([3, page 313] hence it is $\sqrt{n}$ degenerate and we are stuck.

3 Chromatic number of C_{2k}-free graphs

In order to obtain a better estimate for C_4-free graphs (and the girth 5 as well) we shall need the following deep result of Alon [1], a theorem of Shearer [7] and a result of the author [4] repeated in [9, page 124].

Theorem A. *Let G be a graph on n vertices with average degree $d \geq 1$ in which for every vertex $v \in V$ the induced subgraph on the set of all neighbors of v is r-colorable. Then G contains an independent set of size at least $\frac{1}{640\log(r+1)} \cdot \frac{n \log d}{d}$.* □

The theorem of Shearer is:

Theorem B. *If G is triangle free, then $\alpha(G) \geq \frac{|G|(\log d - 1)}{d}$, where $d \geq 1$ and $\alpha(G)$ is the independence number of G.* □

Lastly we shall prove a slightly more general result than that of [4] and [9, page 124] in a form suitable for precise computation.

The greedy coloring lemma. *Let $\mathcal{F}$ be a class of graphs that is closed under taking induced subgraphs. Suppose further that $\alpha(G) \geq f(|V(G)|)$ for every $G \in \mathcal{F}$ for which $|G| \geq m$ for some positive integer $m \geq 2$. Assume further that $f\colon [m, \infty) \to (0, \infty)$ is a positive, nondecreasing continuous function and that for every $G \in \mathcal{F}$ for which $|V(G)| \leq m$ it holds that $\chi(G) \leq t$ for some positive integer t. Then for every $G \in \mathcal{F}$, $\chi(G) \leq t + 1 + \int_m^{\max(n,m)} \frac{1}{f(x)} dx$, where $|V(G)| = n$.*

Proof: By induction on n. For $n \leq m$ $\chi(G) \leq t < t+1$. Hence this is obviously true. Suppose $|V(G)| = n+1$ and let X be an independent set that realizes $\alpha(G)$. If $|X| \geq n+1-m$ then $|V(G\backslash X)| \leq m$. Hence $\chi(G) \leq \chi(G\backslash X)+1 \leq t+1$ and we are done. Otherwise $|V(G\backslash X)| \geq m$ and by induction $\int_m^n \frac{1}{f(x)} dx = \int_m^{n-\alpha(G)} \frac{1}{f(x)} dx + \int_{n-\alpha(G)}^n \frac{1}{f(x)} dx \geq \chi(G\backslash X) - (t+1) + \int_{n-\alpha(G)}^n \frac{1}{f(n)} dx = \chi(G\backslash X) - (t+1) + \alpha(G)/f(n) \geq \chi(G) - 1 - (t+1) + 1 = \chi(G) - (t+1)$, completing the proof. □

The function in Theorem A is not suitable for application in conjunction with the greedy coloring lemma as long as $0 \leq d \leq 3$, say, because of monotonicity and positivity violation as well as because d is independent of n, and the same holds true for the function in Theorem B for $0 < d \leq 7$. However by Turan Theorem $\alpha(G) \geq \frac{n}{d+1}$ and hence we can modify these functions as follows:

Set $h(n,d) = \frac{1}{640 \log(2k-1)} \cdot \frac{n \log(d+3)}{(d+3)} \leq \begin{cases} \frac{n}{d+1} & d \leq 3 \\ \frac{1}{640 \log(2k-1)} \frac{n \log d}{d} & d \geq 3 \end{cases}$

Set $g(n,d) = \frac{n \log(d+3)}{2(d+3)} \leq \begin{cases} \frac{n}{d+1} & d \leq 7 \\ \frac{n(\log d - 1)}{d} & d \geq 7 \end{cases}$

Observe that both functions are monotone decreasing with d.

Now we prove the main result of this paper.

Theorem 1. *Let $k \geq 2$ be a positive integer and G be a C_{2k}-free graph on n vertices.*

(1) There exists a constant $c(k)$ that depends on k only, such that $\chi(G) \leq c(k)^{k-1}\sqrt{f(G)}/\log n$.

(2) There exists a constant $c(k)$, depending on k only, such that $\chi(G) \leq c(k) n^{1/k}/\log n$.

(3) If girth $(G) \geq 5$ and $|G| \geq e^{49}$ then $\chi(G) \leq f(G)$.

Proof: If G is C_{2k}-free then the neighborhood of every vertex v contains no path of length $2k-1$. Hence by a theorem of Gallai and Roy [3, page 220] every such neighborhood is $2k-2$ colorable. Applying Theorem A we infer that $\alpha(G) \geq h(|G|, d)$ where $h(n,d)$ is the smoothing function defined above. The Bondy-Simonovits [2] theorem implies that $d \leq 180kn^{1/k}$. By observation above $\alpha(G) \geq h(n, d) \geq h(n, 180kn^{1/k})$.

Applying the greedy coloring lemma with $m = 3$, $t + 1 = 4$ and with $h(n, 180kn^{1/k})$, which satisfies the conditions of the greedy coloring lemma, we get that $\chi(G) \leq 4 + c_1(k) \int_3^n \frac{1}{X^{(1-1/k)} \log x} dx \leq c(k)n^{1/k} / \log n$ for some suitable (large) constant $c(k)$. Applying the method of Proposition 2 for this bound of $\chi(G)$ and with the degeneracy $d \leq 180kn^{1/k}$ we infer that $\chi(G) \leq c(k)^{k-1}\sqrt{f(G)} / \log n$ proving (1) and (2).

To prove (3) we apply $g(n, \sqrt{n-1})$ in the greedy lemma with $m = e^{16}$ and $t = e^8 + 1$ (as such graphs are $\sqrt{n-1}$ degenerate) to obtain $\chi(G) \leq e^8 + 2 + \int_m^n \frac{1}{g(x, \sqrt{x-1})} dx \leq 8.02 \int_m^n \frac{(1 - \frac{2}{\log x})}{2\sqrt{x} \log x} dx < e^8 + 2 + 8.02\sqrt{n} / \log n$, where the factor 8.02 is the result of approximating the left integral by the right one.

Now $\chi(G) \leq e^8 + 2 + 8.02\sqrt{n} / \log n$ and by Proposition 2, $f(G) \geq \frac{1}{6}\sqrt{n}$, equating we get that $\chi(G) \leq f(G)$ if $|G| = n \geq e^{49}$, completing the proof. □

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