$30 \leq$ R(3,3,4) ≤ 31

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ABSTRACT. With the help of computer algorithms, we improve the upper bound on the classical three color Ramsey number $R(3,3,4)$, and thus we show that the exact value of this number is 30 or 31. We also present computer enumeration of all 3-colorings of edges on at least 14 vertices without monochromatic triangles.

1 Introduction and Notation

An $(r_1, r_2, \ldots, r_k)$ *coloring*, $r_i \geq 1$ for $1 \leq i \leq k$, is an assignment of one of k colors to each edge in a complete graph, such that it does not contain monochromatic complete subgraph K_{r_i} in color i, for $1 \leq i \leq k$. Similarly, an $(r_1, r_2, \ldots, r_k; n)$ *coloring* is an $(r_1, \ldots, r_k)$ coloring of K_n. Let $\mathcal{R}(r_1, \ldots, r_k)$ and $\mathcal{R}(r_1, \ldots, r_k; n)$ denote the set of all $(r_1, \ldots, r_k)$ and $(r_1, \ldots, r_k; n)$ colorings, respectively. The Ramsey number $R(r_1, \ldots, r_k)$ is defined to be the least $n > 0$ such that $\mathcal{R}(r_1, \ldots, r_k; n)$ is empty.

A coloring using k colors will be also called a k-coloring. Clearly, any 2-coloring can be considered as a graph. In this paper we will study k-colorings only for $k = 2$ or $k = 3$. In the former case we will also use standard graph theory terminology. A regularly updated survey of the

*Supported in part by a grant from the NSA number MDA904-94-H-2009.

most recent results on the best known bounds on multicolor ($k > 2$) and graph ($k = 2$) Ramsey numbers can be found in [Rad].

In 1966 Kalbfleisch [Kalb] constructed a cyclic $(3, 3, 4; 29)$ coloring, which to date gives the best known lower bound $R(3, 3, 4) \geq 30$. Recently, Piwakowski [Piw] obtained an upper bound $R(3, 3, 4) \leq 32$. In this paper we reduce this bound to 31, and we conjecture that the exact value of this Ramsey number is 30. We note that the only known nontrivial value of a multicolor classical (i.e. avoiding monochromatic complete subgraphs) Ramsey number is $R(3, 3, 3) = 17$ [GG], and at the moment the only other multicolor case which seems not hopeless is $R(3, 3, 4)$.

Two k-colorings are *isomorphic* if there exist a bijection between the vertices of the underlying complete graphs preserving all the colors of edges, and they are *weakly isomorphic* if there exists a bijection between vertices which preserves the relation of two edges having the same color. It is convenient to think of a weak isomorphism as a composition of permutation of colors with an isomorphism; for example any graph G, seen as a 2-coloring, is weakly isomorphic to its complement $\bar{G}$.

Let C be a k-coloring and G be a simple undirected graph. Throughout this paper we will also use the following notation.

$deg_G(x)$	— the degree of vertex x in graph G
$n(G)$, $e(G)$	— the number of vertices and edges in graph G
$V(G)$, $E(G)$	— the vertex and edge sets of graph G
$N_G(x)$	— the neighborhood of vertex x in graph G
$C[i]$	— the graph formed by edges of color i in coloring C
C_x^i	— the coloring induced in C by vertices in $N_{C[i]}(x)$
$C_x^i[j]$	— the graph formed by edges of color j in coloring C_x^i
$\mathcal{R}(a, b, c; \geq n)$	— $\bigcup_{k \geq n} \mathcal{R}(a, b, c; k)$

2 (3, 3, 3) Colorings

The main result of this paper is that $(3, 3, 4; 31)$ colorings don't exist. If C is a $(3, 3, 4; m)$ coloring then, clearly, for every vertex x, C_x^3 is a $(3, 3, 3; deg_{C[3]}(x))$ coloring, so the latter are important in the study of $\mathcal{R}(3, 3, 4)$. Independently, such colorings are interesting by themselves, and this section gathers basic information about them.

In 1968 Kalbfleisch and Stanton [KS] proved that there are exactly two nonisomorphic $(3, 3, 3; 16)$ colorings; let us denote them by KS_1 and KS_2. Since $R(3, 3) = 6$, by merging two colors in any $(3, 3, 3; n)$ coloring, we obtain a $(3, 6; n)$ graph. It is interesting that all six $(3, 6; 16)$ graphs obtained in this way from $\mathcal{R}(3, 3, 3; 16)$ are isomorphic, yet the 3-colorings KS_1 and KS_2 are not even weakly isomorphic. Both KS_1 and KS_2 are vertex transitive, and the common underlying $(3, 6; 16)$ graph can be defined over $GF(2^4)$ with two vertices being adjacent if and only if their difference is a cube.

Deleting one point from the vertex transitive colorings KS_1 and KS_2 leads to exactly two nonisomorphic $(3,3,3;15)$ colorings, with 35 edges in each color. In a technically very complicated paper, but without using a computer, Heinrich [Hein] was able to prove that no other such colorings exist, i.e. $|\mathcal{R}(3,3,3;15)| = 2$.

Our computations related to $\mathcal{R}(3,3,3;n)$ confirmed all the above results, generated all such colorings on $n \geq 14$ vertices, and enumerated all possible $(3,6;n)$ graphs which form a single color, for all n.

We found that there are exactly 651 nonisomorphic $(3,3,3;14)$ colorings, which is reduced to only 115 up to weak isomorphism. The number of edges in any single color ranges from 28 to 35. Table 1 presents the numbers of these colorings, divided into classes with the same edge count in three colors. Note that the ratio between the last two columns is at most 3!, and it is equal to 6 iff each coloring in this class under all permutations of 3 colors leads to 6 nonisomorphic colorings.

Having at our disposition a data base of all $(3,6;n)$ graphs [RK], which were verified and used in few previous Ramsey related projects, we have checked for each $(3,6;n)$ graph whether its complement can be split into two colors yielding a $(3,3,3;n)$ coloring. The cumulative data is gathered in Table 2. These splittable graphs are exactly those which can form a single color subgraph in any member of $\mathcal{R}(3,3,3)$. For example, only 66 graphs can form a single color in a $(3,3,3;14)$ coloring. For this work, all colorings in $\mathcal{R}(3,3,3;\geq 14)$ were generated independently by each of the authors and compared. The computer time needed for obtaining and verifying these results was very small.

counts of edge colors	nonisomorphic colorings	weakly nonisomorphic colorings
31 30 30	249	45
31 31 29	126	23
32 30 29	138	23
32 31 28	60	10
33 29 29	24	5
33 30 28	36	6
34 29 28	12	2
35 28 28	6	1
all	651	115

Table 1. Statistics for $\mathcal{R}(3,3,3;14)$ colorings

n	number of (3, 6, n) graphs	splittable	nonsplittable
1	1	1	0
2	2	2	0
3	3	3	0
4	7	7	0
5	14	14	0
6	37	37	0
7	100	100	0
8	356	355	1
9	1407	1395	12
10	6657	6444	213
11	30395	26034	4361
12	116792	58538	58254
13	275086	21921	253165
14	263520	66	263454
15	64732	1	64731
16	2576	1	2575
17	7	0	7
all	761692	114919	646773

Table 2. (3, 6) graphs splittable to (3, 3, 3) colorings

3 Algorithm and Computations

The following simple lemma is the basis of the skeleton of our computations showing that $R(3,3,4) \leq 31$.

Lemma 1. *In any* $(3,3,4;31)$ *coloring* C *there are three vertices* x, y *and* z *such that* xyz *forms a triangle in the third color, and* $C_x^3, C_y^3, C_z^3 \in \mathcal{R}(3,3,3;\geq 14)$.

Proof: $R(3,4) = 9$ implies that the minimum degree in the graph $C[3]$ is at least $14 = 31-8-8-1$. Since $R(3,3,3) = 17$, C must have triangles in the third color. Thus a triangle, as required by the lemma, exists. □

Given an arbitrary $(3,3,3;n)$ coloring C, consider the set of colorings obtained by distinguishing each of the $n(n-1)/2$ edges in C. Let M denote the set of all such weakly nonisomorphic colorings with a distinguished edge, obtained from $\mathcal{R}(3,3,3;\geq 14)$, where weak isomorphisms preserve the distinguished edges (a distinguished edge can be considered as having color 4). With computer help we have found that M consists of exactly 5670 (3, 3, 3) colorings with a distinguished edge: 3 on 16 vertices, 13 on 15 vertices, and 5654 on 14 vertices. We will write $(C, e) \in M$ to denote that the edge e has been distinguished in coloring C.

Assume C and xyz are as in Lemma 1. Clearly, each of $(C^3_x, \{y,z\})$, $(C^3_y, \{x,z\})$, and $(C^3_z, \{x,y\})$ must be weakly isomorphic to some coloring in the set M. The colorings in M form the starting points of the main algorithm constructing $(3,3,4;m)$ colorings from a triangle in color 3 supporting three overlapping elements of M. The following algorithm was executed for all triples, repetitions allowed, of $(3,3,3)$ colorings with a distinguished edge in M.

The Algorithm

Step 1: Let (C_1, e_1), (C_2, e_2), $(C_3, e_3) \in M$. The distinguished edges e_1, e_2, e_3 have color 3. Produce 16 starting configurations X by exchanging colors 1 and 2 in C_2 and C_3 (4 possibilities) and, independently, by assembling a triangle T from the edges e_1, e_2, e_3 by identifying their endpoints in pairs (4 possibilities).

Step 2: Without loss of generality, assume that $X = (C_1, \{y,z\}), (C_2, \{x,z\}), (C_3, \{x,y\})$ with $T = \{x,y,z\}$. Reject X if

$$deg_{C_1[3]}(y) \neq deg_{C_2[3]}(x),$$

$$deg_{C_1[3]}(z) \neq deg_{C_3[3]}(x),$$

or

$$deg_{C_2[3]}(z) \neq deg_{C_3[3]}(y).$$

Otherwise, find all possible embedding of X into the situation as in Lemma 1 by permuting vertices in $N_{C_2[3]}(x)$, $N_{C_3[3]}(x)$ and $N_{C_3[3]}(y)$, while keeping the vertices of T fixed. Identify $N_{C_1[3]}(y)$ with $N_{C_2[3]}(x)$, $N_{C_1[3]}(z)$ with $N_{C_3[3]}(x)$, and $N_{C_2[3]}(z)$ with $N_{C_3[3]}(y)$. If the identified neighborhoods induce identical colorings in all three cases, then such relabeled X can be considered a partial coloring on

$$m = n(C_1) + n(C_2) + n(C_3) - deg_{C_1[3]}(y) - deg_{C_1[3]}(z) - deg_{C_2[3]}(z)$$

vertices. Reject X if it contains a triangle in color 1 or 2. After this step, the colored edges of X are only those which where taken from some C_i and the edges in color 3 between T and $V(X) - T$.

Step 3: For every partial coloring on m vertices obtained in Step 2, assign three possible colors to each uncolored edge, and iterate the following process. For each edge with more than one possible color, delete colors which lead to a forbidden clique or violate obvious degree restrictions. Terminate if some edge has no possible colors. If there is no edge for which the deletion of possible colors is enforced, then go to Step 4. There is no branching in this step.

Step 4: Perform the exhaustive search for all possible extensions of partial colorings obtained in Step 3 to full $(3,3,4)$ colorings on m vertices.

The algorithm outlined above did not produce any full $(3,3,4)$ coloring on the input set M. A large number of partial colorings were produced at Steps 2 and 3. These were generated by two independent implementations by each of the two authors, and they always agreed up to isomorphism. In addition, the Step 4 was tested on other data known to produce nonempty results.

Theorem 1. *There does not exist a $(3,3,4)$ coloring C with a triangle xyz in color 3, such that $deg_{C[3]}(x)$, $deg_{C[3]}(y)$, $deg_{C[3]}(z) > 13$.*

Proof: The computations described above showed that no such coloring exists. □

Two excellent programs, written by Brendan McKay, were used in this work: *nauty* [McK] for testing isomorphism of edge colorings, and *autoson* for distributing a large number of small tasks over a local area network. The total time required for all computations was about 0.5 CPU years, mostly on Sun Sparcstations and SGI Indys. This was achieved in a reasonable amount of time by employing a number of computers simultaneously.

4 New Bound

Theorem 2. $30 \leq R(3,3,4) \leq 31$.

Proof: The lower bound follows from a cyclic coloring of the edges of K_{29} with vertex set Z_{29}, in which $\{1,4,10,12\}$, $\{2,5,6,14\}$ and $\{3,7,8,9,11,13\}$ are the vertex distances in Z_{29} of the first, second, and third color, respectively. This coloring was found by Kalbfleisch [Kalb]. Lemma 1 and Theorem 1 give the upper bound. □

Theorem 3. $R(3,3,4) = 31$ *if and only if there exists a $(3,3,4;30)$ coloring C such that every triangle $T \subset C[3]$ has a vertex $x \in T$ with $deg_{C[3]}(x) = 13$, and furthermore C has at least 14 vertices v such that $deg_{C[1]}(v) = deg_{C[2]}(v) = 8$ and $deg_{C[3]}(v) = 13$.*

Proof: By Theorem 2 it is sufficient to show that any $(3,3,4;30)$ coloring C has the properties as required on the right hand side. $R(3,4) = 9$ implies that for every vertex v, $deg_{C[3]}(v) \geq 13 = 30-8-8-1$. Thus the first part is an immediate consequence of Theorem 1. For the second part, assume that C has at most 13 vertices with the required degree distribution. Then, for at least 17 vertices $deg_{C[3]}(v) \geq 14$, and since $R(3,3,3) = 17$ we conclude further that some three of them form a triangle in color 3. This contradicts Theorem 1, and completes the proof. □

It appears that our current approach is not efficient enough to proceed similarly with all $(3,3,3;13)$ colorings, though such computations would lead to the determination of the exact value of $R(3,3,4)$. We conjecture that $R(3,3,4) = 30$. The evidence to support it consists of the intermediate results of the computations completed for this work, which showed that the known $(3,3,4; \geq 28)$ colorings are quite exceptional. In addition, we have also performed a large number of heuristic searches for such colorings, without finding any new ones. The only known $(3,3,4;29)$ coloring is the one found by Kalbfleisch (described above), and the only known $(3,3,4;28)$ coloring is obtained from it by the deletion of one vertex.

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