Edge-Irreducible Quartic Graphs

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Dedicated to Professor Tatsuo Homma
on the occasion of his seventieth birthday

ABSTRACT. The edge-reduction of a simple regular graph is an operation which removes two vertices and preserves the regularity. It has played an important role in the study of cubic graphs [6,7,8]. Our main purpose is to study the structure of edge-irreducible quartic graphs. All edge-irreducible quartic graphs are determined from a constructive view point. Then a unique decomposition theorem for edge-irreducible quartic graphs is obtained.

1 Introduction

A graph $G = (V, E)$, with vertex set V and edge set E, means a simple (undirected) graph without loops and multi-edges. For avoiding confusion, $\langle u, v\rangle$ is used to denote an edge with endpoints u and v. To the contrary, $\{u, v\}$ means a set of vertices u and v.

A graph is said to be r-regular if r edges are incident to each of its vertices. For a graph $G = (V, E)$ and a subset $X \subset V$, $A(X) = \{u \in V - X | \langle u, v\rangle \in E, v \in X\}$ is said to be an *adjacent set* of vertices for X. We write $A(x)$ instead of $A(\{x\})$. $\langle A\rangle$ denotes the maximal subgraph of G with vertex set A.

For cubic (= 3-regular) graphs, an *edge-reduction* (H-reduction and X-reduction in their papers) was introduced by Johnson[1,3] and Kötzig[2], (Figure 1).

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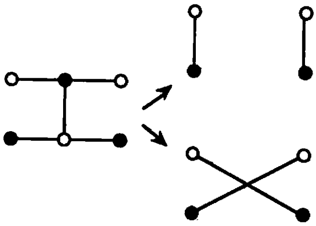


Figure 1. Edge-reductions

Theorem JK. ([1][2]) *K_4 and $K_{3,3}$ are the only edge-irreducible connected cubic graphs.*

Tsukui has introduced $\widetilde{S}$- and $\widetilde{X}$- transformations for 3-regular graphs (Figure 2), and proved:

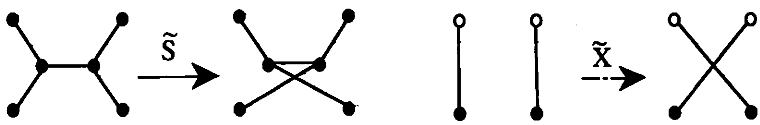


Figure 2. $\widetilde{S}$- and $\widetilde{X}$- transformations

Theorem Ts. (Tsukui [6]) *If G and H are connected cubic graphs with the same order, then G can be obtained from H by a finite sequence of $\widetilde{S}$-transformations.*

For quartic(=4-regular) graphs, Toida has shown a Johnson type theorem as follows.

Theorem Td. (Toida [4]) *Any connected quartic graph reduces to K_5 by two types of reductions (edge-reduction and vertex-reduction)(definition 1.1, 5.1).*

In the study of cubic graphs ([1],[2],[4],[6],[7],[8]), "edge-reduction" has played an important role.

A purpose of this paper is to make clear the structure of connected edge-irreducible quartic graphs. The main result is to classify all connected edge-irreducible quartic graphs.

Theorem 1. *If G is a connected edge-irreducible quartic graph, then G is a finite combination of minimal 4-blocks of $\mathcal{B} = \{B_{6,1}, B_{6,2}, B_{6,4}, B_{7,2}, B_{8,4}\}$.*

Further, the uniqueness of the above "combination" is proved.

Theorem 2. *For any connected edge-irreducible quartic graph G, all decomposition of G into minimal 4-blocks, with respect to combination, are unique up to order and $3B_{6,4} = 2B_{8,4}$.*

In §2, neighbourhoods of irreducible edges are investigated. In §3, five minimal 4-blocks are introduced and the concept of combinations is defined for classifying the results of §2. §4 is devoted to proving the main theorems. In the last section, the same techniques as in §4 are applied for "vertex-irreducible" quartic graphs. Then a counter example is shown to the theorem Td for 6-regular graphs.

Definition 1.1. *Let $G = (V, E)$ be an r-regular graph and $e = \langle u, v\rangle \in E$. Then, $P = \{(u_{i1}, v_{j1}), \ldots, (u_{ik}, v_{jk}), \ldots, (u_{i,r-1}, v_{j,r-1})\}$ is said to be a* pairing *for $\langle u, v\rangle$, if*

(1) $A(v) - \{u\} = \{u_{i1}, \ldots, u_{ik}, \ldots, u_{i,r-1}\}$,

(2) $A(u) - \{v\} = \{v_{j1}, \ldots, v_{jk}, \ldots, v_{j,r-1}\}$, and

(3) $u_{ik} \neq u_{ih}, v_{ik} \neq v_{ih}$ $(k \neq h)$.

Further, a pairing P is said to be proper *if*

(4) $u_{ik} \neq v_{jk}$ $(k = 1, 2, \ldots, r-1)$,

(5) $\{u_{ik}, v_{jk}\} \neq \{u_{ih}, v_{jh}\}$ $(k \neq h)$, and

(6) $\langle u_{ik}, v_{jk}\rangle \notin E$ $(k = 1, 2, \ldots, r-1)$.

For a proper pairing $P = \{(u_{ik}, v_{jk})\}$ for $e = \langle u, v\rangle$,

$$G/\!\!/(e; P) = (G - \{u, v\}) \cup \{\langle u_{ik}, v_{ik}\rangle | k = 1, 2, \ldots, r-1\}$$

is said to be obtained from G by an edge-reduction *of $e = \langle u, v\rangle$ by P. $G/\!\!/(e; P)$ is an r-regular graph with order=ord.$G - 2$.*

Definition 1.2. *An edge e of an r-regular graph G is said to be* edge-reducible *if there exists a proper pairing for e.*

When all edges of an r-regular graph G are irreducible, we say that G is irreducible. An edge e is said to be free *if any pairing for e is a proper pairing. We say that G is* not free *if G has no free edge.*

Definition 1.3. *A connected graph B is said to be an* r-block, *if*

(1) deg$(v) \leq r$ *and* deg$(v) \neq \pm 1 (\mod r)$ *for any vertex v of B,*

(2) *there exists a vertex of* deg $\neq r$, *and*

(3) *for any r-regular graph G and any embedding of B into G, all edges of B are edge-irreducible in G.*

Definition 1.4. *Any vertex in an r-block of degree $\neq r$ is called* white. *If a connected graph K is obtained from r-blocks $B_1, \ldots, B_s (s \geq 1)$ by identifying some white vertices, $u_1^i, u_2^i, \ldots, u_{i_k}^i$ with $\sum_j \deg(u_j^i) = r$, then K is an irreducible r-regular graph or an r-block. Then, K is called a* combination *of $B_1, \ldots, B_s (s \geq 1)$ $(K = B_1 \oplus \cdots \oplus B_s)$.*

An r-block B is minimal *if B is not a combination of r-blocks $B_1, \ldots, B_s$ $(s \geq 2)$.*

2 Quartic graph

Definition 2.1. *Let $e = \langle u, v \rangle$ be an edge of a quartic graph $G = (V, E)$ and K_V a complete graph with vertex set V. A set $O(e)$ of edges of $E(K_V)$ is called an* obstruction *(set) for an edge e in G, if $O(e) \subset E(G)$ implies that e is edge-irreducible in G. An obstruction $O(e)$ for an edge e is* minimal *if any proper subset of $O(e)$ is not an obstruction for e.*

Suppose that $e = \langle u, v \rangle$ is an irreducible edge in a quartic graph $G = (V, E)$. Then $3 \leq \#A(\{u, v\}) \leq 6$.

2.2. *For each case of $\#A(\{u, v\})$, we show all possible edge-reductions and minimal obstructions for e.*

case #3 ($\#A(\{u, v\}) = 3$) In this case only one reduction (one pairing) is possible (Figure 3-1). Hence there exist three obstructions for e which are essentially the same (Figure 3-2).

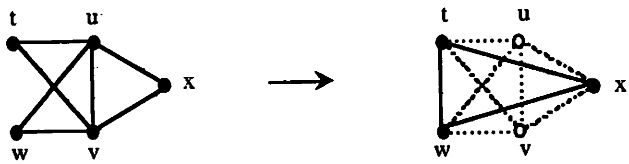
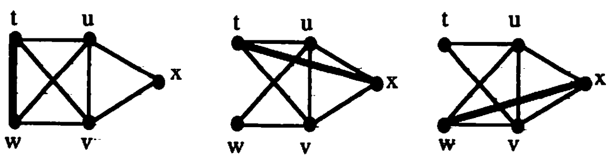


Figure 3-1. Case #3



(#3-1) (#3-2) (#3-3)

Figure 3-2. Minimal obstructions for #3

case #4 ($\#A(\{u, v\}) = 4$) Two edge-reductions can be obtained (Figure 4-1). Taking account of symmetry, there are three minimal obstructions for e (Figure 4-2).

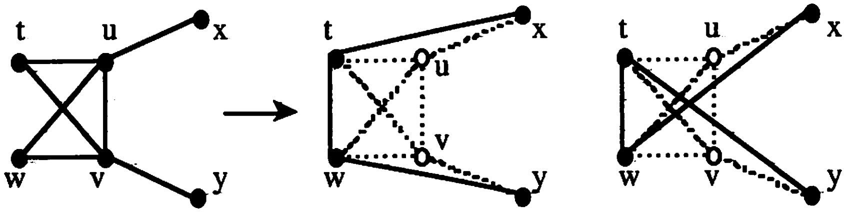


Figure 4-1. Case #4

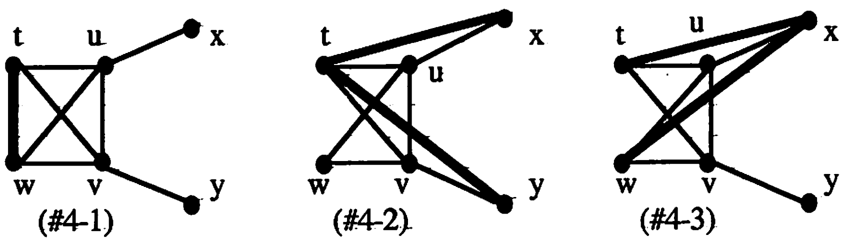


Figure 4-2. Minimal obstructions for #4

case #5 (#$A(\{u,v\})$ = 5) We have four reduced graphs (Figure 5-1). There are 11 minimal obstructions, but because of symmetry we need consider only 4 minimal obstructions (Figure 5-2).

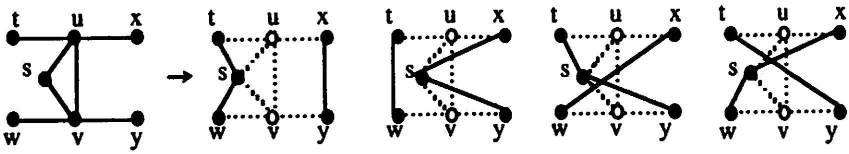


Figure 5-1. Case #5

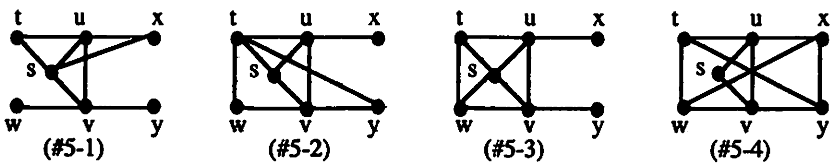
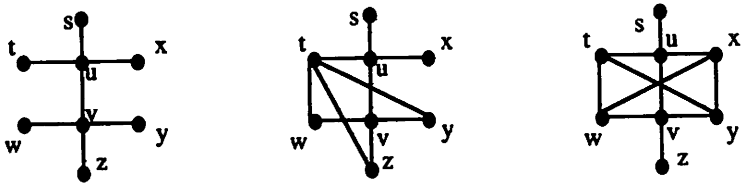


Figure 5-2. Minimal obstructions for #5

case #6 (#$A(\{u,v\}) = 6$) Since for this case any pairing may be a proper pairing, there are 6 reductions and $15 (= 3\times 2+3\times 3)$ minimal obstructions.

But it is sufficient for us to consider just two minimal obstructions by symmetry (Figure 6).



(#6-1) (#6-2)

Figure 6. Minimal obstructions for #6

3 4-blocks

Example 3.1 *There exist two irreducible graphs with order* ≤ 6, K_5 *and* $\overline{K}_2 + C_4$, *which are not combinations of two or more blocks (Figure 7).*

Each graph of $\mathcal{B} = \{B_{6,1}, B_{6,2}, B_{6,4}, B_{7,2}, B_{8,4}\}$ *is a minimal 4-block (Figure 8).* K_5 *is a combination of* $B_{6,2}$, *and* $\overline{K}_2 + C_4$ *is a combination of* $B_{7,2}$ *which is also a combination of* $B_{8,4}$.

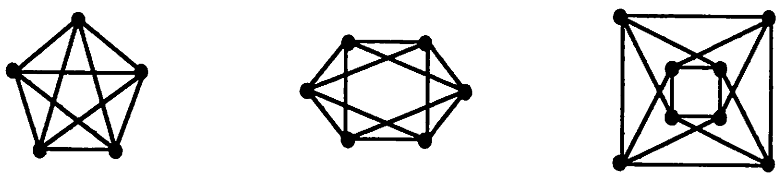
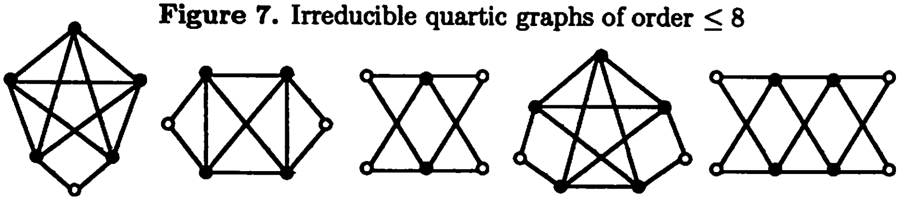


Figure 7. Irreducible quartic graphs of order ≤ 8



$B_{6,1}$ $B_{6,2}$ $B_{6,4}$ $B_{7,2}$ $B_{8,4}$

Figure 8. Five minimal 4-blocks

Lemma 3.2. *Let* $G = (V, E)$ *be an irreducible quartic graph having* K_4 *as a subgraph and ord.*$G = \#V \geq 6$. *Then, there exists a block* $B_{6,1}$ *or* $B_{6,2}$, *in* G, *which contains* K_4.

Proof: Let $U = \{t, u, v, w\} \subset V$ and K_4 be a complete graph in G with $V(K_4) = U$ (Figure 9). Hence $1 \leq \#A(U) \leq 4$. If $\#A(U) = 1$ then $G \cong K_5$. Since $\#A(U) \geq 3$ implies G is reducible, $\#A(U) = 2$. Let $A(U) = \{x, y\}$ so that

(1) $x \sim t, x \sim w, y \sim u, y \sim v$, or

(2) $x \sim t, x \sim w, x \sim u, y \sim v$ (Figure 9).

(1) implies that $B_{6,2}$ contains K_4 in G. For the case (2), since G is irreducible, $x \sim y$. Hence, there exists $B_{6,1}$ in G which contains K_4.

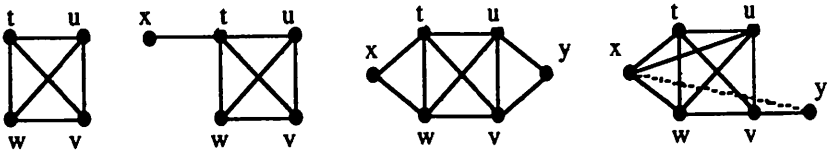


Figure 9. Blocks containing K_4

Lemma 3.3. *Let $G = (V, E)$ be an irreducible quartic graph having $\overline{K}_3 + K_2$ as a subgraph and ord.$G \geq 6$. Then, there exists a block $B_{6,1}$ or $B_{6,2}$, in G, which contains $\overline{K}_3 + K_2$.*

Proof: Let $\overline{K}_3 + K_2$ be a subgraph of G with $V(\overline{K}_3 + K_2) = \{t, u, v, w, x\}$ (Figure 3-1). From case #3 there exists an edge of $F = \{\langle t, w\rangle, \langle t, x\rangle, \langle w, x\rangle\}$ in G, $1 \leq \#(F \cap E) \leq 3$. If $\#(F \cap E) = 3$, it follows that $G \cong K_5$. Hence, as in lemma 3.2, there exists $B_{6,2}$ or $B_{6,1}$, respectively, according as $\#(F \cap E) = 1$ or 2, which contains $\overline{K}_3 + K_2$.

Lemma 3.4. *Let $G = (V, E)$ be an irreducible quartic graph having $K_1 + C_4$ as a subgraph and ord.$G \geq 7$. Hence, there exists a block $B_{7,2}$, in G which contains $K_1 + C_4$.*

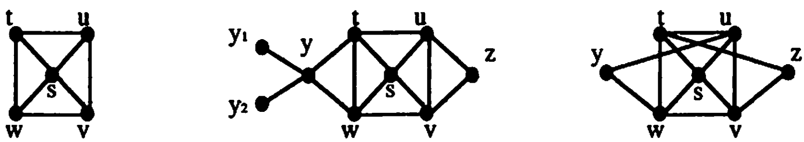


Figure 10. Blocks containig $K_1 + C_4$

Proof: Let $K_1 + C_4$ be a subgraph of G with $V(K_1 + C_4) = \{s, t, u, v, w\} = U$ as in Figure 10. $1 \leq \#A(U) \leq 4$. $\#A(U) = 1$ implies $G \cong \overline{K}_2 + C_4$ with ord.$G = 6 < 7$. As in the proof of lemma 3.2, we may assume $\#A(U) = 2$ and there are two cases as in Figure 10. For the center case of Figure 10, $\{(y_1, s), (y_2, w), (w, u)\}$ is a proper pairing for $\langle y, t\rangle$. The last case shows $B_{7,2}$.

Lemma 3.5. *Let $G = (V, E)$ be an irreducible quartic graph having $\overline{K}_2 + C_4 - K_3$ as a subgraph. Then $G \cong \overline{K}_2 + C_4$.*

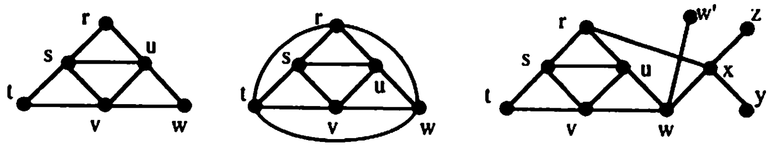


Figure 11. Blocks containig $\overline{K} + C_4 - K_3$

Proof: Let $\overline{K}_2 + C_4 - K_3$ be a subgraph of G with $V(\overline{K}_2 + C_4 - K_3) = \{r, s, t, u, v, w\} = U$, as in Figure 11. If $\#A(U) = 0$ then $G \cong \overline{K}_2 + C_4$. Suppose now $\#A(U) \neq 0$, then there exists a vertex $x \in V - U$ with $1 \leq \#(A(x) \cap U) \leq 3$. Let $w \sim x$. If $\#(A(x) \cap U) = 1$, $\langle w, x \rangle$ is obviously free. Suppose $2 \leq \#(A(x) \cap U) \leq 3$ and $A(x) = \{w, r, y, z\}$. Then $\{\langle v, r \rangle, \langle w', z \rangle, \langle u, y \rangle\}$ is a proper pairing for $\langle w, x \rangle$, even if $r = w'$ and/or $t = y$, where $A(w) = \{u, v, x, w'\}$. Hence $G \cong \overline{K}_2 + C_4$.

Lemma 3.6. *Let $G = (V, E)$ be an irreducible quartic graph having $\overline{K}_2 + K_2$ as a subgraph and ord.$G \geq 7$. Then there exists a block $B_{6,1}$, $B_{6,2}$, or $B_{7,2}$, in G, which contains $\overline{K}_2 + K_2$.*

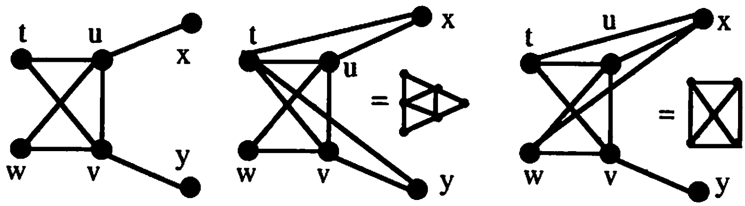


Figure 12. Blocks containing $\overline{K}_2 + K_2$

Proof: Let $\overline{K}_2 + K_2$ be a subgraph of G with $V(\overline{K}_2 + K_2) = \{t, u, v, w\} = U$ as in Figure 12. If $t \sim w$ or $x = y$, the argument of lemma 3.2 or lemma 3.3 applies, and $\overline{K}_2 + K_2$ is contained in $B_{6,1}$ or $B_{6,2}$. Hence suppose that $t \not\sim w$ and $x \neq y$. From case #4, only #4-2 and #4-3 can occur (Figure 4-2). In case #4-2 G is isomorphic to $\overline{K}_2 + C_4 - K_3$. For the case #4-3 it has $K_1 + C_4$ as a subgraph. This completes the proof by lemma 3.4 and lemma 3.5.

Lemma 3.7. *Let $G = (V, E)$ be an irreducible quartic graph having K_3 as a subgraph and ord.$G \geq 7$ then K_3 is contained in one of $B_{6,1}, B_{6,2}$ and $B_{7,2}$.*

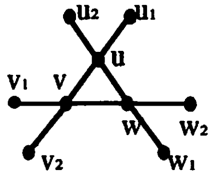


Figure 13

Proof: Let $U = \{u, v, w\}$ and $K_3 = K_U$, then $1 \leq \#A(U) \leq 6$. Suppose $\#A(U) = 6$ as in Figure 13. Since $\langle u, v\rangle$ is irreducible and $\#A(U) = 6$, it follows from #5 that $u_i \sim v_j (i, j = 1, 2)$ (#5-4).

From the same observation for $\langle v, w\rangle$ and $\langle w, u\rangle, v_i \sim w_j, w_i \sim u_j$ for any $i, j = 1, 2$. This contradicts G being quartic. Hence, $\#A(U) < 6$ and there exists $\overline{K}_2 + K_2$ which contains K_3. The proof is completed by lemma 3.6.

4 Proof of the theorems

Theorem 4.1. *Let $G = (V, E)$ be an irreducible quartic graph with $ordG \geq 7$. For any edge e of G, there exists a minimal block B in $\mathcal{B}$, as a subgraph, which containes e.*

Proof: Let $e = \langle u, v\rangle$ be an edge of G. If $\#A(\langle u, v\rangle) \leq 5$, e is contained in some K_3. From lemma 3.7, e is contained in one of $B_{6,1}, B_{6,2}$, and $B_{7,2}$.

Suppose now $\#A(\{u, v\}) = 6$. Since e is irreducible, there is a minimal obstruction which is #6-1 or #6-2 in Figure 6. For the case #6-1, $\langle\{t, u, v, w, y, z\}\rangle = B_{6,4}$.

Now suppose #6-2 occurs (Figure 14). Let $U = \{t, u, v, w, x, y\}$.

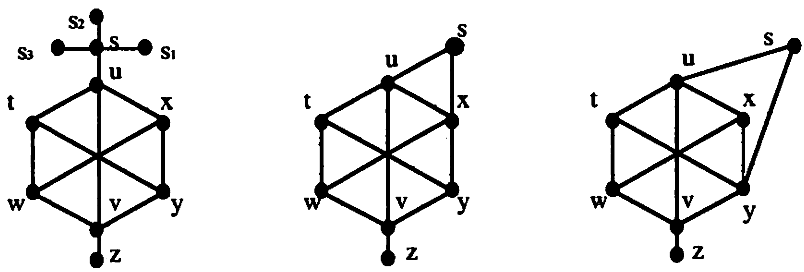


Figure 14.

(4.1.1 assertion) *There exists no K_3 containing $\langle s, u\rangle$.*

Suppose now that there exists K_3 which contains $\langle s, u\rangle$, and say $s \sim x$. By lemma 3.7 $K_3 = K_{\{s,u,x\}}$ is contained in a minimal block B, which is one of $B_{6,1}, B_{6,2}$ and $B_{7,2}$. In any case, the number of white vertices in $\{s, u, x\}$ is at most one. If not all of $\{s, u, x\}$ are white, there exists K_4 or $K_1 + C_4$ which contains K_3. This is a contradiction.

If there exists a white vertex in $\{s, u, x\}$, then $B = B_{6,2}$. If s is a white vertex then $u \sim y$ or $u \sim w$. If u (or x) is a white vertex, $w \sim y$ (or $t \sim v$). These contradict G being quartic.

(4.1.2 assertion) *If $\#(A(s) \cap U) \geq 2$ then $e = \langle u, v\rangle$ is contained in $B_{6,4}$.*

Suppose $s \sim y$, then $\langle u, v\rangle$ is contained in $\langle\{s, t, u, v, x, y\}\rangle = B_{6,4}$ as in Figure 14.

(4.1.3) *Finally, suppose that $\#(A(s) \cap U) = 1$.*

Let $A(s) = \{u, s_1, s_2, s_3\}$ (the left case of Figure 14). Since $\langle s, u\rangle$ is irreducible, say, $s_1 \sim t, s_1 \sim x$, and $s_1 \sim v$. Then $\langle\{s, s_1, t, u, v, x\}\rangle = B_{6,4}$.

Hence, if $\#A(\{u, v\}) = 6$, $e = \langle u, v\rangle$ is in $B_{6,4}$. This completes the proof.

Corollary 4.2. *Let $K = (V, E)$ be a 4-block. For any edge e of K, there exists a manimal block $B \in \mathcal{B}$, as a subgraph of K, which contains e.*

Proof: For an edge which is not incident to a white vertex, the proof is the same as that of Theorem 4.1. It is noted that there is no edge both of whose end points are white.

Now suppose that $e = \langle u, v\rangle \in E$ and one of u and v is a white vertex in K. Since u or v is white, $\#A(\langle u, v\rangle) = 3$ or 4. Hence only (#5-1) without $\{w, y\}$ and (#6-1) without $\{s, x\}$ can occur (Figure 5-2 and Figure 6). For those cases $B_{6,2}$ and $B_{6,4}$ contain e according as $\#A(\{u, v\}) = 3$ and $\#A(\{u, v\}) = 4$.

Definition 4.3. *Let $G = (V, E)$ be an edge-irreducible quartic graph or a 4-block. For any edge e of G, by the proof of (4.1) and (4.2), there exists a subgraph B_e which contains e and is isomorphic to one of $\mathcal{B} - \{B_{8,4}\}$. Since $B_{6,4}$ is a subgraph of $B_{6,1}$ and of $B_{7,2}$, we define the above B_e as follows.*

If there are two minimal block subgraphs B_1 and B_2 which contain e and B_1 is a proper subgraph of B_2, then we choose B_2 as B_e. Generally, for the case $B_e \cong B_{6,4}$, a subgraph B_e is not determined uniquely.

Let $\beta : E \to (\mathcal{B} - \{B_{8,4}\})$ be a map so that $\beta(e)$ is isomorphic to a maximal subgraph B_e of G which contains e, in the above sense.

Then, the following holds.

Proposition 4.4. *Let $G = (V, E)$ be an edge-irreducible quartic graph or a 4-block. Suppose e and f are different edges of G, where $\beta(e) \not\cong B_{6,4}$. Then $B_e = B_f$, or $B_e \cap B_f$ consists of at most two vertices.*

Proof: Let $W(B_e)$ denote the set of all white vertices of a minimal block subgraph B_e. Since $B_e \cong \beta(e) \not\cong B_{6,4}$, $\#W(B_e) \leq 2$. If $W(B_e) = W(B_f)$ then $B_e = B_f$ or $B_e \cap B_f = W(B_e)$ and the proposition is true.

Suppose now $B_e \neq B_f$ and $B_e \cap B_f \neq \emptyset$ for edges e and f of G. It can not happen that $W(B_e) \subset V(B_f) - W(B_f)$. We may assume that a white vertex, say u, of B_e is in $V(B_f) - W(B_f)$. Hence $B_f - \{u\}$ is disconnected. This is a contradiction to the fact that any minimal 4-block is 2-connected.

Example 4.5 *Let $V_{2n} = \{v_0, v_1, \ldots, v_{2n-1}\}$ be a set, and let*

$$F_{2n,4} = (V_{2n}, \{\langle v_i, v_{i+1}\rangle, \langle v_{n+i}, v_{n+i+1}\rangle, \langle v_i, v_{n+i+1}\rangle, \langle v_{n+i}, v_{i+1}\rangle | i = 0, \ldots, n-2\})$$

be a 4-block with $2n$ vertices ($n \geq 3$).

Denote by $S_{2n+1,2}$ the 4-block obtained from $F_{2(n+1),4}$ by identifying $v_0 = v_{n-1} (n \geq 3)$.

Let R_{2n} be the irreducible quartic graph obtained from $F_{2(n+1),4}$ by identifying $v_0 = v_{n-1}$ and $v_n = v_{2n-1}$. It is noted that $F_{6,4} = B_{6,4}, F_{8,4} = B_{8,4}$ and $S_{7,2} = B_{7,2}$. $F_{2n,4}(n \geq 5)$, $S_{2m+1,2}(m \geq 4)$ and $R_{2m}(m \geq 4)$ are combinations of $B_{6,4}$'s and $B_{8,4}$'s, as follows.

$$F_{2n,4} = \begin{cases} \frac{(n-1)}{2} B_{6,4} & (n : \text{odd} \geq 3) \\ \frac{(n-4)}{2} B_{6,4} \oplus B_{8,4} & (n : \text{even} \geq 4) \end{cases}$$

$$S_{2m+1,2} = \begin{cases} \frac{(m-3)}{2} B_{6,4} \oplus B_{8,4} & (m : \text{odd} \geq 5) \\ \frac{m}{2} B_{6,4} & (m : \text{even} \geq 4) \end{cases}$$

$$R_{2m} = \begin{cases} \frac{(m-3)}{2} B_{6,4} \oplus B_{8,4} & (m : \text{odd} \geq 5) \\ \frac{m}{2} B_{6,4} & (m : \text{even} \geq 4) \end{cases}$$

Definition 4.6. *Let $G = (V, E)$ be an irreducible quartic graph or a 4-block. We will define a map*

$$\beta^* : E \to \{B_{6,1}, B_{6,2}, F_{2n,4}, S_{2n+1,2}, R_{2n} | n \geq 3\}$$

as follows.

For any edge e of G, $\beta(e) \in \{B_{6,1}, B_{6,2}, F_{2n,4}, S_{2n+1,2} | n = 3\}$ by (4.3), since e is irreducible. Let

$$\beta^*(e) = \begin{cases} \beta(e), & \text{if } \beta(e) \in \{B_{6,1}, B_{6,2}, S_{7,2}\}, \\ G, & \text{if } G \cong R_{2n} \text{ for some } n \geq 3, \\ S_{2n+1,2}, & \text{if some subgraph } S_{2n+1,2} \text{ contains } e, \\ F_{2n,4}, & \text{if some subgraph } F_{2n,4} \text{ contains } e \text{ with maximal } n \geq 3. \end{cases}$$

B_e^* denotes a subgraph of G containing e and $B_e^* \cong \beta^*(e)$.

Proposition 4.7. (1) *For any different edges e and f of G, in (4.6), $B_e^* = B_f^*$ or $E(B_e^*) \cap E(B_f^*) = \emptyset$,*
(2) *for any edge f of B_e^*, $B_f^* = B_e^*$.*

Proposition 4.8. *Let $G = (V, E)$ be a 4-block with ord.$G \geq 7$ which is not minimal. Then there exist a block B of $\mathcal{B}$ and blocks K_i so that $G = B \oplus K_1 \oplus \cdots \oplus K_s (s \geq 1)$.*

Proof: Since G is a 4-block, there exists a vertex, say u, of degree 2. Let $A(u) = \{u_1, u_2\}$. From the fact that ord.$G \geq 7$, $B = B_{\langle u,u_1 \rangle} \not\cong B_{6,1}$.

If $B \cong B_{6,2}$ or $B \cong B_{7,2}$, let v be another white vertex of B. Hence we have $G = B \oplus K$, where $K = G - \{V(B) - \{v\}\} - \{u\}$ and $B \cap K = \{v\}$. Let f be an edge incident to v in K. By (4.3) there is a minimal block B_f so that $B_f \cap B = \{v\}$. Hence K is a 4-block with *ord.*$K \leq$ *ord.*$G - 5$.

Suppose now $B = B_{\langle u,u_1 \rangle} \cong B_{6,4}$. By (4.6) $B^* = B_{\langle u,u_1 \rangle} \cong S_{2n+1,2}$ or $B^* \cong F_{2n,4}$ for some $n \geq 3$. Since G is not minimal block, $n \geq 5$. Hence $G = B \oplus K_1 \oplus \cdots \oplus K_s$, where $B \cong B_{6,4}$ or $B \cong B_{8,4}$.

Theorem 1. *Let G be a connected irreducible quartic graph with ord.$G \geq 7$. Then G is a finite combination of minimal 4-blocks of $\mathcal{B} = \{B_{6,1}, B_{6,2}, B_{6,4}, B_{7,2}, B_{8,4}\}$.*

Proof: If $\beta(e) = B_{6,4}$ for any edge e of G, then $G \cong R_{2n}$ for some $k \geq 3$, or $\beta^*(e) = S_{2n+1,2} (n \geq 4)$ or $\beta^*(e) = F_{2n,4} (n \geq 5)$. Hence, in any case, G is a combination of $B_{6,4}$'s and $B_{8,4}$'s.

Suppose there exists an edge e of G so that $\beta(e) \neq B_{6,4}$. Now we can see that $G = B_e \oplus K$, where $K = G - \{V(B_e) - W(B_e)\}$ is a 4-block. Hence by (4.8) G is a combination of elements of $\mathcal{B}$.

Now the following is a corollary of the above theorem.

Theorem 2. *For any connected edge-irreducible quartic graph G, all decompositions of G into minimal 4-blocks, with respect to combination, are unique up to order and $3B_{6,4} = 2B_{8,4}$.*

Theorem 3. *Let $G = (V, E)$ be a connected reducible quartic graph. Then there exists a finite sequence $(G_0, G_1, \ldots, G_n)$ of connected quartic graphs, in which G_i edge-reduces to G_{i+1}, $(i = 0, 1, \ldots, n-1)$, $G_0 = G$ and G_n is edge-irreducible.*

Proof: It is sufficient to prove the theorem that if G is connected and reducible then there exists (another) edge e whose edge-reduction preserves connectedness.

Suppose $G = (V, E)$ is reducible connected quartic graph, and reduction of an edge $e = \langle u, v \rangle$ causes G to be disconnected. Assume that

$$G_1 \cup G_2 = G /\!\!/ e = (G - \{u, v\}) \cup \{\langle u_1, v_1 \rangle, \langle u_2, v_2 \rangle, \langle u_3, v_3 \rangle\}$$

is a disjoint union of two connected quartic graphs G_1 and G_2. If $\#A(\{u,v\}) \leq 4$ then $G /\!\!/ e$ is connected.

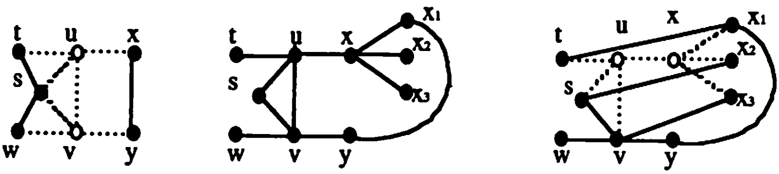


Figure 15

case A $(\#A(\{u,v\}) = 5)$

We can suppose that Figure 15 is the case without loss of generality. Let $G_1 \supset \{s,t,w\}$, $G_2 \supset \{x,y\}$ and $A(x) = \{u, x_1, x_2, x_3\}$ (as in Figure 15). Since G_2 is connected and quartic, there exists a path connecting y and one of x_1, x_2 and x_3, say x_1, in $G_2 - \langle x,y\rangle$. Hence $\{(x_1,t),(x_2,s),(x_3,v)\}$ is a proper pairing for $\langle u,x\rangle$ in G. Obviously $G /\!\!/ \langle u,x\rangle$ is connected.

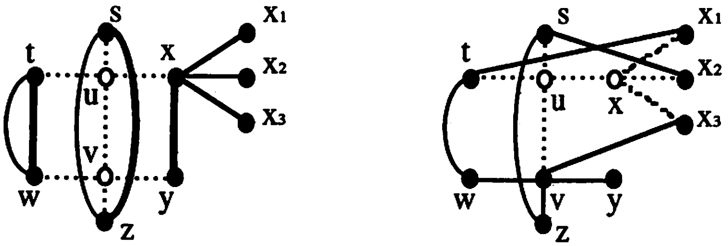


Figure 16.

case B $(\#A(\langle u,v\rangle) = 6)$

We may assume that $P = \{(t,w),(s,z),(x,y)\}$ is a proper pairing for $e = \langle u,v\rangle$, $G(t,w) \cup G(s,z) \cup G(x,y) = G /\!\!/ e; P$ and $G(x,y)$, at least, is disconnected from the other part. Where, $G(t,w)$ is a connected component containing t and w, and we may assume $G(t,w) = G(s,z)$ and $A(x) = \{u, x_1, x_2, x_3\}$ in G (Figure 16).

By the same argument as in case A,

(1) t and w are connected by a path in $G /\!\!/ e - \langle t,w\rangle$,

(2) s and z are connected by a path in $G /\!\!/ e - \langle s,z\rangle$, as in Figure 16 (left).

Hence $P' = \{(x_1,t),(x_2,s),(x_3,v)\}$ is a proper pairing for $\langle u,x\rangle$ and $G /\!\!/ \langle u,v\rangle; P'$ is connected as in Figure 16 (right). This completes the proof of the theorem.

5 v-reduction

Definition 5.1. *Let $G = (V, E)$ be a 2n-regular graph and $A(v) = \{v_1, v_2, \ldots, v_{2n}\}$ be the adjacent set of a vertex v of G. $P(v) = \{\{v_{i1}, v_{j1}\}, \ldots, \{v_{in}, v_{jn}\}\}$ is said to be an unordered pairing for v if $\{v_{i1}, \ldots, v_{in}, v_{j1}, \ldots, v_{jn}\} = A(v)$ and to be proper if v_{ik} and v_{jk} are not adjacent in G, $(k = 1, 2, \ldots, n)$.*

For a proper unordered pairing $P(v)$ for v,

$$G/\!\!/(v; P) = \{G - v\} \cup \{\langle v_{i1}, v_{j1}\rangle, \ldots, \langle v_{in}, v_{jn}\rangle\}(= G/\!\!/v)$$

is called a vertex-reduction *(abbreviated by* v-reduction*) of G at v along P.*

Definition 5.2. *A connected graph $\mathbb{B}$ is a* 4-v-block, *if*

(1) $3 \leq \deg(v) \leq 4$, *for any vertex v of $\mathbb{B}$,*

(2) *there exists a vertex of degree 3, and*

(3) *for any quartic graph G and any embedding of $\mathbb{B}$ into G, all vertices of $\mathbb{B}$ are vertex-irreducible in G.*

Definition 5.3. *If a connected graph $\mathbb{K}$ is obtained from 4-v-blocks $\mathbb{B}_1, \ldots, \mathbb{B}_s (s \geq 2)$ by adding edges which join vertices of degree 3, then $\mathbb{K}$ is a vertex-irreducible quartic graph or a 4-v-block. Then $\mathbb{K}$ is called a* v-combination *of $\mathbb{B}_1, \ldots, \mathbb{B}_s$. A 4-v-block $\mathbb{B}$ is* minimal *if $\mathbb{B}$ can not be a* v-combination *of $n \geq 2$ v-blocks.*

Example 5.4 $\mathbb{B}_{5,2}$ and $\mathbb{B}_{4,4}$, in Figure 17, are minimal 4-v-blocks.

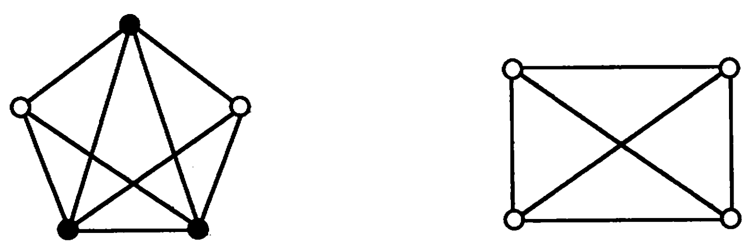


Figure 17. Minimal 4-v-blocks $\mathbb{B}_{5,2}$ and $\mathbb{B}_{4,4}$

The following is obtained in the same way as in section 4.

Proposition 5.5. *Any v-irreducible connected quartic graph, other than K_5, is a v-combination of finite number of copies of $\mathbb{B}_{5,2}$ and $\mathbb{B}_{4,4}$.*

Proposition 5.6. *There are infinitely many connected regular graphs which are edge-irreducible and vertex-irreducible.*

Proposition 5.6 is shown by a 6-regular graph in Figure 18, which was made by a student H. Takasu in 1991.

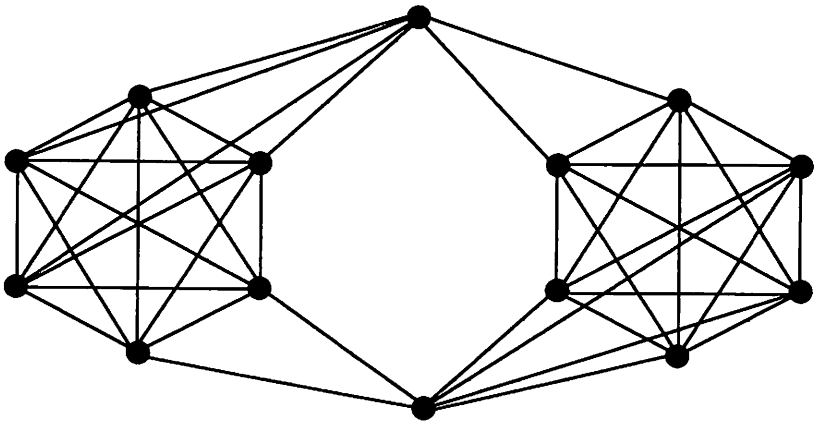


Figure 18. An edge-irreducible and vertex-irreducible 6-regular graph

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