The first eigenvalue of the discrete Dirichlet problem for a graph

Atsushi Katsuda
Department of Mathematics
Faculty of Science
Okayama University
Tsushima-naka 3-1-1
Okayama, 700
Japan
email: katsuda@math.okayama-u.ac.jp

Hajime Urakawa
Mathematics Laboratories
Graduate School of Information Sciences
Tohoku University
Katahira 2-1-1
Sendai, 980-77
Japan
email: urakawa@math.is.tohoku.ac.jp

ABSTRACT. We give a graph theoretic analogue of the celebrated Faber-Krahn inequality, that is, the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet problem for a bounded domain Ω in the Euclidean space $\mathbf{R}^n$ satisfies, $\lambda_1(\Omega) \geq \lambda_1(\mathbf{B})$ if $\mathrm{vol}(\Omega) = \mathrm{vol}(\mathbf{B})$, and equality holds only when Ω is a ball $\mathbf{B}$. The first eigenvalue $\lambda_1(G)$ of the Dirichlet problem of a graph $G = (V, E)$ with boundary satisfies, if the number of edges equals m, $\lambda_1(G) \geq \lambda_1(L_m)$, and equality holds only when G is the linear graph L_m.

1 Introduction

In this paper, we give a discrete analogue of the celebrated Faber-Krahn inequality (see [1]) for the first eigenvalue of the Dirichlet eigenvalue probelm for a bounded domain in the Euclidean space. Let $\lambda_1(\Omega)$ be the first eigenvalue of the Dirichlet eigenvalue problem for a given bounded domain

JCMCC 27 (1998), pp. 217-225

Ω in the n-dimensional Euclidean space $\mathbf{R}^n$:

$$\begin{cases} \Delta u = \lambda u & (\text{on } \Omega) \\ u = 0 & (\text{on } \partial\Omega). \end{cases}$$

The Faber-Krahn inequality says that if the volume of Ω is equal to the one of a ball $\mathbf{B}$ in $\mathbf{R}^n$, then

$$\lambda_1(\Omega) \geq \lambda_1(\mathbf{B}),$$

and equality holds only when Ω is a ball $\mathbf{B}$.

In graph theory, one can also introduce a graph with boundary, $G = (V, E) = (\dot{V} \cup \partial V, \dot{E} \cup \partial E)$ (see [4 and Sect. 2) and consider the Dirichlet eigenvalue problem of the combinatorial Laplacian Δ of G:

$$\begin{cases} \Delta u = \lambda u & (\text{on } \dot{V}) \\ u = 0 & (\text{on } \partial V). \end{cases}$$

Let us denote the eigenvalues of this problem by

$$0 < \lambda_1(G) < \lambda_2(G) \leq \cdots \leq \lambda_k(G),$$

where $k = \#(\dot{V})$ is the number of vertices in $\dot{V}$ (see for example, Lemma 1.9 in [3]).

Now we give here an example: we denote by white (resp. black) circles, vertices in $\dot{V}$ (resp. ∂V) and solid (resp. dotted) lines, edges in $\dot{E}$ (resp. ∂E). L_m will stands for the graph in Figure 1.

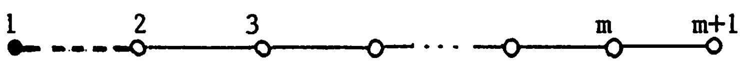

Figure 1

Our main result is stated as follows.

Theorem 1.1. *Let $G = (V, E) = (\dot{V} \cup \partial V, \dot{E} \cup \partial E)$ be a connected graph with boundary. Assume that the number of edges satisfies $\#(E) = \#(\dot{E} \cup \partial E) = m$. Then*

$$\lambda_1(G) \geq \lambda_1(L_m),$$

and equality holds if and only if G is equal to L_m.

In our previous paper [5], we treated a graph with boundary satisfying the additional condition that any vertex which has exactly one edge belongs to ∂V. In this case, the lower bound of the first eigenvalue is achieved by the graph of kite type illustrated in Figure 2, and can be proved alternatively by the new method of this paper.

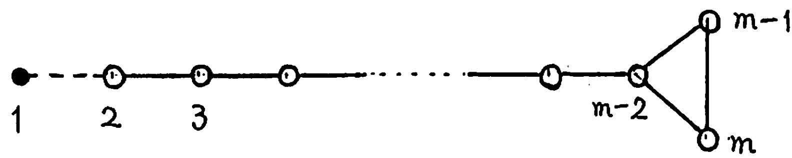

Figure 2

2 Preliminaries

In this section, we collect notions and results following [3] or [4] that will be used in the sequel.

A graph $G = (V, E)$ is a collection of the set V of vertices and the set E of edges connecting two vertices. We always assume that G is connected in this paper. A graph $G = (V, E) = (\dot{V} \cup \partial V, \dot{E} \cup \partial E)$ is the one with boundary by definition (see for instance [3], [4], [6]), i.e.,

(1) each edge in $\dot{E}$ has both end vertices in $\dot{V}$ and vice versa, and

(2) each edge in ∂E has exactly one end vertex in $\dot{V}$ and one in ∂V, and vice versa.

We call vertices in $\dot{V}$ (resp. ∂V) the *interior* (resp. *boundary*) vertices, and similarly for the edges. The *combinatorial Laplacian* Δ acting on the space $C(V)$ of real-valued functions on V is by definition

$$\Delta f(x) = f(x) - \frac{1}{m(x)} \sum_{y \sim x} f(y), \quad x \in V, \quad f \in C(V),$$

where $y \sim x$ means that x and y are connected by an edge in $E = \dot{E} \cup \partial E$, and for each $x \in V = \dot{V} \cup \partial V$, $m(x)$ is its degree, i.e., the number of edges in $E = \dot{E} \cup \partial E$ incident to x. Let $C_0(V)$ be the subspace of $C(V)$ consisting of functions vanishing on ∂V. A real number λ is an *eigenvalue* of Δ on $C_0(V)$ if there exists a non-vanishing function $f \in C_0(V)$ (called an *eigenfunction*) satisfying $\Delta f = \lambda f$. This means that f and λ satisfy the Dirichlet eigenvalue problem:

$$\begin{cases} \Delta f = \lambda f & (\text{on } \dot{V}), \\ f = 0 & (\text{on } \partial V). \end{cases}$$

The eigenvalues are labelled as in Sect. 1.

Here we should mention the relation between our Laplacian Δ and the one $\mathcal{L}$ in a recent book by Chung (cf. p. 3, [2]): let D be the diagonal matrix with the (v, v)-th entry having value $m(v)$, $v \in V$, i.e.,

$$Df(x) = m(x)f(x), \quad x \in V, \quad f \in C(V).$$

Then it holds that
$$\Delta = D^{-1/2}\,\mathcal{L}\,D^{1/2}.$$
Therefore, both the Dirichlet problems have the same eigenvalues (see Sect. 8.4 in [2]).

We state here for later use, the well-known characterization of the eigenvalues by the *Rayleigh quotient* (see [4] for instance) which is defined by
$$\mathcal{R}(f) = \frac{(\Delta f, f)}{(f,f)} = \frac{(df, df)}{(f,f)}, \quad 0 \not\equiv f \in C_0(V),$$
where the inner products are given by
$$(f_1, f_2) = \sum_{x\in V} m(x)\, f_1(x) f_2(x), \quad f_1,\ f_2 \in C_0(V),$$
and
$$(\psi_1, \psi_2) = \sum_{e\in E} \psi_1(e)\psi_2(e),$$
for two real valued functions ψ_1, ψ_2 on E. Moreover, for $f \in C_0(V)$, df is a function on E defined by
$$df(e) = f(t(e)) - f(o(e)), \quad e = (o(e), t(e)) \in E,$$
where an orientation on E is fixed once in advance and $o(e)$ (resp. $t(e)$) is the origin (resp. the terminal) of each edge e. It is well-known (cf. Lemma 1.9 in [3], Theorem 2.3 in [4]) that

Lemma 2.1. *The k-th eigenvalue of the Dirichlet eigenvalue problem for a graph $G = (V, E) = (\dot V \cup \partial V, \dot E \cup \partial E)$ with boundary, is given by*
$$\lambda_k(G) = \inf\left\{\mathcal{R}(f);\, 0 \not\equiv f \in C_0(V), \quad (f, \varphi_i) = 0\, (\forall i = 1, \cdots, k-1)\right\},$$
where φ_i is the i-th eigenfunction of the Dirichlet eigenvalue problem.

Moreover, the first eigenfunction φ_1 is positive every where on $\dot V$ or negative everywhere on $\dot V$.

Example 2.2: The first eigenvalue of the Dirichlet problem for the graph of type L_m is calculated as follows: $\lambda_1(L_1) = 1$, $\lambda_1(L_2) = 1 - 1/\sqrt{2}$, $\lambda_1(L_3) = 1 - \sqrt{3}/2$. In general, $\lambda_1(L_m)$ is a least positive zero of the equation $P_n(\lambda) = 0$, where $P_n(\lambda)$ is a polynomial in λ of order n given by
$$P_n(\lambda) = (\lambda - 1)Q_n(\lambda) - \frac{1}{2}Q_{n-1}(\lambda) \quad (n \geq 2),$$
and
$$Q_n(\lambda) = \prod_{j=1}^{n-1}\left(\lambda - 1 + \cos\left(\frac{j\pi}{n}\right)\right) \quad (n \geq 2); \quad Q_1(\lambda) = 0.$$

In the sequel, we always take the sign of the first eigenfunction φ_1 is positive. Then

Lemma 2.3. *The first eigenfunction $\varphi_1(x)$, $x \in V_{L_m}$ of the graph of type L_m is strictly monotone increasing, that is,*

$$\varphi_1(i) < \varphi_1(j) \quad (1 \le i < j \le m+1).$$

Proof: We first show φ_1 attains a maximum, say M, at $m+1$. Indeed, if not, there exists an integer i with $1 < i < m+1$ satisfying $\varphi_1(i) = M > \varphi_1(m+1)$. Define a function g on V_{L_m} by

$$g(k) = \begin{cases} \varphi_1(k), & 1 \le k \le i, \\ M, & \text{otherwise}, \end{cases}$$

for $k \in V_{L_m}$. Then we have

$$(g, g) > (\varphi_1, \varphi_1), \qquad (dg, dg) \le (d\varphi_1, d\varphi_1),$$

hence we obtain

$$\mathcal{R}(g) < \mathcal{R}(\varphi_1) = \lambda_1(L_m),$$

which contradicts Lemma 2.1.

Second, we show that $\varphi_1(x)$ is never a local convex function in $x \in V_{L_m}$. Assume that φ_1 is strictly locally convex, that is, there exist two integers $1 \le i < i+1 < j \le m+1$ such that φ_1 can be extended continuously to a strictly convex function on the interval $[i, j]$ in the real line $\mathbf{R}$. Taking a linear function h on $\mathbf{R}$ satisfying $h(i) = \varphi_1(i)$ and $h(j) = \varphi_1(j)$, define a test function f on V_{L_m} by

$$f(k) = \begin{cases} h(k), & i \le k \le j, \\ \varphi_1(k), & \text{otherwise}, \end{cases}$$

for $k \in V_{L_m}$. Then we have immediately

$$(f, f) > (\varphi_1, \varphi_1), \quad (df, df) < (d\varphi_1, d\varphi_1),$$

so that

$$\mathcal{R}(f) < \mathcal{R}(\varphi_1) = \lambda_1(L_m),$$

which contradicts Lemma 2.1.

Together with the fact that φ_1 attains a maximum at $m+1$, we have, in particular, that $\varphi_1(i) \le \varphi_1(j)$ for $1 \le i < j \le m+1$.

Lastly, we show that φ_1 is strict increasing. In order to see this, we only have to see that the following two cases never occur:

(1) there exists i such that $1 < i < m$ and

$$\varphi_1(i) = \varphi_1(i+1) < \varphi(i+2),$$

(2) or there exists i such that $1 < i < m$ and

$$\varphi_1(i) = \varphi_1(i+1) = \cdots = \varphi_1(m+1).$$

The first case (1) never occurs, because if so, φ_1 is locally convex. Assume that the second case occurs. Then we take for a sufficiently small $\epsilon > 0$, the following test function:

$$f_\epsilon(k) = \begin{cases} \varphi_1(k), & (1 \le k \le m), \\ \varphi_1(m+1) + \epsilon, & (k = m+1). \end{cases}$$

Then we have

$$(f_\epsilon, f_\epsilon) = (\varphi_1, \varphi_1) + 2\epsilon\varphi_1(m+1) + \epsilon^2,$$

and

$$(df_\epsilon, df_\epsilon) = (d\varphi_1, d\varphi_1) + \epsilon^2.$$

We denote the Rayleigh quotient of f_ϵ by $\mathcal{Q}(\epsilon)$ for $(\epsilon > 0)$: $\mathcal{Q}(\epsilon) = \mathcal{R}(f_\epsilon)$. Then

$$\mathcal{Q}(0) = \mathcal{R}(\varphi_1) = \lambda_1(L_m),$$

and its derivative in ϵ satisfies

$$\mathcal{Q}'(\epsilon) = \frac{2\epsilon(\varphi_1, \varphi_1) + 4\epsilon^2\varphi_1(m+1) - 2\epsilon(d\varphi_1, d\varphi_1) - 2\varphi_1(m+1)(d\varphi_1, d\varphi_1)}{\{(\varphi_1, \varphi_1) + 2\epsilon\varphi_1(m+1) + \epsilon^2\}^2}$$
$$< 0,$$

for all sufficiently small $\epsilon > 0$, since $\varphi_1(m+1)(d\varphi_1, d\varphi_1) > 0$. Therefore, there exists $\epsilon > 0$ satisfying

$$\mathcal{R}(\epsilon) < \mathcal{R}(0) = \lambda_1(L_m),$$

which is a contradiction. □

3 Plantation Technique

To show Theorem 1.1, we use the plantation technique which is defined as follows:

Definition 3.1: (1) The *plantation* is the method to produce the graph of type L_m from any graph with boundary. Let $G = (V, E) = (\dot{V} \cup \partial V, \dot{E} \cup \partial E)$ be a graph with boundary. Let φ_1 be the first eigenfunction of the Dirichlet eigenvalue problem for $G = (V, E)$. Let p be a vertex of G at which φ_1 attains a maximum, say M. Taking a boundary vertex q, let c be a geodesic connecting q and p. Recall that a geodesic in a graph G is by definition a shortest path connecting two vertices and a geodesic passes only once through each vertex of the geodesic.

Now the plantation is an operation of choosing the geodesic c, cutting each edge in $E - c$ and pasting it successively in a row from the vertex p, and adding new vertices if necessary. Then we obtain the graph of type L_m with $m = \#(E) = \#(\dot{E} \cup \partial E)$. Let us denote by P a plantation from G to L_m. See Figure 3.

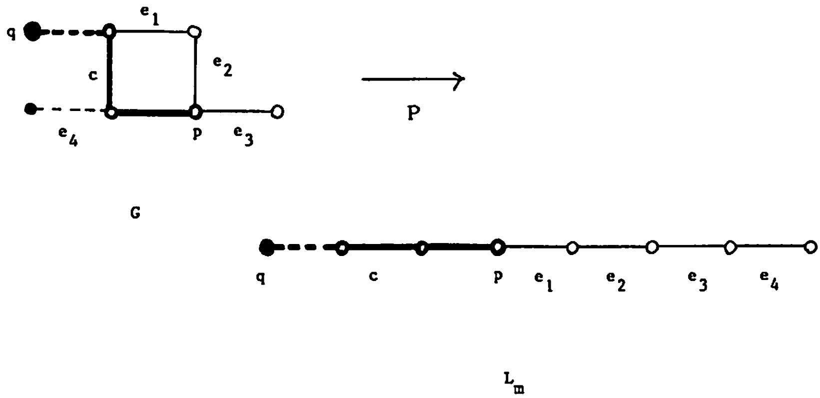

Figure 3

(2) Using the plantation P, for the eigenfunction φ_1 of G, let us define the function $\widetilde{\varphi}$ on L_m by

$$\widetilde{\varphi}(x) = \begin{cases} \varphi_1(x), & x \in c \\ M, & x \notin c, \end{cases}$$

and the multi-valued function Φ_1 on L_m by

$$\Phi_1(P(x)) = \varphi_1(x), \quad \text{and} \quad \Phi_1(P(y)) = \varphi_1(y),$$

for each $e = (x, y) \in E$ with end vertices x and y, where P is the plantation in (1) obtaining L_m from $G = (V, E)$.

Then we obtain:

Lemma 3.2. *We have:*

$$(\Phi_1, \Phi_1)_{L_m} = (\varphi_1, \varphi_1)_G, \quad (d\Phi_1, d\Phi_1)_{L_m} = (d\varphi_1, d\varphi_1)_G \tag{1}$$

$$(\widetilde{\varphi}, \widetilde{\varphi})_{L_m} \geq (\Phi_1, \Phi_1)_{L_m}, \quad (d\widetilde{\varphi}, d\widetilde{\varphi})_{L_m} \leq (d\Phi_1, d\Phi_1)_{L_m} \tag{2}$$

Proof: For (1), the proof is obtained by

$$\begin{aligned}(\Phi_1, \Phi_1)_{L_m} &= \sum_{e=(x,y)\in E} (\Phi_1(P(x))^2 + (\Phi_1(P(y))^2) \\ &= \sum_{x\in V} m(x)\varphi_1(x)^2 \\ &= (\varphi_1, \varphi_1)_G,\end{aligned}$$

and

$$\begin{aligned}(d\Phi_1, d\Phi_1)_{L_m} &= \sum_{e\in E} d\Phi_1(P(e)), d\Phi_1(P(e)) \\ &= \sum_{e\in E} d\varphi_1(e)d\varphi_1(e) \\ &= (d\varphi_1, d\varphi_1)_G.\end{aligned}$$

For (2), we have

$$\begin{aligned}(\Phi_1, \Phi_1)_{L_m} &= \sum_{e=(x,y)\in E} (\Phi_1(P(x))^2 + \Phi_1(P(y))^2) \\ &= \sum_{x\in c} 2\varphi_1(x)^2 + \sum_{x\notin c,\, P(x)\neq m+1} 2\varphi_1(x)^2 + \varphi_1(m+1)^2 \\ &\leq \sum_{x\in c} 2\varphi_1(x)^2 + \sum_{x\notin c,\, P(x)\neq m+1} 2M^2 + M^2 \\ &= (\widetilde{\varphi}, \widetilde{\varphi})_{L_m}.\end{aligned}$$

The last inequality follows from the fact that $\widetilde{\varphi}$ equals the constant M on $L_m - c$ and $\widetilde{\varphi}$ coincides with Φ_1 on c. □

4 Proof of the Main Theorem

We are now in position to give a proof of Theorem 1.1.

By Lemma 3.2, we have

$$\begin{aligned}\lambda_1(G) &= \frac{(d\varphi_1, d\varphi_1)_G}{(\varphi_1, \varphi_1)_G} \\ &= \frac{(d\Phi_1, d\Phi_1)_{L_m}}{(\Phi_1, \Phi_1)_{L_m}} \\ &\geq \frac{(d\widetilde{\varphi}, d\widetilde{\varphi})_{L_m}}{(\widetilde{\varphi}, \widetilde{\varphi})_{L_m}} \\ &\geq \lambda_1(L_m),\end{aligned}$$

which is the desired inequality.

Furthermore, assume that $G = (V, E)$ is not L_m. In order to show that $\lambda_1(G) > \lambda_1(L_m)$, we only have to see

$$\lambda_1(G) \geq \frac{(d\widetilde{\varphi}, d\widetilde{\varphi})_{L_m}}{(\widetilde{\varphi}, \widetilde{\varphi})_{L_m}} > \lambda_1(L_m).$$

By the assumption that G is not L_m, there exists a vertex in $G - c$. Therefore, $\widetilde{\varphi}$ is not strictly increasing by its construction, i.e., there exists $1 < i < m$ such that

$$\widetilde{\varphi}(i) = \widetilde{\varphi}(i+1) = \cdots = \widetilde{\varphi}(m+1).$$

By Lemma 2.3, $\widetilde{\varphi}$ is never the first eigenfunction of L_m. Hence, we obtain

$$\frac{(d\widetilde{\varphi}, d\widetilde{\varphi})_{L_m}}{(\widetilde{\varphi}, \widetilde{\varphi})_{L_m}} > \lambda_1(L_m).$$

We obtain Theorem 1.1. □

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