Maximum packings of bowtie designs

Elizabeth J. Billington*
Centre for Combinatorics
Department of Mathematics
The University of Queensland
Queensland 4072
Australia

C.C. Lindner†
Department of Discrete and Statistical Sciences
120 Math Annex
Auburn University
Auburn, Alabama 36849-5307
U.S.A.

ABSTRACT. A bowtie is a simple graph on 5 vertices with 6 edges, which consists of a pair of edge disjoint triangles having one common vertex. A bowtie design of order n is an edge disjoint decomposition of the complete undirected graph K_n into bowties. These exist if and only if $n \equiv 1$ or 9 (mod 12). For any $n \geq 5$, a maximum packing of the complete undirected graph K_n with bowties is a collection of edge-disjoint bowties picked from K_n, of maximum cardinality. The unused edges of K_n in this decomposition, if any, form the leave of the packing, which is necessarily a set with cardinality as small as possible.

In this paper a maximum packing of K_n with bowties is found, for all $n \geq 5$ and for all possible leaves.

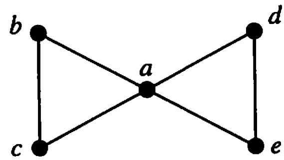
1 Introduction

A *bowtie* is a simple graph on 5 vertices with 6 edges, which consists of a pair of edge disjoint triangles having one common vertex. A *bowtie design*

*Research supported by the Australian Research Council.
†Research supported by NSF grant DMS-9300959

of order n is a pair (S, B), where B is a collection of edge disjoint bowties which partition the edge set of K_n, which has vertex set S. It is easy to see that a necessary condition for the existence of a bowtie design (S, B) of order n is $n \equiv 1$ or 9 (mod 12), $n \geq 5$, and in such a design the number of bowties is $|B| = n(n-1)/12$. Now given a bowtie design (S, B), if we denote by T the collection of $n(n-1)/6$ triangles making up the bowties in B, then (S, T) is a Steiner triple system. In 1988 Horák and Rosa [6] proved that *any* Steiner triple system of order $n \equiv 1$ or 9 (mod 12) can be partitioned into bowties, so that the *spectrum* (or set of achievable orders n) for bowtie designs is precisely the set of all $n \equiv 1$ or 9 (mod 12).

In the following we shall denote the bowtie:



by $\{\{a, b, c\}, \{a, d, e\}\}$, or, more frequently, by $\{abc, ade\}$ for short.

A bowtie design is an example of a G-design, that is, a decomposition of K_n into edge-disjoint copies of a simple graph G. In the case that no such decomposition exists, the problem of packing as many edge-disjoint copies of G as possible into K_n can instead be considered. The unused edges of K_n in such a packing are known as the *leave*; these should involve as few edges of K_n as possible, and in the case that a G-design exists, the leave is empty.

Maximum packings of K_n with certain graphs G have been considered previously. For $G = K_3$, packings of K_n with triples was done in [8] (see also [10]). For $G = C_4$, a cycle of length four, see [9], [5]; for $G = K_4$, see [2], and for $G = K_4 - e$, see [4].

The following table lists the smallest possible expected leaves when K_n is packed with bowties.

n (mod 12)	leave
1, 9	$\emptyset$
3, 7	K_3
5	C_4
0, 2, 6, 8	F, a 1-factor
4, 10, $n \geq 16$	X_i, $1 \leq i \leq 22$
11	Y_i, $1 \leq i \leq 4$
$n = 10$	X_i, $i = 1, 2, 4, 6, 7, 8, 12$–18

Here K_3 is of course a triangle, C_4 is a cycle of length 4, F denotes a 1-factor of K_n, and X_i, Y_i are given in the following figures. Note that each X_i is a spanning subgraph of odd degree, with four edges more than a 1-factor, while each Y_i has 7 edges and is of even degree.

The case $n = 10$ needs separate treatment from the cases $n \equiv 4, 10 \pmod{12}$, $n \geq 16$, because nine of the X_i involve *more* than 10 vertices, and so they cannot arise as a leave in a packing of K_{10} with bowties.

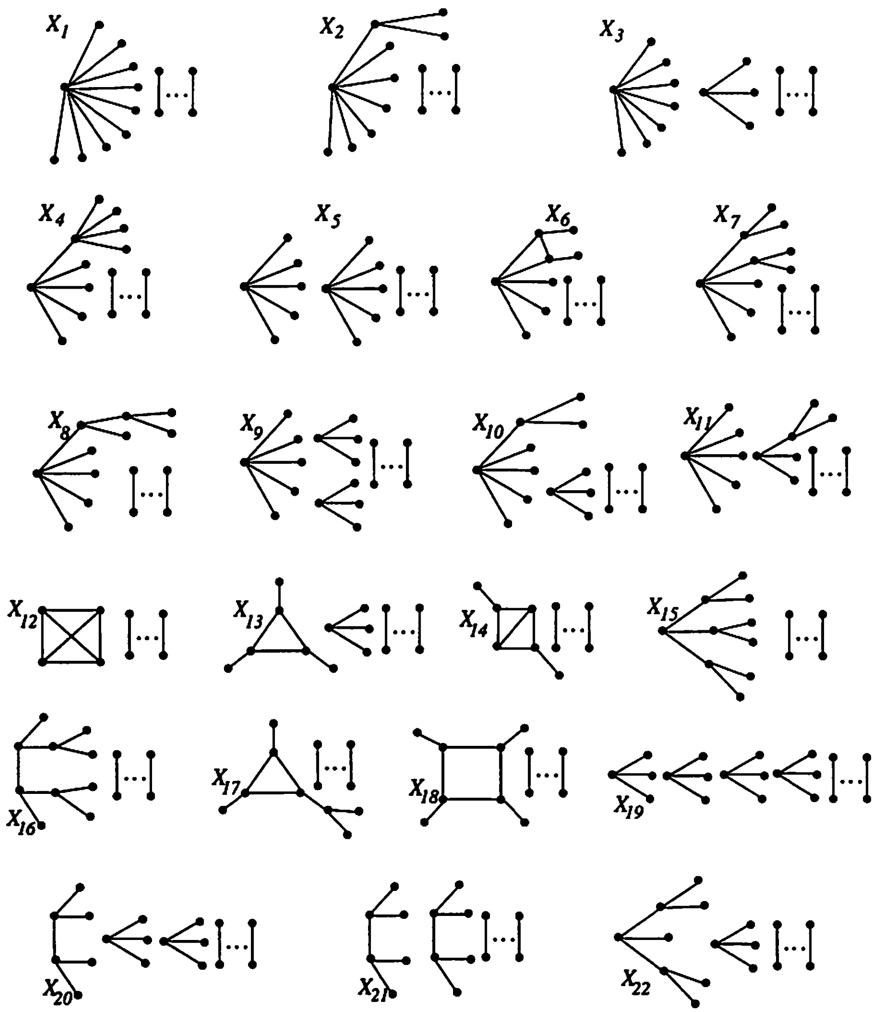


Figure 1. The leaves X_i, $1 \leq i \leq 22$, for $n \equiv 4, 10 \pmod{12}$, $n \geq 16$. (For $n = 10$, leaves are X_i, for $i = 1, 2, 4, 6, 7, 8$ and $12 \leq i \leq 18$.)

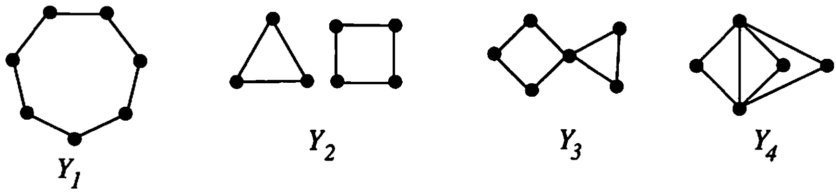


Figure 2. The leaves Y_i, $1 \le i \le 4$, for $n \equiv 11 \pmod{12}$.

The cases 1 and 9 (mod 12), with empty leave, were done in [6], while the cases 3 and 7 (mod 12), when the leave is K_3, were done in [1]. (In that paper the bowtie decompositions also had the extra property of being 2-perfect, which we do not consider here.)

In the remainder of this paper we show that there exists a maximal packing of K_n with bowties in all cases, and all the possible minimal leaves listed above can be achieved. In our constructions, we shall use a bowtie-decomposition of $K_{2,2,2}$.

Example 1.1

Let the vertex set of $K_{2,2,2}$ be $\{1,2\} \cup \{3,4\} \cup \{5,6\}$. Then the two bowties $\{135, 146\}$, $\{236, 245\}$ form an edge-disjoint decomposition of $K_{2,2,2}$.

We also need the definition of a *group divisible design*, GDD, on a set V. Let $\mathcal{G}$ be a partition of V into subsets called *groups*, and let $\mathcal{B}$ be a set of blocks (subsets of V) of sizes in a set K, such that each pair of points of V from different groups is in one block, while pairs of points from the same group are in no block. Then if there are t_i groups in $\mathcal{G}$ of size g_i, $1 \le i \le s$, we say the GDD is a K-GDD of type $g_1^{t_1} \dots g_s^{t_s}$. Here we shall have $K = \{3\}$, and we abbreviate to 3-GDD.

2 The cases $n \equiv 0, 2, 5, 6$ or 8 (mod 12)

When $n \equiv 0$ or 2 (mod 6), a packing of K_n with triangles has leave a 1-factor, while when $n \equiv 5 \pmod 6$, a packing of K_n with triangles has leave a cycle of length four, C_4. In the former case, with $n \equiv 0$ or 2 (mod 6), the number of triangles is even, while in the latter case, if $n \equiv 5 \pmod{12}$ the number of triangles is even, and if $n \equiv 11 \pmod{12}$, there is an odd number of triangles. In those cases where the packing of K_n by triangles contains an *even* number of triangles, an algorithm due to P. Boling ([7]) suffices to show that a pairing of triangles into bowties can be achieved. We include this for completeness.

So let (S, P) denote a packing of K_n with triangles; thus S is the vertex set of K_n and P is an edge-disjoint set of triangles with edges in K_n. Let

$u \in S$, $u \notin C_4$ in the case $n \equiv 5 \pmod{12}$. Start by taking all the triples in P *not* containing u, and pair off as many of these as possible into bowties. When no more will pair off, what remains is a *partial parallel class*, and also all the triples containing u. Let the triples in the partial parallel class be $t_1, t_2, \ldots, t_s$. Form sets of triples T_i, $1 \le i \le s$, by placing in T_i those triples containing u which intersect t_i. Each set T_i will contain either two or three triples.

Now we claim there exists a System of Distinct Representatives (SDR) in $T_1, T_2, \ldots, T_s$, by Hall's theorem. Form a bipartite graph with vertex set $V_1 \cup V_2$ where $V_1 = \{t_1, t_2, \ldots, t_s\}$ and V_2 is the set of all triples containing u, and with an edge between vertex t_i and triple $\{u, a, b\}$ if and only if t_i contains a or b. Then vertices in V_1 all have degree 2 or 3, while those in V_2 all have degree 0, 1 or 2. Thus there exists a matching $\{\{t_i, \{u, a_i, b_i\}\} \mid 1 \le i \le s\}$ where $\{t_i, \{u, a_i, b_i\}\}$ is an edge in the bipartite graph. This matching gives us s bowties, since either $a_i \in t_i$ or $b_i \in t_i$. The remaining triples, not yet in bowties, all contain u, and are necessarily even in number, and so pair off into bowties.

The above method of forming bowties clearly works for cases when the packing of K_n by triples contains an *odd* number of triples. However, in such cases, we need to show existence of *all* the possible leaves and corresponding packings. Once a suitable leave has been achieved, the remaining even number of triples can easily be formed into bowties, using the above algorithm. However, in the following sections, we generally give the bowties directly, since it is just as easy to do this.

3 The case $n \equiv 11 \pmod{12}$

Here the leave is one of Y_i, $1 \le i \le 4$. First we deal with the case K_{11}.

Example 3.1

Let K_{11} have vertex set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T\}$, and let

$$X = \{\{123, 156\}, \{147, 189\}, \{248, 260\}, \{259, 27T\}, \{340, 36T\}, \{358, 379\}\}.$$

K_{11} **with leave** Y_1: Bowties are: $X \cup \{\{469, 45T\}, \{01T, 078\}\}$.
The leave is a 7-cycle, $\{57, 76, 68, 8T, T9, 90, 05\}$.
K_{11} **with leave** Y_2: Bowties are: $X \cup \{\{469, 45T\}, \{08T, 057\}\}$.
The leave is $\{01, 09, 1T, 9T, 67, 68, 78\}$.
K_{11} **with leave** Y_3: Bowties are: $X \cup \{\{057, 45T\}, \{678, 08T\}\}$.
The leave is $\{01, 09, 1T, 9T, 49, 46, 69\}$.
K_{11} **with leave** Y_4: Bowties are: $X \cup \{\{469, 45T\}, \{678, 057\}\}$.
The leave is $\{01, 08, 09, 0T, 1T, 8T, 9T\}$.

Next we also deal separately with the case K_{23}.

Example 3.2
Let the vertex set for K_{23} be the same as that for K_{11} in Example 3.1 above, namely $S = \{0,1,2,3,4,5,6,7,8,9,T\}$, together with another 12 vertices, say W. On the set W take a 1-factorization, into 11 1-factors. On the set S we may place any one of the above four decompositions given in Example 3.1. Then we choose an arbitrary pairing of the eleven 1-factors with the eleven points in S, and form six triples from each 1-factor: if a 1-factor is $\{ab, cd, ef, gh, ij, kl\}$, and if this is paired with, say, the point x from S, then the six triples $\{abx, cdx, efx, ghx, ijx, klx\}$ are formed, and these form three bowties (since x is common to them all).

The result is a packing of K_{23} with bowties, and with leave the same as any one of the four leaves given in Example 3.1 above.

Now we give a construction for the general case $n = 12m+11$ when $m \geq 2$. Let the vertex set of K_n be $\{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 6m+5,\ j = 1,2\}$. Then bowties are taken as follows:

(1) On the set $\{\infty\} \cup \{(i,j) \mid 6m+1 \leq i \leq 6m+5,\ j = 1,2\}$, place a packing of K_{11}; see Example 3.1.

(2) On the set $\{\infty\} \cup \{(i,j) \mid i = 3s-2, 3s-1, 3s,\ j = 1,2\}$ for each $s = 1,2,\ldots,2m$, place a packing of K_7 (see [1]), ensuring that the triangle leave contains the point ∞. Thus there are $2m$ such triangles from these $2m$ packings, and they form m bowties to adjoin to our set.

(3) There exists a 3-GDD of type $5^1 3^{2m}$ on the set $\{1,2,\ldots,6m+5\}$; for each block abc in this GDD, on the set $\{(a,1),(a,2)\} \cup \{(b,1),(b,2)\} \cup \{(c,1),(c,2)\}$, place a decomposition of $K_{2,2,2}$ into bowties (Example 1.1).

The result is a packing of K_n with bowties, where $n = 12m+11$, and with leave whatever leave was chosen for the packing of K_{11} in (1) above.

This completes the case $n \equiv 11 \pmod{12}$.

4 The case $n = 10$

As pointed out in the introduction, only 13 of the 22 X_i leaves are possible here, as 9 of them involve more than 10 vertices.

Let the vertex set of K_{10} be $S = \{0,1,2,3,4,5,6,7,8,9\}$.

X_1 **leave:** Take any STS(9) on the set $S \setminus \{0\}$; then with the set S, the leave is $\{0i \mid 1 \leq i \leq 9\}$. It is easy to partition the 12 triples in the STS(9) into six bowties.

X_2 **leave:** Bowties are $\{\{123,147\},\{159,168\},\{089,369\},\{456,258\},$ $\{267,249\},\{348,357\}\}$; the leave is $\{01,02,03,04,05,06,07,78,79\}$.

X_4 **leave:** Bowties are $\{\{125,345\},\{138,179\},\{089,067\},\{146,369\},$ $\{237,478\},\{249,268\}\}$; the leave is $\{01,02,03,04,05,56,57,58,59\}$.

X_6 **leave:** Bowties are $\{\{013,159\},\{024,269\},\{147,168\},\{235,278\},$ $\{346,379\},\{458,567\}\}$; the leave is $\{05,06,07,08,09,89,38,49,12\}$.

X_7 leave: Bowties are $\{\{013, 168\}, \{024, 269\}, \{127, 149\}, \{238, 379\}, \{345, 478\}, \{567, 589\}\}$; the leave is $\{05, 06, 07, 08, 09, 15, 25, 36, 46\}$.

X_8 leave: Bowties are $\{\{058, 067\}, \{128, 169\}, \{135, 147\}, \{236, 257\}, \{348, 379\}, \{249, 456\}\}$; the leave is $\{01, 02, 03, 04, 09, 59, 89, 68, 78\}$.

X_{12} leave: Bowties are $\{\{024, 035\}, \{139, 016\}, \{148, 157\}, \{258, 269\}, \{237, 368\}, \{459, 467\}\}$; the leave is $\{12, 34, 56, 07, 08, 09, 78, 79, 89\}$.

X_{13} leave: Bowties are $\{\{012, 156\}, \{138, 179\}, \{239, 257\}, \{035, 367\}, \{046, 268\}, \{459, 478\}\}$; the leave is $\{14, 24, 34, 07, 08, 09, 58, 69, 89\}$.

X_{14} leave: Bowties are $\{\{014, 159\}, \{138, 167\}, \{237, 258\}, \{035, 369\}, \{026, 249\}, \{456, 478\}\}$; the leave is $\{12, 34, 07, 08, 09, 57, 68, 79, 89\}$.

X_{15} leave: Bowties are $\{\{012, 189\}, \{135, 146\}, \{236, 258\}, \{034, 379\}, \{056, 678\}, \{249, 457\}\}$; the leave is $\{07, 08, 09, 17, 27, 38, 48, 59, 69\}$.

X_{16} leave: Bowties are $\{\{047, 068\}, \{126, 137\}, \{149, 158\}, \{035, 569\}, \{245, 279\}, \{238, 346\}\}$; the leave is $\{01, 02, 09, 39, 89, 78, 48, 57, 67\}$.

X_{17} leave: Bowties are $\{\{013, 024\}, \{056, 157\}, \{168, 348\}, \{149, 259\}, \{278, 467\}, \{236, 379\}\}$; the leave is $\{07, 08, 09, 69, 89, 58, 35, 45, 12\}$.

X_{18} leave: Bowties are $\{\{013, 179\}, \{028, 045\}, \{146, 158\}, \{235, 247\}, \{269, 567\}, \{349, 368\}\}$; the leave is $\{06, 07, 09, 59, 89, 78, 37, 48, 12\}$.

5 The cases $n \equiv 4, 10 \pmod{12}$, $n \geq 16$

Here there are 22 possible different leaves X_i, $1 \leq i \leq 22$.

5.1 $n = 12m + 4$

First we need to deal with K_{16}.

Example 5.1

Let K_{16} have vertex set $\{1, 2, \ldots, 16\}$. We list the leaves and corresponding bowties:

X_1 leave: Leave is

$$\{\{7,16\},\{8,16\},\{9,16\},\{10,16\},\{11,16\},\{12,16\},\\ \{13,16\},\{14,16\},\{15,16\},\{1,2\},\{3,4\},\{5,6\}\}.$$

Bowties are

{1 6 11, 1 3 9}, {3 8 13, 3 5 11}, {4 9 14, 4 2 10}, {4 5 8, 4 6 12},
{6 7 10, 6 8 14}, {8 7 11, 8 1 10}, {9 8 12, 9 2 11}, {10 11 14, 10 3 12},
{11 4 13, 11 12 15}, {1 7 14, 1 12 13}, {15 2 8, 15 3 14}, {16 14, 16 2 5},
{5 1 15, 5 9 10}, {3 2 7, 3 6 16}, {7 4 15, 7 9 13}, {15 6 9, 15 10 13},
{12 5 7, 12 2 14}, {13 5 14, 13 2 6}.

X_2 **leave**: Leave is

$$\{\{7,16\},\{8,16\},\{9,16\},\{10,16\},\{11,16\},\{12,16\},\\ \{15,16\},\{13,15\},\{14,15\},\{1,2\},\{3,4\},\{5,6\}\}.$$

Bowties are

{6111, 623}, {7212, 7610}, {8313, 845}, {319, 3511},
{6412, 6814}, {7811, 7513}, {9812, 91013}, {13112, 13214},
{1018, 10414}, {1514, 1528}, {2911, 2510}, {3715, 31012},
{11413, 111214}, {1417, 1459}, {1615, 16613}, {15512, 151011},
{16314, 1624}, {9615, 947}.

X_3 **leave**: Leave is

$$\{\{9,16\},\{10,16\},\{11,16\},\{12,16\},\{13,16\},\{14,16\},\\ \{15,16\},\{5,8\},\{6,8\},\{7,8\},\{1,2\},\{3,4\}\}.$$

Bowties are

{6111, 6710}, {3813, 31012}, {9812, 91013}, {10515, 101114},
{139, 1415}, {4210, 4612}, {11213, 1810}, {1449, 14213},
{457, 41113}, {1417, 14512}, {325, 3614}, {15814, 15311},
{1128, 11712}, {21215, 279}, {9511, 9615}, {13715, 1356},
{1615, 1637}, {1626, 1648}.

X_4 **leave**: Leave is

$$\{\{11,16\},\{12,16\},\{13,16\},\{14,16\},\{15,16\},\{7,15\},\\ \{8,15\},\{9,15\},\{10,15\},\{1,2\},\{3,4\},\{5,6\}\}.$$

Bowties are

{1611, 1415}, {236, 2712}, {8313, 845}, {9414, 9812},
{7610, 7811}, {111014, 111215}, {14213, 14315}, {1024, 10316},
{5311, 5215}, {379, 3112}, {5910, 51214}, {131012, 1319},
{6412, 61315}, {7513, 7114}, {8614, 8110}, {1129, 11413},
{1615, 1647}, {1628, 1669}.

X_5 **leave**: Leave is

$$\{\{11,16\},\{12,16\},\{13,16\},\{14,16\},\{15,16\},\{8,10\},\\ \{8,9\},\{5,8\},\{6,8\},\{7,8\},\{1,2\},\{3,4\}\}.$$

Bowties are as for case X_3 above, except remove the bowties {9812, 91013} and {11213, 1810}, and replace them with {91016, 91213} and {1812, 11013}.

X_6 **leave**: Leave is

$$\{\{9,16\},\{10,16\},\{11,16\},\{14,16\},\{15,16\},\{12,14\},\\ \{13,15\},\{14,15\},\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}.$$

Bowties are

{1 6 11, 1 7 14},	{2 7 12, 2 4 10},	{3 8 13, 3 10 12},	{6 7 10, 6 4 12},
{9 8 12, 9 10 13},	{14 2 13, 14 10 11},	{5 7 13, 5 10 15},	{8 6 14, 8 1 10},
{7 9 15, 7 4 16},	{5 1 9, 5 8 11},	{3 6 15, 3 7 11},	{2 3 5, 2 6 9},
{14 3 9, 14 4 5},	{15 2 11, 15 4 8},	{11 4 9, 11 12 13},	{1 12 15, 1 4 13},
{16 5 12, 16 6 13},	{16 1 3, 16 2 8}.		

X_7 **leave:** Leave is

$$\{\{7,16\},\{8,16\},\{9,16\},\{14,16\},\{15,16\},\{10,14\},\{11,14\},\{12,15\},\{13,15\},\{1,2\},\{3,4\},\{5,6\}\}.$$

Bowties are

{1 6 11, 1 7 14},	{2 7 12, 2 4 10},	{3 8 13, 3 2 6},	{6 7 10, 6 4 12},
{5 10 15, 5 12 14},	{14 4 9, 14 6 8},	{5 4 8, 5 3 11},	{7 8 11, 7 5 13},
{13 9 10, 13 11 2},	{14 2 13, 14 3 15},	{4 7 15, 4 11 13},	{6 9 15, 6 13 16},
{9 1 5, 9 3 7},	{1 3 10, 1 8 15},	{11 2 15, 11 9 12},	{8 2 9, 8 10 12},
{16 2 5, 16 10 11},	{16 1 4, 16 3 12}.		

X_8 **leave:** Leave is

$$\{\{1,16\},\{6,16\},\{7,16\},\{8,16\},\{9,16\},\{1,2\},\{1,5\},\{2,3\},\{2,4\},\{10,11\},\{12,13\},\{14,15\}\}.$$

Bowties are

{1 3 7, 1 4 6},	{1 8 15, 1 9 14},	{2 6 15, 2 7 13},	{2 9 11, 2 10 14},
{5 3 11, 5 4 12},	{3 8 10, 3 9 15},	{8 4 14, 8 5 6},	{4 10 15, 4 11 13},
{9 5 10, 9 6 12},	{7 6 10, 7 8 11},	{13 3 6, 13 14 16},	{11 6 14, 11 15 16},
{4 7 9, 4 3 16},	{15 5 13, 15 7 12},	{12 2 8, 12 1 11},	{14 3 12, 14 5 7},
{13 8 9, 13 1 10},	{16 2 5, 16 10 12}.		

X_9 **leave:** Leave is

$$\{\{11,16\},\{12,16\},\{13,16\},\{14,16\},\{15,16\},\{5,8\},\{6,8\},\{7,8\},\{3,4\},\{9,4\},\{10,4\},\{1,2\}\}.$$

Bowties are obtained from the X_3 leave bowties by removing {9 8 12, 9 10 13}, {4 2 10, 4 6 12}, {14 4 9, 14 2 13} and replacing them with {12 8 9, 12 4 6}, {9 10 16, 9 13 14}, {2 4 14, 2 10 13}.

X_{10} **leave:** Leave is

$$\{\{11,16\},\{12,16\},\{13,16\},\{14,16\},\{15,16\},\{9,11\},\{10,11\},\{5,8\},\{6,8\},\{7,8\},\{3,4\},\{1,2\}\}.$$

Bowties are

{6111, 6710}, {8313, 8912}, {11213, 139}, {1246, 12514}, {1018, 10312}, {523, 5613}, {8211, 81415}, {12215, 12711}, {13411, 131015}, {9213, 91016}, {724, 759}, {10214, 1045}, {11314, 11515}, {1517, 1536}, {14713, 1414}, {9415, 9614}, {1615, 1637}, {1626, 1648}.

X_{11} leave: Leave is

$$\{\{11,16\},\{12,16\},\{13,16\},\{14,16\},\{15,16\},\{5,8\},\{6,8\},\{7,8\},\{7,9\},\{7,10\},\{3,4\},\{1,2\}\}.$$

Bowties are

{8313, 8912}, {1449, 14213}, {1024, 1018}, {14512, 1417}, {13411, 13715}, {1637, 1626}, {8416, 81415}, {1128, 11712}, {15212, 1569}, {1619, 16510}, {315, 329}, {725, 746}, {1545, 15310}, {11115, 1412}, {6113, 6511}, {1359, 131012}, {11910, 11314}, {6312, 61014}.

X_{12} leave: Leave is

$$\{\{1,16\},\{6,16\},\{11,16\},\{1,6\},\{1,11\},\{6,11\},\{3,12\},\{8,13\},\{4,9\},\{5,14\},\{10,15\},\{2,7\}\}.$$

Bowties are

{12 13 5, 12 14 1}, {1 4 6, 1 5 9}, {16 13 4, 16 9 14}, {2 3 9, 2 4 6}, {7 8 14, 7 9 11}, {13 14 2, 13 15 1}, {16 5 10, 16 15 2}, {5 2 8, 5 4 7}, {11 14 4, 11 15 5}, {6 7 15, 6 8 12}, {6 9 13, 6 10 14}, {1 2 10, 1 3 7}, {10 7 13, 10 9 12}, {8 9 15, 8 10 11}, {11 12 2, 11 13 3}, {3 4 10, 3 5 6}, {12 13 5, 12 14 1}, {15 12 4, 15 14 3}.

X_{13} leave: Leave is

$$\{\{1,16\},\{11,16\},\{6,16\},\{5,12\},\{5,13\},\{12,13\},\{5,14\},\{3,12\},\{8,13\},\{10,15\},\{4,9\},\{2,7\}\}.$$

Bowties are obtained from the X_{12} Leave bowties above by deleting {12 13 5, 12 14 1} and adding the bowtie {1 12 14, 1 6 11}.

X_{14} leave: Leave is

$$\{\{1,16\},\{2,16\},\{3,16\},\{1,2\},\{1,6\},\{2,6\},\{6,9\},\{4,7\},\{5,8\},\{10,13\},\{11,14\},\{12,15\}\}.$$

Bowties are

{16410, 16713}, {16511, 16814}, {16612, 16915}, {1413, 1710}, {2514, 2811}, {3615, 3912}, {159, 1812}, {11115, 1314}, {429, 4512}, {4815, 4311}, {6414, 6711}, {7212, 7515}, {738, 7914}, {21015, 2313}, {1035, 1068}, {10911, 101214}, {1356, 1389}, {131112, 131415}.

X_{15} leave: Leave is

$$\{\{14,15\},\{12,13\},\{10,11\},\{16,1\},\{16,2\},\{16,3\},$$
$$\{1,4\},\{1,5\},\{2,6\},\{2,7\},\{3,8\},\{3,9\}\}.$$

Bowties are

{16 4 5, 16 6 7},	{16 8 9, 16 11 12},	{16 13 14, 16 10 15},	{1 2 11, 1 3 6},
{4 6 12, 4 7 8},	{5 7 13, 5 2 9},	{1 10 14, 1 8 13},	{4 11 15, 4 9 14},
{5 10 12, 5 3 15},	{2 3 14, 2 8 10},	{6 8 15, 6 9 11},	{7 9 10, 7 3 12},
{13 2 4, 13 6 10},	{14 5 6, 14 7 11},	{15 1 7, 15 2 12},	{3 4 10, 3 11 13},
{8 5 11, 8 12 14},	{9 11 2, 9 13 15}.		

X_{16} leave: Leave is

$$\{\{1,16\},\{4,16\},\{5,16\},\{1,2\},\{1,6\},\{2,3\},$$
$$\{2,7\},\{3,8\},\{3,9\},\{10,11\},\{12,13\},\{14,15\}\}.$$

Bowties are

{4 2 14, 4 5 8},	{7 5 6, 7 10 12},	{10 3 13, 10 4 6},	{15 5 13, 15 3 12},
{6 9 13, 6 2 16},	{7 8 11, 7 3 16},	{16 8 9, 16 10 15},	{16 11 12, 16 13 14},
{1 3 4, 1 5 10},	{1 7 13, 1 8 12},	{1 9 15, 1 11 14},	{2 5 12, 2 8 13}
{2 9 10, 2 11 15},	{3 5 14, 3 6 11},	{4 7 15, 4 9 12},	{11 4 13, 11 5 9},
{6 8 15, 6 12 14},	{14 7 9, 14 8 10}.		

X_{17} leave: Leave is

$$\{\{1,16\},\{11,16\},\{6,16\},\{1,8\},\{1,4\},\{4,8\},$$
$$\{4,9\},\{8,13\},\{5,14\},\{10,15\},\{3,12\},\{2,7\}\}.$$

Bowties are obtained from the X_{12} leave bowties above by deleting {1 4 8, 1 5 9} and adding the bowtie {1 5 9, 1 6 11}.

X_{18} leave: Leave is

$$\{\{7,16\},\{11,16\},\{6,16\},\{2,11\},\{11,12\},\{3,12\},$$
$$\{6,12\},\{6,8\},\{9,10\},\{5,13\},\{1,14\},\{4,15\}\}.$$

Bowties are obtained from the X_{12} leave bowties above by deleting the six bowties containing 12, and replacing them with {8 3 16, 8 13 12}, {7 6 15, 7 10 13}, {3 11 13, 3 14 15}, {1 16 12, 1 6 11}, {12 4 9, 12 14 5}, {12 10 15, 12 2 7}.

X_{19} leave: Leave is

$$\{\{13,16\},\{14,16\},\{15,16\},\{5,8\},\{6,8\},\{7,8\},$$
$$\{3,4\},\{4,11\},\{4,12\},\{1,2\},\{2,9\},\{2,10\}\}.$$

Bowties are

{1611, 139}, {8313, 8912}, {1449, 141011}, {13112, 13214},
{1415, 1810}, {14512, 1417}, {1615, 1637}, {1626, 1648},
{523, 5710}, {1436, 14815}, {1128, 11315}, {15212, 15713},
{16910, 161112}, {13410, 13511}, {7911, 724}, {6913, 61015},
{5915, 546}, {1267, 12310}.

X_{20} **leave**: Leave is

$$\{\{13,16\},\{14,16\},\{15,16\},\{9,13\},\{10,13\},\{5,8\},\\ \{6,8\},\{7,8\},\{4,12\},\{4,11\},\{3,4\},\{1,2\}\}.$$

Bowties are

{3813, 3614}, {1449, 1417}, {141011, 14213}, {1626, 1648},
{16910, 161112}, {51113, 546}, {151213, 1589}, {14812, 14515},
{10312, 1057}, {729, 7413}, {1615, 1637}, {61015, 6712},
{1169, 11715}, {1613, 1912}, {1315, 1811}, {359, 3211},
{1014, 1028}, {2512, 2415}.

X_{21} **leave**: Leave is

$$\{\{13,16\},\{14,16\},\{15,16\},\{9,13\},\{10,13\},\{5,8\},\\ \{6,8\},\{7,8\},\{7,11\},\{7,12\},\{3,4\},\{1,2\}\}.$$

Bowties are the same as the first 8 listed above for X_{20}, together with:

{9611, 9112}, {8111, 8210}, {3211, 379}, {15411, 1516},
{2715, 2412}, {529, 5312}, {10612, 10315}, {1613, 1657},
{1510, 1413}, {7410, 7613}.

X_{22} **leave**: Leave is

$$\{\{13,16\},\{14,16\},\{15,16\},\{9,13\},\{10,13\},\{11,15\},\\ \{12,15\},\{5,8\},\{6,8\},\{7,8\},\{3,4\},\{1,2\}\}.$$

Bowties are the same as the first 6 listed above for X_{20}, together with:

{1615, 1637}, {10615, 1057}, {1267, 12310}, {927, 9611},
{1613, 1912}, {935, 9815}, {1123, 1147}, {13412, 13715},
{1315, 1811}, {1014, 1028}, {14812, 14515}, {2512, 2415}.

We also need to deal with the isolated cases K_{28} and K_{40}.

Example 5.2
For the 13 leaves X_i that arise in a bowtie packing of K_{10}, we use the following construction of a packing of K_{28} with a packing of K_{10} embedded in it. For this we need an idempotent commutative quasigroup $(Q, \circ)$ where $Q = \{1,2,3,4,5,6,7,8,9\}$, which contains a subquasigroup on $\{1,2,3\}$:

$\circ$	1	2	3	4	5	6	7	8	9
1	1	3	2	5	6	7	8	9	4
2	3	2	1	6	7	8	9	4	5
3	2	1	3	7	8	9	4	5	6
4	5	6	7	4	9	1	2	3	8
5	6	7	8	9	5	4	3	1	2
6	7	8	9	1	4	6	5	2	3
7	8	9	4	2	3	5	7	6	1
8	9	4	5	3	1	2	6	8	7
9	4	5	6	8	2	3	1	7	9

Let the vertex set of K_{28} be $\{\infty\} \cup \{(i,j) \mid 1 \le i \le 9,\ 1 \le j \le 3\}$. Place a bowtie packing of K_{10} on $\{\infty\} \cup \{(i,j) \mid 1 \le i \le 3,\ 1 \le j \le 3\}$ with any of the 13 leaves for K_{10}. Adjoin the nine edges

$$\{\{(4,1),(4,2)\},\{(4,3),(5,1)\},\{(5,2),(5,3)\},\{(6,1),(6,2)\},\{(6,3),(7,1)\},\\ \{(7,2),(7,3)\},\{(8,1),(8,2)\},\{(8,3),(9,1)\},\{(9,2),(9,3)\}\}.$$

to the leave. Then take a further 108 triples; nine contain ∞:

$$\{\infty,(4,2),(4,3)\},\{\infty,(5,1),(5,2)\},\{\infty,(5,3),(6,1)\},\\ \{\infty,(6,2),(6,3)\},\{\infty,(7,1),(7,2)\},\{\infty,(7,3),(8,1)\},\\ \{\infty,(8,2),(8,3)\},\{\infty,(9,1),(9,2)\},\{\infty,(9,3),(4,1)\}.$$

Ninety-nine more are obtained from the above quasigroup:

$$\{(x,1),(y,1),(x\circ y,2)\},\ \{(x,2),(y,2),(x\circ y,3)\},\ \{(x,3),(y,3),(\alpha(x\circ y),1)\},$$

where α is the permutation $(4\ 5\ 6\ 7\ 8\ 9)$, and where $x, y \in \{1,2,\ldots,9\}$ but x and y are not both chosen from $\{1,2,3\}$.

It is tedious but straightforward to pair these $9+99=108$ triples into 54 bowties, which are put with the bowties from the packing of order 10.

The nine leaves $X_3, X_5, X_9, X_{10}, X_{11}, X_{19}, X_{20}, X_{21}, X_{22}$ remain for K_{28}. We deal with these now. We use a bowtie packing of K_{28} with leave X_2, and then switch a few bowties. So first we list the packing with X_2 leave:

Let the vertex set for K_{28} from now on be $\{1,2,\ldots,28\}$. The X_2 leave will be

$$\{\{1,28\},\{2,28\},\{3,28\},\{10,28\},\{\{11,28\},\{12,28\},\{19,28\},\\ \{19,20\},\{\{19,21\},\{4,13\},\{5,22\},\{14,23\},\{\{6,15\},\\ \{7,24\},\{16,25\},\{8,17\},\{9,26\},\{18,27\}\}.$$

Then a bowtie packing is given by:

{1 2 3, 1 10 19}, {1 11 21, 1 12 20}, {21 20 28, 21 3 12}, {11 10 12, 11 2 20},
{2 12 19, 2 10 21}, {3 10 20, 3 11 19}, {28 5 14, 28 6 23}, {28 15 24, 28 7 16},
{28 8 25, 28 17 26}, {22 13 28, 22 9 27}, {18 9 28, 18 13 26}, {4 27 28, 4 9 17},
{1 4 14, 1 5 15}, {10 13 23, 10 14 24}, {19 6 22, 19 7 23}, {1 6 16, 1 7 17},
{10 15 25, 10 16 26}, {19 8 24, 19 9 25}, {1 8 18, 1 9 13}, {10 17 27, 10 18 22},
{19 4 26, 19 5 27}, {2 4 15, 2 5 16}, {11 13 24, 11 14 16}, {20 7 22, 20 8 23},
{2 6 17, 2 7 18}, {11 15 26, 11 16 27}, {20 9 24, 20 4 25}, {2 8 13, 2 9 14},
{11 17 22, 11 18 23}, {20 5 26, 20 6 27}, {3 4 16, 3 5 17}, {12 13 25, 12 14 26},
{21 8 22, 21 9 23}, {3 6 18, 3 7 13}, {12 15 27, 12 16 22}, {21 4 24, 21 5 25},
{3 8 14, 3 9 15}, {12 17 23, 12 18 24}, {21 6 26, 21 7 27}, {4 5 18, 4 6 10},
{13 14 27, 13 15 19}, {22 23 4, 22 24 1}, {4 7 11, 4 8 12}, {13 16 20, 13 17 21},
{22 25 2, 22 26 3}, {5 6 13, 5 7 12}, {14 15 22, 14 16 21}, {23 24 5, 23 25 3},
{5 8 10, 5 9 11}, {14 17 19, 14 18 20}, {23 26 1, 23 27 2}, {7 8 15, 7 9 10},
{16 17 24, 16 18 19}, {25 26 7, 25 27 1}, {6 7 14, 6 9 12}, {15 16 23, 15 18 21},
{24 25 6, 24 27 3}, {8 6 11, 8 9 16}, {17 15 20, 17 18 25}, {26 24 2, 26 27 8}.

For the leave X_3, remove the bowties

{20 7 22, 20 8 23}, {20 9 24, 20 4 25}, {13 16 20, 13 17 21}, {16 17 24, 16 18 19},
{15 16 23, 15 18 21}, {8 6 11, 8 9 16}, {17 15 20, 17 18 25}

from the X_2 leave case above, and replace them with

{17 18 25, 17 13 16}, {20 7 22, 20 16 19}, {21 18 19, 21 15 17}, {15 16 18, 15 20 23},
{16 8 23, 16 9 24}, {8 6 11, 8 9 20}, {20 4 25, 20 17 24}.

Next we give the case with leave X_{19}, and then use this case for the remaining leaves, X_5, X_9, X_{10}, X_{11}, X_{20}, X_{21} and X_{22}.

The X_{19} leave is

$$\{\{4,11\},\{4,18\},\{4,25\},\{5,12\},\{5,19\},\{5,26\},\{6,13\},\{6,20\},\{6,27\},$$
$$\{7,14\},\{7,21\},\{7,28\},\{1,24\},\{2,23\},\{3,22\},\{8,17\},\{9,10\},\{15,16\}\}.$$

Then a bowtie packing is given by:

{1 2 3, 1 8 9}, {1 10 15, 1 16 17}, {23 1 22, 23 16 24}, {2 8 10, 2 9 16},
{2 15 22, 2 17 24}, {3 8 24, 3 10 16}, {3 15 17, 3 9 23}, {8 15 23, 8 16 22},
{9 15 24, 9 17 22}, {10 17 23, 10 22 24}, {18 11 25, 18 7 17}, {20 13 27, 20 17 12},
{26 12 19, 26 14 23}, {28 14 21, 28 23 19}, {1 4 5, 1 11 26}, {1 18 12, 1 25 19},
{8 4 26, 8 11 12}, {8 18 19, 8 25 5}, {15 4 12, 15 11 19}, {15 18 5, 15 25 26},
{22 4 19, 22 11 5}, {22 18 26, 22 25 12}, {1 6 7, 1 13 28}, {1 20 14, 1 27 21},
{8 6 28, 8 13 14}, {8 20 21, 8 27 7}, {15 6 14, 15 13 21}, {15 20 7, 15 27 28},
{22 6 21, 22 13 7}, {22 20 28, 22 27 14}, {2 4 6, 2 11 27}, {2 18 13, 2 25 20},
{9 4 27, 9 11 13}, {9 18 20, 9 25 6}, {16 4 13, 16 11 20}, {16 18 6, 16 25 27},
{23 4 20, 23 11 6}, {23 18 27, 23 25 13}, {2 5 7, 2 12 28}, {2 19 14, 2 26 21},
{9 5 28, 9 12 14}, {9 19 21, 9 26 7}, {16 5 14, 16 12 21}, {16 19 7, 16 26 28},
{23 5 21, 23 12 7}, {3 4 7, 3 11 28}, {3 18 14, 3 25 21}, {10 4 28, 10 11 14},
{10 18 21, 10 25 7}, {17 4 14, 17 11 21}, {17 25 28, 17 5 13}, {17 19 6, 17 26 27},
{3 5 6, 3 12 27}, {3 19 13, 3 26 20}, {10 5 27, 10 12 13}, {10 19 20, 10 26 6},
{24 4 21, 24 11 7}, {24 18 28, 24 25 14}, {24 5 20, 24 12 6}, {24 19 27, 24 26 13}.

We give the remaining cases in terms of removing triples from the X_{19} case, and replacing them with appropriate different triples, in the process changing the leave accordingly. We do not explicitly list the bowties, as the triples can easily be so formed. (If necessary, apply the algorithm described in Section 2 above!)

X_5 **leave**: The leave is

$$\{\{4,11\},\{4,18\},\{4,25\},\{4,13\},\{4,20\},\{5,12\},\{5,19\},\{5,26\},\{5,14\},$$
$$\{5,21\},\{6,27\},\{7,28\},\{1,24\},\{2,23\},\{3,22\},\{8,17\},\{9,10\},\{15,16\}\}.$$

Remove the following triples from the case with leave X_{19}

16 4 13	23 4 20	16 5 14	23 5 21	16 23 24	8 13 14	8 20 21	24 5 20
8 27 7	1 27 21	1 13 28	1 8 9	9 5 28	3 9 23	17 5 13	3 8 24
3 26 20	3 19 13	10 19 20	17 19 6	17 12 20	10 12 13	13 20 27	24 26 13

and replace them with the following triples:

4 16 23	13 14 16	20 21 23	5 16 24	6 13 20	8 20 27	7 8 14	7 21 27
1 13 27	1 8 21	1 9 28	5 9 23	5 13 28	3 8 9	3 23 24	8 13 24
20 24 26	3 13 26	3 19 20	5 17 20	10 12 20	12 13 17	10 13 19	16 17 19

X_9 **leave**: The leave is

$$\{\{6,13\},\{6,20\},\{6,27\},\{6,21\},\{6,28\},\{7,14\},\{4,11\},\{4,18\},\{4,25\},$$
$$\{5,12\},\{5,19\},\{5,26\},\{1,24\},\{2,23\},\{3,22\},\{8,17\},\{9,10\},\{15,16\}\}.$$

Remove the following triples from the case with leave X_{19}

22 6 21	8 6 28	8 27 7	22 13 7	8 16 22	13 20 27	8 20 21	16 12 21
16 4 13	23 4 20	23 5 21	1 4 5	1 16 17	17 12 20	17 5 13	

and replace them with the following triples:

7828	72122	6822	71327	82027	81621	131622	1416
121617	122021	52123	1517	131720	4513	42023	

X_{10} **leave:** The leave is

$$\{\{6,13\},\{6,20\},\{6,27\},\{6,26\},\{5,6\},\{5,12\},\{5,19\},\{7,21\},\{14,28\},\{4,11\},\{4,18\},\{4,25\},\{1,24\},\{2,23\},\{3,22\},\{8,17\},\{9,10\},\{15,16\}\}.$$

Remove the following triples from the case with leave X_{19}

356	10266	142128	10527	31016	16514	257	21228
161221	162527	222512	21127	222714	91214	2916	16514

and replace them with the following triples:

51026	3610	3516	5714	2728	122128	141621	101627
121625	222527	121422	2912	91416	2516	51427	21127

X_{11} **leave:** The leave is

$$\{\{4,11\},\{4,18\},\{4,25\},\{4,13\},\{4,20\},\{5,12\},\{5,19\},\{5,26\},\{12,14\},\{12,21\},\{6,27\},\{7,28\},\{1,24\},\{2,23\},\{3,22\},\{8,17\},\{9,10\},\{15,16\}\}.$$

Remove the following triples from the case with leave X_{19}

16413	23420	91214	161221	16186	162324	24520	16514
2916	257	142128	9528	356	21813	22621	162628
11328	167	347	145				

and replace them with the following triples:

41623	202324	51624	5714	91216	141621	91428	259
5620	2721	212628	21626	21318	61618	131628	1613
367	147	1528	345				

X_{20} **leave:** The leave is

$$\{\{4,11\},\{4,18\},\{4,25\},\{19,25\},\{25,26\},\{5,12\},\{6,13\},\{6,20\},\{6,27\},\{7,14\},\{7,21\},\{7,28\},\{1,24\},\{2,23\},\{3,22\},\{8,17\},\{9,10\},\{15,16\}\}.$$

Remove the following triples from the case with leave X_{19}

152526	12519	145	8426	189	11015	10266	167	9267

and replace them with the following triples:

1519	4526	148	11525	101526	1610	179	6726	8926

X_{21} **leave:** The leave is

$$\{\{4,18\},\{4,25\},\{4,11\},\{11,20\},\{11,27\},\{5,12\},\{5,19\},\{5,26\},\{12,21\},\{12,28\},\{6,13\},\{7,14\},\{1,24\},\{2,23\},\{3,22\},\{8,17\},\{9,10\},\{15,16\}\}.$$

Remove the following triples from the case with leave X_{19}

16 11 20	2 11 27	16 12 21	2 12 28	8 11 12	8 20 21	2 5 7	2 8 10
2 4 6	23 4 20	10 12 13	8 27 7	2 18 13	10 5 27	13 20 27	23 25 13
11 18 25	23 11 6	2 25 20	16 18 6	16 4 13			

and replace them with the following triples:

11 12 16	16 20 21	2 7 28	2 8 11	2 6 27	8 10 12	7 8 21	2 12 13
5 7 27	2 5 10	8 20 27	10 13 27	11 23 25	2 4 20	2 18 25	6 11 18
13 16 18	4 6 16	6 20 23	13 20 25	4 13 23			

X_{22} **leave:** The leave is

$$\{\{4,18\},\{4,25\},\{4,11\},\{11,12\},\{11,19\},\{13,25\},\{20,25\},\{5,26\},\{6,27\},\{7,14\},\{7,21\},\{7,28\},\{1,24\},\{2,23\},\{3,22\},\{8,17\},\{9,10\},\{15,16\}\}.$$

Remove the following triples from the case with leave X_{19}

8 11 12	15 11 19	23 25 13	2 25 20	8 25 5	11 18 25	15 18 5	23 11 6
8 6 28	22 20 28	2 15 22	15 25 26	8 4 26	8 16 22	16 4 13	22 13 7
15 20 7	3 26 20	3 4 7					

and replace them with the following triples:

5 8 12	11 15 18	5 15 19	5 18 25	6 13 23	11 23 25	6 8 11	6 20 28
2 20 22	2 15 25	8 25 26	8 22 28	4 8 16	13 16 22	7 15 22	15 20 26
3 7 20	3 4 26	4 7 13					

This completes the packing of K_{28} with triples, for any X_i leave, $1 \le i \le 22$. Bowties can be easily formed for each of the above packings, using the algorithm described in Section 2 if necessary!

Example 5.3

A 3-GDD of type $8^1 4^3$ exists on 20 points. So let the vertices of K_{40} be $\{(i,j) \mid 1 \le i \le 20,\ j = 1,2\}$, and suppose that the 3-GDD has groups $\{1,2,3,4\}$, $\{5,6,7,8\}$, $\{9,10,11,12\}$, $\{13,14,\dots,20\}$. Then on $\{(i,j) \mid 1 \le i \le 4,\ j = 1,2\}$, $\{(i,j) \mid 5 \le i \le 8,\ j = 1,2\}$ and $\{(i,j) \mid 9 \le i \le 12,\ j = 1,2\}$, place a bowtie packing of K_8 (with 1-factor leave). On $\{(i,j) \mid 13 \le i \le 20,\ j = 1,2\}$, place a bowtie packing of K_{16} (with any one of the 22 leaves X_i possible). Finally, for each block $\{a,b,c\}$ of the 3-GDD, place a bowtie packing of $K_{2,2,2}$ on the vertex set $\{(a,j) \mid j =$

$1,2\} \cup \{(b,j) \mid j=1,2\} \cup \{(c,j) \mid j=1,2\}$. The result is a bowtie packing of K_{40}, with leave any one of the X_i, $1 \leq i \leq 22$.

Now we can give a general construction. Let $n = 12m+4$, $m \geq 4$, and let K_n have vertex set $\{(i,j) \mid 1 \leq i \leq 6m+2,\ j=1,2\}$. Take bowties as follows:

(1) On $\{(i,j) \mid 6m-5 \leq i \leq 6m+2,\ j=1,2\}$, place a packing of K_{16} with bowties, with *any* of the 22 chosen leaves.

(2) On $\{(i,j) \mid 6s-5 \leq i \leq 6s,\ j=1,2\}$, for $s=1,2,\ldots,m-1$, place a packing of K_{12} with bowties (and with leave a 1-factor).

(3) On $\{1,2,\ldots,6m+2\}$, take a 3-GDD of type $8^1 6^{m-1}$, which exists for $m \geq 4$. For each block abc in this GDD, on the set $\{(a,1),(a,2)\} \cup \{(b,1),(b,2)\} \cup \{(c,1),(c,2)\}$, place a decomposition of $K_{2,2,2}$ into bowties (Example 1.1).

The result is a packing of K_n with bowties, in the case $n = 12m+4$, $m \geq 4$, and with each of the 22 leaves X_i possible.

5.2 $n = 12m+10$

In Section 5 the case K_{10} was dealt with; 13 of the 22 leaves X_i are possible. For the general construction in this case we also need to obtain maximum packings of K_{22}. Note that for the leaves X_i, $i = 1,2,4,6,7,8,12$–18, we may use the results on K_{10} in Section 5 above as follows:

Take the vertex set of K_{22} to be

$$\{0,1,2,3,4,5,6,7,8,9\} \cup \{a,b,c,d,e,f,g,h,i,j,k,l\}.$$

On the set $\{0,1,2,3,4,5,6,7,8,9\}$ place a packing of K_{10} with any of its 13 possible leaves. On the set $\{a,b,c,d,e,f,g,h,i,j,k,l\}$, take a 1-factorisation of K_{12}, which has 11 1-factors, and pair ten of these 1-factors with the ten points $0,1,2,3,4,5,6,7,8,9$ in some order. (The remaining 1-factor forms part of the final leave.) Next, for each 1-factor such as $\{ab, cd, ef, gh, ij, kl\}$, if this is paired with vertex 0, then the bowties

$$\{0\ a\ b,\ 0\ c\ d\}, \{0\ e\ f,\ 0\ g\ h\}, \{0\ i\ j,\ 0\ k\ l\}$$

are adjoined to the maximum packing. The result is a maximum packing of K_{22}, with the same X_i leave as was chosen for K_{10}.

The next example deals with the other leaves for K_{22}.

Example 5.4

Let K_{22} have vertex set $\{1,2,3,\ldots,21,22\}$.

X_3 leave: Leave is
$\{\{15,22\},\{16,22\},\{17,22\},\{18,22\},\{19,22\},\ \{20,22\},\ \{21,22\},\ \{11,14\},$

{12,14}, {13,14}, {9,10}, {7,8}, {5,6}, {3,4}, {1,2}}.
Bowties are

{16 15 20, 16 3 14},	{19 14 15, 19 3 20},	{11 10 15, 11 20 21},	{7 6 11, 7 10 16},
{18 7 20, 18 6 21},	{21 12 15, 21 16 17},	{12 2 17, 12 4 20},	{5 2 11, 5 17 20},
{14 6 20, 14 2 4},	{10 2 21, 10 14 18},	{21 7 14, 21 1 5},	{1 8 15, 1 3 11},
{3 10 17, 3 2 7},	{4 11 18, 4 5 9},	{12 5 19, 12 11 16},	{13 8 9, 13 3 5},
{15 5 7, 15 4 17},	{19 9 11, 19 6 17},	{18 2 19, 18 5 16},	{1 9 20, 1 7 19},
{3 9 21, 3 6 12},	{8 19 21, 8 12 18},	{11 8 17, 11 13 22},	{8 5 14, 8 3 22},
{6 1 10, 6 2 22},	{15 3 18, 15 6 13},	{13 4 21, 13 2 20},	{22 1 12, 22 5 10},
{22 9 14, 22 4 7},	{9 7 12, 9 17 18},	{17 7 13, 17 1 14},	{1 13 18, 1 4 16},
{13 16 19, 13 10 12},	{4 6 8, 4 10 19},	{8 2 16, 8 10 20},	{9 2 15, 9 6 16}.

X_5 leave: Leave is
{{17,22},{18,22},{19,22}, {20,22}, {21,22}, {11,14}, {12,14}, {13,14}, {14,15}, {14,16}, {9,10}, {7,8}, {5,6}, {3,4}, {1,2}}.
Bowties are obtained from those for X_3 above, by removing {16 15 20, 16 3 14}, {19 14 15, 19 3 20} and adding {15 16 22, 15 19 20}, {3 14 19, 3 16 20}.

X_9 leave: Leave is
{{17,22},{18,22},{19,22}, {20,22}, {21,22}, {11,14}, {12,14}, {13,14}, {10,15}, {10,16}, {9,10}, {7,8}, {5,6}, {3,4}, {1,2}}.
Bowties are obtained from those for X_3 above, by removing {16 15 20, 16 3 14}, {11 10 15, 11 20 21}, {7 10 16, 7 6 11}, {18 7 20, 18 6 21}. and adding {16 15 22, 16 3 14}, {20 7 16, 20 11 15}, {7 10 11, 7 6 18}, {21 6 11, 21 18 20}.

X_{10} leave: Leave is
{{17,22},{18,22},{19,22}, {20,22}, {21,22}, {15,21}, {16,21}, {13,14}, {11,14}, {12,14}, {9,10}, {7,8}, {5,6}, {3,4}, {1,2}}.
Bowties are obtained from those for X_3 above, by removing

{16 15 20, 16 3 14}	{21 16 17, 21 12 15}	{12 4 20, 12 2 17}	{18 7 20, 18 6 21}
{5 17 20, 5 2 11}	{14 2 4, 14 6 20}	{10 14 18, 10 2 21}	{11 20 21, 11 10 15}

and adding

{16 15 22, 16 3 14}	{15 10 11, 15 12 20}	{21 2 11, 21 12 17}	{20 7 18, 20 16 17}
{4 14 20, 4 2 12}	{5 2 17, 5 11 20}	{6 14 18, 6 20 21}	{10 2 14, 10 18 21}

X_{11} leave: Leave is
{{17,22},{18,22},{19,22}, {15,22}, {16,22}, {11,14}, {12,14}, {13,14}, {13,20}, {13,21}, {9,10}, {7,8}, {5,6}, {3,4}, {1,2}}.
Bowties are obtained from those for X_3 above, by removing

{11 10 15, 11 20 21}	{5 2 11, 5 17 20}	{1 8 15, 1 3 11}	{4 11 18, 4 5 9}
{13 8 9, 13 3 5}	{13 4 21, 13 2 20}	{1 13 18, 1 4 16}	

and adding

{20 21 22, 20 5 17}	{11 2 20, 11 10 15}	{5 2 13, 5 3 11}	{4 11 21, 4 13 18}
{1 11 18, 1 3 13}	{9 4 5, 9 8 13}	{1 4 16, 1 8 15}	

X_{19} leave: Leave is
{{19,22},{20,22},{21,22}, {11,14}, {12,14}, {13,14}, {10,15}, {10,16}, {9,10}, {8,17}, {8,18}, {7,8}, {5,6}, {3,4}, {1,2}}.
Bowties are

{211719, 21816}, {4815, 41819}, {151419, 151622}, {111322, 111619},
{91422, 91319}, {21714, 2115}, {17310, 171620}, {2319, 2915},
{14620, 1424}, {11520, 11218}, {21210, 2139}, {549, 51219},
{3111, 3513}, {3612, 3822}, {22112, 2226}, {5715, 51022},
{151221, 15318}, {16314, 16518}, {7119, 7422}, {1610, 11318},
{4116, 41321}, {20412, 201019}, {101418, 10411}, {17212, 17713},
{201821, 2089}, {6718, 61121}, {6916, 61315}, {171822, 17514},
{6417, 6819}, {12918, 12811}, {121316, 12710}, {258, 2716},
{7911, 7320}, {13220, 13810}, {1719, 171115}, {1814, 11520}.

X_{20} leave: Leave is
{{19,22},{20,22},{21,22}, {17,19}, {18,19}, {11,14}, {12,14}, {13,14}, {9,10}, {10,15}, {10,16}, {7,8}, {5,6}, {3,4}, {1,2}}.
Bowties are obtained from those for X_{19} above, by removing

{211719, 21816} {4815, 41819} {151419, 151622}
{111322, 111619} {91422, 91319}

and adding

{81721, 81516} {4818, 41519} {161921, 161122}
{221415, 22913} {19914, 191113}

X_{21} leave: Leave is
{{19,22},{20,22},{21,22}, {17,19}, {18,19}, {11,14}, {12,14}, {13,14}, {11,15}, {11,16}, {9,10}, {7,8}, {5,6}, {3,4}, {1,2}}.
Bowties are obtained from those for X_{20} above, by removing

{2319, 2915} {11520, 11218} {5715, 51022} {16314, 16518}
{201821, 2089} {171822, 17514} {258, 2716} {7911, 7320}
{1719, 171115} {81721, 81516} {161921, 161122}

and adding

{1719, 171122} {161015, 161921} {16314, 161822} {8516, 81517}
{18520, 181721} {20821, 20911} {928, 9715} {5215, 5711}
{2319, 21118} {51022, 51417} {7216, 7320}

X_{22} leave: Leave is
{{19,22},{20,22},{21,22}, {17,19}, {18,19}, {15,20}, {16,20}, {11,14}, {12,14}, {13,14}, {9,10}, {7,8}, {5,6}, {3,4}, {1,2}}.
Bowties are obtained from those for X_{21} above, by removing

{17310, 171620} {3111, 3513} {17212, 17713} {1814, 11520}
{16314, 161822} {51022, 51417} {7216, 7320}

and adding

$\{1\,11\,15,\ 1\,8\,14\}$ $\{3\,1\,20,\ 3\,10\,17\}$ $\{7\,3\,13,\ 7\,17\,20\}$ $\{2\,7\,16,\ 2\,12\,17\}$
$\{16\,3\,11,\ 16\,14\,17\}$ $\{5\,3\,14,\ 5\,13\,17\}$ $\{22\,5\,10,\ 22\,16\,18\}$

In the general case, if $n = 12m + 10$ and $m \geq 3$, take the vertex set $\{(i,j) \mid 1 \leq i \leq 6m+5,\ j = 1,2\}$. A 3-GDD of type $11^1 3^{2m-2}$ exists for $m \geq 3$; take this on the set $\{1, 2, \ldots, 6m+5\}$. Now we take bowties as follows.

(1) On $\{(i,j) \mid 6m-5 \leq i \leq 6m+5,\ j = 1,2\}$, place a bowtie packing of K_{22} with any of the 22 X_i leaves.

(2) On $\{(i,j) \mid i = 3s-2, 3s-1, 3s;\ j = 1,2\}$, for each $s = 1, 2, \ldots, 2m-2$, place a bowtie packing of K_6 (which has leave a 1-factor).

(3) For each block abc in the 3-GDD above, on the set $\{(a,1),(a,2)\} \cup \{(b,1),(b,2)\} \cup \{(c,1),(c,2)\}$, place a decomposition of $K_{2,2,2}$ into bowties (Example 1.1).

The result is a bowtie packing of K_n for $n = 12m+10$, $m \geq 3$, and with each of the 22 leaves X_i possible.

The case K_{34} remains. Note that K_{16} has the same 22 leaves X_i as we expect here. So we construct a bowtie packing of order 34 with one of order 16 embedded in it.

Let the vertex set of K_{34} be $\{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 11,\ j = 1,2,3\}$. On the set $S = \{\infty\} \cup \{(i,j) \mid 1 \leq i \leq 5,\ 1 \leq j \leq 3\}$ we may place a bowtie packing of order 16 with any one of the 22 leaves X_i. We then have a further 144 triples, which can easily be paired into bowties, and also a further leave of nine edges:

$$\{\{(6,1),(6,2)\},\{(6,3),(7,1)\},\{(7,2),(7,3)\},\{(8,1),(8,2)\},\{(8,3),(9,1)\},\\ \{(9,2),(9,3)\},\{(10,1),(10,2)\},\{(10,3),(11,1)\},\{(11,2),(11,3)\}\}.$$

(These 9 edges are adjoined to the X_i leave from the bowtie packing of order 16 on the set S.) Then 9 of the 144 triples contain ∞:

$$\{\infty,(6,2),(6,3)\},\{\infty,(7,1),(7,2)\},\{\infty,(7,3),(8,1)\},\\ \{\infty,(8,2),(8,3)\},\{\infty,(9,1),(9,2)\},\{\infty,(9,3),(10,1)\},\\ \{\infty,(10,2),(10,3)\},\{\infty,(11,1),(11,2)\},\{\infty,(11,3),(6,1)\}.$$

Next, we use an idempotent commutative quasigroup $(Q, \circ)$ with $Q = \{1, 2, \ldots, 11\}$, and with a subquasigroup of order 5 on $\{1, 2, 3, 4, 5\}$:

∘	1	2	3	4	5	6	7	8	9	10	11
1	1	4	2	5	3	7	8	9	10	11	6
2	4	2	5	3	1	8	9	10	11	6	7
3	2	5	3	1	4	9	10	11	6	7	8
4	5	3	1	4	2	10	11	6	7	8	9
5	3	1	4	2	5	11	6	7	8	9	10
6	7	8	9	10	11	6	1	2	3	4	5
7	8	9	10	11	6	1	7	3	4	5	2
8	9	10	11	6	7	2	3	8	5	1	4
9	10	11	6	7	8	3	4	5	9	2	1
10	11	6	7	8	9	4	5	1	2	10	3
11	6	7	8	9	10	5	2	4	1	3	11

Then, using this quasigroup, 135 ($= 3 \times 45$) more triples are formed as follows:

$$\{(x,1),(y,1),(x \circ y,2)\},\ \{(x,2),(y,2),(x \circ y,3)\},\ \{(x,3),(y,3),(\alpha(x \circ y),1)\},$$

where α is the permutation (6 7 8 9 10 11), and where $x, y \in \{1,2,\ldots,11\}$ but x and y are not both chosen from $\{1,2,3,4,5\}$.

It is tedious but straightforward to pair these $9 + 135 = 144$ triples into 72 bowties.

This completes the case of order 34, and consequently the case $n = 12m + 10$.

6 Summary

We have now shown that all expected smallest leaves can be achieved. We restate this as follows.

Theorem 6.1 *A maximum packing of K_n with bowties exists for all $n \geq 5$ and with leave as in the following table, where X_i and Y_i are defined as in Figures 1 and 2 of Section 1 above.*

n (mod 12)	leave
1, 9	$\emptyset$
3, 7	K_3
5	C_4
0, 2, 6, 8	F, a 1-factor
4, 10, $n \geq 16$	X_i, $1 \leq i \leq 22$
11	Y_i, $1 \leq i \leq 4$
$n = 10$	X_i, $i = 1,2,4,6,7,8,12$–18

References

[1] E.J. Billington and C.C. Lindner, The spectrum for 2-perfect bowtie systems, *Discrete Mathematics* **135** (1994), 61–68.

[2] A.E. Brouwer, Optimal packings of K_4's into a K_n, *J. Combinatorial Theory (Ser A)* **26** (1979), 278–297.

[3] C.J. Colbourn, D.G. Hoffman and R. Rees, A new class of group divisible designs with block size three, *J. Combinatorial Theory (Ser A)* **59** (1992), 73–89.

[4] D.G. Hoffman, C.C. Lindner, M.J. Sharry and A.P. Street, Maximum packings of K_n with copies of $K_4 - e$, *Aequationes Mathematicae* **51** (1996), 247–269.

[5] D.G. Hoffman and W.D. Wallis, Packing complete graphs with squares, *Bull. I.C.A.* **1** (1991), 89–92. Corrigendum *ibid.* **3**.

[6] P. Horák and A. Rosa, Decomposing Steiner triple systems into small configurations, *Ars Combinatoria* **26** (1988), 91–105.

[7] Pat Boling, The bowtie algorithm for Steiner triple systems, PhD Thesis, Auburn University, 1997.

[8] J. Schönheim, On maximal systems of k-tuples, *Studia Sci. Math. Hung.* **1** (1966), 363–368.

[9] J. Schönheim and A. Bialostocki, Packing and covering the complete graph with 4-cycles, *Canad. Math. Bull.* **18** (1975), 703–708.

[10] R.G. Stanton and M.J. Rogers, Packings and coverings by triples, *Ars Combinatoria* **13** (1982), 61–69.